MODULI STACKS OF (φ, Γ)-MODULES: ERRATA

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The following errata, clarifications and minor improvements are for the published version of [EG22]. All references are to [EG22] unless otherwise specified.

We begin with some minor corrections.

- In the notation section (and throughout the book), the residue field of K is k.
- In Remark 2.1.13, $K_0(\zeta_{p^\infty})$ should be replaced by $(K_0)_{\mathrm{cyc}}$, and the claim that this condition is equivalent to K being abelian over $\mathbf{Q}_p$ should be deleted.

  The same applies in Definition 3.2.3 and Remark 3.2.4, and the discussion between them. In particular we should define $K^{\mathrm{basic}} := K \cap (K_0)_{\mathrm{cyc}}$.

  (We would like to thank Dat Pham for pointing this out. This issue was also explained to us by Léo Poyeton some years previously, but we unfortunately confused $K_0(\zeta_{p^\infty})$ and $(K_0)_{\mathrm{cyc}}$ in our writeup.)
- We should explicitly assume throughout that the coefficient field E contains K. (This is implicitly assumed at several points, but not always explicitly asserted.)
- The definition of the universal unramified character in §5.3 is incorrect if $k \neq \mathbf{F}_p$; instead, we need to set $\varphi(v) = a'v$ where $a' \in (\mathbf{F} \otimes_k A)^\times$ has norm a. See Dat Pham's [Pha22, §2]. (Note that this construction is one place that uses the assumption mentioned in the previous point.)
- In Hypothesis 7.3.1, we should take $b \geq peh/(p-1)$ rather than $b \geq eh/(p-1)$, because in the following paragraph, the valuation of T' should be given by $pv(p)/(p-1)$ rather than $v(p)/(p-1)$.

  In the second paragraph of the proof of Lemma 7.3.5, the image of T' should be $\zeta_{p^s} - 1$, rather than $\zeta_{p^{s+1}} - 1$. Thus the image of T coincides with the image of the trace of $\zeta_{p^s} - 1$, which is in $K_{\mathrm{cyc},s}$ and thus fixed by $G_{K_{\mathrm{cyc},s}}$.

  (This was again pointed out to us by Dat Pham.)
- Strictly speaking, the proof of Theorem 8.6.2 is incomplete (but the gap is easily filled); see the discussion between the statement and proof of [CEGS22, Thm. 7.6] for a complete proof.
- In the first paragraph of the proof of Theorem 6.3.2, the representation $\overline{\rho}$ (in the deformation ring $R^{\mathrm{crys},\underline{\lambda}}_{\overline{\rho}}$) is equal to $\overline{\rho}_d$. (Thanks to Matteo Tamiozzo for this.)
- Contrary to the claim in Section 5.3, it is not always possible to choose the characters $\psi_{\underline{n}}$ in such a way that if $\underline{n}$, $\underline{n}'$ have $(\psi_{\underline{n}}\psi_{\underline{n}'}^{-1})|_{I_K} = \overline{\epsilon}|_{I_K}$, then in fact $\psi_{\underline{n}}\psi_{\underline{n}'}^{-1} = \overline{\epsilon}$. The only place that this assumption was used was (implicitly) in the definitions of the characters $\omega_{\underline{k},i}$ before Definition 5.5.11; accordingly, we simply replace the $\omega_{\underline{k},i}$ by unramified twists in such a way

that if $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = p-1$ for all $\overline{\sigma}$, then $\omega_{\underline{k},i} = \omega_{\underline{k},i+1}$. (Again, we thank Dat Pham for pointing this out to us.)

We would like to thank Dat Pham for pointing out that it is not clear that the proof of Theorem 5.5.12 is complete. More precisely, without making additional arguments, it is not obvious that our constructions cover all of $\mathcal{X}_{d,\mathrm{red}}(\overline{\mathbf{F}}_p)$. Since the proof of Theorem 5.5.12 is quite involved, rather than attempt to describe the additional arguments needed, we instead give a slightly different and more streamlined complete proof of a slightly stronger result. We would also like to thank Jack Sempliner for pointing out some typos in the formulas at the end of the argument, which we correct here.

In giving this more streamlined argument, we also include some information about extension classes in maximally nonsplit representations, which is exploited in [BCGN23]. In order to do this we introduce the following notation. Suppose that we have a representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ which is maximally nonsplit of niveau one and weight $\underline{k}$, and write $(t_1, \ldots, t_d) \in (\mathbf{G}_m)^d_{\underline{k}}(\overline{\mathbf{F}}_p)$ for the image of $\overline{\rho}$ under the eigenvalue morphism. Then we can can write $\overline{\rho}$ as

$$\begin{pmatrix} \mathrm{ur}_{t_d}\omega_{\underline{k},d} & * & \ldots & * \\ 0 & \mathrm{ur}_{t_{d-1}}\overline{\epsilon}^{-1}\omega_{\underline{k},d-1} & \ldots & * \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \mathrm{ur}_{t_1}\overline{\epsilon}^{1-d}\omega_{\underline{k},1} \end{pmatrix}$$

For each $i = 1, \ldots, d-1$, we view $\mathrm{Ext}^1_{G_K}(\mathrm{ur}_{t_i}\overline{\epsilon}^{i-d}\omega_{\underline{k},i}, \mathrm{ur}_{t_{i+1}}\overline{\epsilon}^{i+1-d}\omega_{\underline{k},i+1})$ as an affine space over $\overline{\mathbf{F}}_p$, and we let

$$\mathrm{Ext}^1_{(t_1,\ldots,t_d),\underline{k}} \subseteq \prod_{i=1}^{d-1} \mathrm{Ext}^1_{G_K}(\mathrm{ur}_{t_i}\overline{\epsilon}^{i-d}\omega_{\underline{k},i}, \mathrm{ur}_{t_{i+1}}\overline{\epsilon}^{i+1-d}\omega_{\underline{k},i+1})$$

be the closed subvariety determined by the condition that for each $i = 1, \ldots, d-2$, the cup product of the classes in $\mathrm{Ext}^1_{G_K}(\mathrm{ur}_{t_i}\overline{\epsilon}^{i-d}\omega_{\underline{k},i}, \mathrm{ur}_{t_{i+1}}\overline{\epsilon}^{i+1-d}\omega_{\underline{k},i+1})$ and $\mathrm{Ext}^1_{G_K}(\mathrm{ur}_{t_{i+1}}\overline{\epsilon}^{i-d}\omega_{\underline{k},i+1}, \mathrm{ur}_{t_{i+2}}\overline{\epsilon}^{i+2-d}\omega_{\underline{k},i+2})$ is zero. (Of course in many cases thes cup products vanish automatically because they land in a vanishing Ext^2 group, in which case this condition is empty.) Note that the successive extensions of characters in $\overline{\rho}$ (i.e. the superdiagonal entries in the matrix above) determine a point of $\mathrm{Ext}^1_{(t_1,\ldots,t_d),\underline{k}}$.

In the proof of Theorem 5.5.12 given here we allow ourselves to refer forwards to Theorem 6.5.1, which strengthens some of the conclusions of Theorem 5.5.12, showing that the stacks $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ in the statement of Theorem 5.5.12 are empty. This requires some justification, because the proof of Theorem 6.5.1 relies on Theorem 5.5.12! However, the proof of Theorem 5.5.12 is by induction on d, and the proof of Theorem 6.5.1 for any particular d only uses Theorem 5.5.12 for that value of d, so we are free to assume in the proof of Theorem 5.5.12 that $\mathcal{X}^{\mathrm{small}}_{d',\mathrm{red},\overline{\mathbf{F}}_p}$ is empty for each $d' < d$. Accordingly, we have also removed $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ from the statement of the theorem.

Theorem (Theorem 5.5.12).

(1) *The Ind-algebraic stack $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack, of finite presentation over $\mathbf{F}$.*

(2) *$\mathcal{X}_{d,\mathrm{red}}$ is equidimensional of dimension $[K : \mathbf{Q}_p]d(d-1)/2$. We can write $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ as a union of irreducible components $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, where each $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is generically maximally nonsplit of niveau one and weight $\underline{k}$.*

More precisely, there is a dense open substack $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ which is maximally nonsplit of niveau one and weight $\underline{k}$. Furthermore, the eigenvalue morphism on $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is dominant (i.e. has dense image in $(\mathbf{G}_m)^d_{\underline{k}}$), and if $(t_1, \dots, t_d)$ is an $\overline{\mathbf{F}}_p$-point in the image of the eigenvalue morphism, then there is a dense Zariski open subset of $\mathrm{Ext}^1_{(t_1,\dots,t_d),\underline{k}}$ such that for any point $(\psi_1, \dots, \psi_{d-1})$ in this subset, there is a point $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}(\overline{\mathbf{F}}_p)$ whose corresponding Galois representation

$$\begin{pmatrix} \mathrm{ur}_{t_d}\omega_{\underline{k},d} & * & \dots & * \\ 0 & \mathrm{ur}_{t_{d-1}}\overline{\epsilon}^{-1}\omega_{\underline{k},d-1} & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{ur}_{t_1}\overline{\epsilon}^{1-d}\omega_{\underline{k},1} \end{pmatrix}$$

realizes the chosen extensions ψ_i.

(3) *If we fix an irreducible representation $\overline{\alpha} : G_K \to \mathrm{GL}_a(\overline{\mathbf{F}}_p)$ (for some $a \geq 1$), then the locus of $\overline{\rho}$ in $\mathcal{X}_{d,\mathrm{red}}(\overline{\mathbf{F}}_p)$ for which $\dim \mathrm{Hom}_{G_K}(\overline{\rho}, \overline{\alpha}) \geq r$ (for any $r \geq 1$) is (either empty, or) of dimension at most*

$$[K : \mathbf{Q}_p]d(d-1)/2 - \lceil r\big((a^2+1)r - a\big)/2 \rceil.$$

(4) *If we fix an irreducible representation $\overline{\alpha} : G_K \to \mathrm{GL}_a(\overline{\mathbf{F}}_p)$ (for some $a \geq 1$), then the locus of $\overline{\rho}$ in $\mathcal{X}_{d,\mathrm{red}}(\overline{\mathbf{F}}_p)$ for which $\dim \mathrm{Ext}^2_{G_K}(\overline{\alpha}, \overline{\rho}) \geq r$ is of dimension at most*

$$[K : \mathbf{Q}_p]d(d-1)/2 - r.$$

Proof. Recall that a closed immersion of reduced algebraic stacks that are locally of finite type over $\mathbf{F}_p$, which is surjective on finite type points, is necessarily an isomorphism. As recalled above, $\mathcal{X}_{d,\mathrm{red}}$ is an inductive limit of such stacks (indeed, by Lemma A.9 , we have $\mathcal{X}_{d,\mathrm{red}} = \varinjlim \mathcal{X}^a_{d,h,s,\mathrm{red}}$, where the $\mathcal{X}^a_{d,h,s}$ are as in Section 3.4), and so if we produce closed algebraic substacks $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ and $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $\mathcal{X}_d$, the union of whose $\overline{\mathbf{F}}_p$-points exhausts those of $\mathcal{X}_{d,\mathrm{red}}$, then $\mathcal{X}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ will in fact be an algebraic stack which is the union of its closed substacks $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ and $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. Thus (1) is an immediate consequence of (2) (where the "union" statement in (2) is now to be understood on the level of $\overline{\mathbf{F}}_p$-points). In fact, it suffices to construct the closed substacks $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$, and to show that the remaining $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{d,\mathrm{red}}$ are contained in a closed substack $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, which is a finite union of finite type algebraic substacks of dimension less than $[K : \mathbf{Q}_p]d(d-1)/2$; indeed, this obviously suffices to prove (1), and it also suffices to prove (2) by an application of Theorem 6.5.1 . (See the discussion before the theorem for an explanation of why this is not a circular argument.)

Claim (4) follows from (3) (with $\overline{\alpha}$ replaced by $\overline{\alpha} \otimes \overline{\epsilon}$) by Tate local duality and the easily verified inequality

$$\lceil r\big((a^2+1)r - a\big)/2 \rceil \geq r.$$

Thus it is enough to prove (2) and (3), which we do simultaneously by induction on d.

As recalled in Remark 5.3.4, there are up to twist by unramified characters only finitely many irreducible $\overline{\mathbf{F}}_p$-representations of G_K of any fixed dimension. Accordingly, we let $\{\overline{\alpha}_i\}$ be a finite set of irreducible continuous representations $\overline{\alpha}_i : G_K \to \mathrm{GL}_{d_i}(\overline{\mathbf{F}}_p)$, such that any irreducible continuous representation of G_K over $\overline{\mathbf{F}}_p$ of dimension at most d arises as an unramified twist of exactly one of the $\overline{\alpha}_i$. We let the 1-dimensional representations in this set be the characters $\psi_{\underline{n}}$ defined in Section 5.3.

Each $\overline{\alpha}_i$ corresponds to a finite type point of $\mathcal{X}_{d_i,\mathrm{red}}$, whose associated residual gerbe is a substack of $\mathcal{X}_{d_i,\mathrm{red}}$ of dimension -1: the morphism $\operatorname{Spec}\overline{\mathbf{F}}_p \to \mathcal{X}_{d_i,\mathrm{red}}$ corresponding to $\overline{\alpha}_i$ factors through a monomorphism $[\operatorname{Spec}\overline{\mathbf{F}}_p/\mathbf{G}_m] \to \mathcal{X}_{d_i,\mathrm{red}}$. It follows from Lemma 5.3.2 that for each $\overline{\alpha}_i$ there is an irreducible closed zero-dimensional algebraic substack of $\mathcal{X}_{d_i,\mathrm{red}}$ of finite presentation over $\overline{\mathbf{F}}_p$ which contains a dense open substack whose $\overline{\mathbf{F}}_p$-points are the unramified twists of $\overline{\alpha}_i$.

In particular, if $d = 1$, then we let $\mathcal{X}^{\underline{k}}_{1,\mathrm{red}}$ be the zero-dimensional stack constructed in the previous paragraph; this satisfies the required properties by definition, so (2) holds when $d = 1$. For (3), note that if $r > 0$ then we must have $a = 1$, and then the locus where $\operatorname{Hom}_{G_K}(\overline{\rho}, \overline{\alpha})$ is non-zero (equivalently, 1-dimensional) is exactly the closed substack of dimension -1 corresponding to $\overline{\alpha}$, so the required bound holds.

We now begin the inductive proof of (2) and (3) for $d > 1$. In fact, it will be helpful to simultaneously prove additional statements (2') and (4'), which we begin by stating. (It is trivial to verify that the discussion above also proves (2') and (4') when $d = 1$.)

(2') is as follows: for each $\underline{k}$, there is a closed irreducible algebraic substack $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ of finite presentation over $\overline{\mathbf{F}}_p$ and dimension $[K:\mathbf{Q}_p]d(d-1)/2 - 1$, which contains a dense open substack $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ which is maximally nonsplit of niveau 1 and of weight $\underline{k}$, and has the property that the corresponding character ν_1 is trivial. Furthermore, the stack $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of (2) is obtained from $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ by unramified twisting, and $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is precisely the closed substack of $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ on which $\nu_1 = 1$. (Note that since $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is maximally nonsplit with $\nu_1 = 1$, both $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ and $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ are indeed twistable.) Finally, the analogue for $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of the rest of (2) should hold: i.e. the eigenvalue morphism on $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ has dense image in the closed subscheme of $(\mathbf{G}_m)^d_{\underline{k}}$ where $\nu_1 = 1$, and for any $\overline{\mathbf{F}}_p$-point $(1 = t_1, \dots, t_d)$ in the image of the eigenvalue morphism on $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, all extension classes in some dense Zariski open subset of $\operatorname{Ext}^1_{(t_1,\dots,t_d),\underline{k}}$ are witnessed by $\overline{\mathbf{F}}_p$-points of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$.

We now turn to the formulation of (4'). To this end, for each character $\overline{\alpha} : G_K \to \overline{\mathbf{F}}_p^{\times}$, and each $\underline{k}$, we let $\mathcal{X}_{\overline{\alpha}}$ be the closed substack consisting of those $\overline{\rho}$ in $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}(\overline{\mathbf{F}}_p)$ for which $\operatorname{Ext}^2_{G_K}(\overline{\alpha}, \overline{\rho}) \neq 0$. (It is closed since, by the compatibility of the formation of H^2 with arbitrary finite type base change, $\mathcal{X}_{\overline{\alpha}}$ is the support of the coherent sheaf $H^2(G_K,\overline{\rho}\otimes\overline{\alpha}^\vee)$, where $\overline{\rho}$ abusively denotes the universal (φ,Γ)-module.) By (4), the dimension of $\mathcal{X}_{\overline{\alpha}}$ is at most $[K:\mathbf{Q}_p]d(d-1)/2-1$. Statement (4') is then formulated as follows: if $\mathcal{X}_{\overline{\alpha}}$ has dimension $[K:\mathbf{Q}_p]d(d-1)/2-1$, then after replacing $\overline{\alpha}$ by an unramified twist, $\mathcal{X}_{\overline{\alpha}}$ contains $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, and the complement $\mathcal{X}_{\overline{\alpha}}\setminus(\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}\cap\mathcal{X}_{\overline{\alpha}})$ has dimension at most $[K:\mathbf{Q}_p]d(d-1)/2-2$.

We now prove the inductive step, so we assume that (2), (2'), (3) and (4') hold in dimension less than d. Let $\underline{k}_{d-1}$ be the Serre weight in dimension $(d-1)$ obtained by deleting the first entry in $\underline{k}$. Set $\overline{\alpha}:=\overline{\epsilon}^{1-d}\omega_{\underline{k},1}$. If $k_{\overline{\sigma},1}-k_{\overline{\sigma},2}=p-1$ for all $\overline{\sigma}$ then we say that we are in the *très ramifiée case*, and we let $\mathcal{U}$ denote $\mathcal{U}^{\underline{k}_{d-1},\mathrm{fixed}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$; otherwise, we let $\mathcal{U}$ denote $\mathcal{U}^{\underline{k}_{d-1}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$. Write $\bar{r}_u$ for the $(d-1)$-dimensional representation corresponding to u. Note that by Tate local duality, $\dim\mathrm{Ext}^2_{G_K}(\overline{\alpha},\bar{r}_u)=\dim\mathrm{Hom}_{G_K}(\bar{r}_u,\overline{\alpha}\otimes\overline{\epsilon})$, so in the très ramifiée case, $\mathrm{Ext}^2_{G_K}(\overline{\alpha},\bar{r}_u)$ is 1-dimensional for each $\overline{\mathbf{F}}_p$-point u of $\mathcal{U}$. If we are not in the très ramifiée case, then after deleting the locus where the unique quotient character of $\bar{r}_u$ is equal to $\overline{\alpha}\otimes\overline{\epsilon}$ from $\mathcal{U}^{\underline{k}_{d-1}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$, we can and do assume that for each $\overline{\mathbf{F}}_p$-point u of $\mathcal{U}$, we have $\mathrm{Ext}^2_{G_K}(\overline{\alpha},\bar{r}_u)=0$.

We let T be an irreducible scheme which smoothly covers $\mathcal{U}$, and we let $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ be the irreducible closed substack of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ constructed as the scheme-theoretic image of the (total space of the) vector bundle V in the notation of Proposition 5.4.4 . Part (2) of that proposition, together with the inductive hypothesis, implies that $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ has the claimed dimension. (Note in particular that if $K=\mathbf{Q}_p$, then condition (2d) of Proposition 5.4.4 holds. Indeed, if we are in the très ramifiée case then condition (2d)(iii) holds on all of T, and otherwise the inductive hypothesis that the image of the eigenvalue morphism is dense in $(\mathbf{G}_m)^{d-1}_{\underline{k}_{d-1}}$ implies that condition (2d)(i) holds on a dense open subscheme of T.) Furthermore, by parts (2) and (3) of Proposition 5.4.4 , there is a dense open substack $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ which is maximally nonsplit of niveau one and weight $\underline{k}$. (To see in the très ramifiée case that the hypotheses of part (3) of Proposition 5.4.4 hold, note that if $k_{\overline{\sigma}_2}-k_{\overline{\sigma}_3}=p-1$ for all $\overline{\sigma}$ then either $\overline{\epsilon}$ is trivial and we satisfy (3)(b)(ii), or $\overline{\epsilon}$ is nontrivial and we satisfy (3)(b)(i); and otherwise after possibly deleting a closed locus in the image of the eigenvalue morphism, we are in case (3)(b)(i).)

By construction (and the inductive hypothesis), the eigenvalue morphism on $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ has dense image in the closed subscheme of $(\mathbf{G}_m)^d_{\underline{k}}$ where $\nu_1=1$. We need to check that for any $\overline{\mathbf{F}}_p$-point $(1=t_1,\dots,t_d)$ in the image of this morphism, the $\overline{\mathbf{F}}_p$-points of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ witness all extension classes in some dense Zariski open subset of $\mathrm{Ext}^1_{(t_1,\dots,t_d),\underline{k}}$. In order to do this we introduce some notation. Let $\mathcal{U}_{(t_2,\dots,t_d)}$ be the fibre over $(t_2,\dots,t_d)$ of the eigenvalue morphism on $\mathcal{U}$, and for each $u\in\mathcal{U}$, write $\bar{s}_u$ for the unique $(d-2)$-dimensional subpresentation of $\bar{r}_u$. We write $\overline{\gamma}_u:=\overline{\epsilon}^{3-d}\omega_{\underline{k},3}$ for unique quotient character of $\bar{s}_u$, and $\overline{\beta}_u:=\overline{\epsilon}^{2-d}\omega_{\underline{k},2}$ for the unique quotient character of $\bar{r}$. Write $c_u\in\mathrm{Ext}^1_{G_K}(\overline{\beta}_u,\overline{\gamma}_u)$ for the class induced by $\bar{r}_u$.

We now examine the construction of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ slightly more closely. As above, Proposition 5.4.4 provides a vector bundle V over a smooth cover of a dense open substack of $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ which parameterises the universal family of extensions of $\overline{\alpha}$ by $\mathcal{U}$, and $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is defined to be the complement of the scheme-theoretic image of a proper subbundle of V. This proper subbundle is the locus where the induced class in $\mathrm{Ext}^1(\overline{\alpha},\overline{\beta}_u)$ is respectively peu ramifiée (in the très ramifiée case) or 0 (in the other cases). (It is shown in the proof of Proposition 5.4.4 that this locus does indeed determine a proper subbundle of V.)

By the definition of $\mathrm{Ext}^1_{(t_1,\dots,t_d),\underline{k}}$, it therefore suffices to show that for each $u\in\mathcal{U}$, the image of the induced map

$$\mathrm{Ext}^1_{G_K}(\overline{\alpha},\bar{r}_u)\to\mathrm{Ext}^1_{G_K}(\overline{\alpha},\overline{\beta}_u)$$

is precisely the kernel of the map

$$\tilde{c}:\mathrm{Ext}^1_{G_K}(\overline{\alpha},\overline{\beta}_u)\xrightarrow{\cup c_u}\mathrm{Ext}^2_{G_K}(\overline{\alpha},\overline{\gamma}_u).$$

To see this, note firstly that $\tilde{c}$ can also be written as the composite

$$\mathrm{Ext}^1_{G_K}(\overline{\alpha},\overline{\beta}_u)\to\mathrm{Ext}^2_{G_K}(\overline{\alpha},\bar{s}_u)\to\mathrm{Ext}^2(\overline{\alpha},\overline{\gamma}_u),$$

where the first map is a boundary map in the long exact sequence

$$\cdots\to\mathrm{Ext}^1_{G_K}(\overline{\alpha},\bar{r}_u)\to\mathrm{Ext}^1_{G_K}(\overline{\alpha},\overline{\beta}_u)\to\mathrm{Ext}^2_{G_K}(\overline{\alpha},\bar{s}_u)\cdots\to$$

In particular the image of $\mathrm{Ext}^1_{G_K}(\overline{\alpha},\bar{r}_u)\to\mathrm{Ext}^1_{G_K}(\overline{\alpha},\overline{\beta}_u)$ is the kernel of $\mathrm{Ext}^1_{G_K}(\overline{\alpha},\overline{\beta}_u)\to\mathrm{Ext}^2_{G_K}(\overline{\alpha},\bar{s}_u)$, so it suffices to check that the natural morphism $\mathrm{Ext}^2_{G_K}(\overline{\alpha},\bar{s}_u)\to\mathrm{Ext}^2_{G_K}(\overline{\alpha},\overline{\gamma}_u)$ is an isomorphism. But this morphism is Tate dual to the natural morphism $\mathrm{Hom}_{G_K}(\overline{\gamma}_u,\overline{\alpha}\otimes\bar{\epsilon})\to\mathrm{Hom}_{G_K}(\bar{s}_u,\overline{\alpha}\otimes\bar{\epsilon})$, which in turn is an isomorphism because $\bar{s}_u$ is maximally nonsplit, so we have completed the verification of (2').

We then let $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ be the substack obtained from $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ by twisting by unramified characters, which has the claimed dimension (i.e. $[K:\mathbf{Q}_p]d(d-1)/2$) by Lemma 5.3.2 , and we let $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ be a dense open substack which is maximally nonsplit of niveau one and weight $\underline{k}$. We can furthermore arrange that in $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}\cap\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is nonempty; indeed, if this is not the case, we can arrange it replacing $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ by its twist by a constant unramified character.

It will be helpful in what follows to arrange things so that $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is precisely the closed substack $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ on which $\nu_1=1$. (Given the inherent ambiguity in the various choices of open substacks that we have made, so far neither of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ nor $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ need be contained in the other.) Thus we now modify our constructions in order to achieve this. To begin with, we consider the closed substack $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. This has dimension $[K:\mathbf{Q}_p]d(d-1)/2-1$. By construction $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}\cap\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is a non-empty open, and hence dense, substack of the irreducible stack $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, which thus has dimension equal to the dimension of this latter stack, namely $[K:\mathbf{Q}_p]d(d-1)/2-1$; it is also contained in $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. Thus its closure in $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is an irreducible component of this latter stack; let $\mathcal{Z}$ denote the union of all the *other* irreducible components of $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$.

We now replace $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ by $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p} \setminus \mathcal{Z}$. Then, by construction, $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p} \cap \mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is dense in $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. In particular, since $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is contained in the closed substack $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, we see that $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is contained in $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. We now redefine $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ to be $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. By construction, this latter locus is dense in $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, and thus (being itself closed in $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$) is equal to $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p} \cap \mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$.

In summary we have now constructed irreducible closed substacks $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ and $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$, of dimensions $[K:\mathbf{Q}_p]d(d-1)/2-1$ and $[K:\mathbf{Q}_p]d(d-1)/2]$ respectively; and a dense open substack $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ which is maximally non-split and on which the eigenvalue morphism to $(\mathbf{G}_m)^d_{\underline{k}}$ is defined and dominant, and such that the intersection $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p} := \mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p} \cap \mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is non-empty (and thus dense in $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$), and coincides with $\mathcal{U}^{\underline{k},\nu_1=1}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. Furthermore, we have established the claimed genericity property of the successive extension classes in the fibres of the eigenvalue morphism. In other words, the stacks $\mathcal{X}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red}}$, $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ and $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ satisfy the properties required of them by (2) and (2') (other, of course, than the claim that the $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ exhaust the irreducible components of $\mathcal{X}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$).

To complete the proof of (2), we need to construct $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. Let $\mathcal{X}^{\mathrm{small}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$ denote the union of the closed substacks

$$\mathcal{X}^{\underline{k}_{d-1}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p} \setminus \mathcal{U}^{\underline{k}_{d-1}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}.$$

By Proposition 5.4.4 , Tate local duality, upper semi-continuity of the fibre dimension, and the inductive hypothesis, we see that for each $1 \le i \le d$, each $\overline{\alpha}_i$ (of dimension a_i, say), and each $s \ge 0$ there is a finitely presented closed algebraic substack $\mathcal{X}_{s,\overline{\alpha}_i,\overline{\mathbf{F}}_p}$ of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$, whose $\overline{\mathbf{F}}_p$-points contain all the representations of the form $0 \to \overline{\rho}_{d-a_i} \to \overline{\rho} \to \overline{\alpha}_i \to 0$ for which $\dim_{\overline{\mathbf{F}}_p} \mathrm{Ext}^2_{G_K}(\overline{\alpha}_i, \overline{\rho}_{d-a_i}) = s$, and whose dimension is at most

$$[K:\mathbf{Q}_p](d-a_i)(d-a_i-1)/2 - \lceil s(({a_i}^2+1)s-a_i)/2\rceil + [K:\mathbf{Q}_p]a_i(d-a_i) + s - 1.$$

Furthermore, if $a=1$, then the locus where $\overline{\rho}_{d-1}$ is an $\overline{\mathbf{F}}_p$-point of $\mathcal{X}^{\mathrm{small}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$ is of dimension strictly less than this. (Here we use (4'), which shows in particular that the locus in $\mathcal{X}^{\mathrm{small}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$ of points where $\dim_{\overline{\mathbf{F}}_p} \mathrm{Ext}^2_{G_K}(\overline{\alpha}_i, \overline{\rho}_{d-1}) = 1$ has dimension at most $[K:\mathbf{Q}_p](d-1)(d-2)/2-2$.)

These stacks are only nonzero for finitely many values of s. For fixed a_i, we see that as a function of s, this quantity is maximised by $s=0$, as well as by $s=1$ when $a_i = 1$. (To see this, we have to maximise the quantity $s - \lceil s(({a_i}^2+1)s-a_i)/2\rceil$. Suppose firstly that $a_i > 1$. Then if $s=0$ we have 0, while if $s>0$ we have $s - \lceil s(({a_i}^2+1)s-a_i)/2\rceil \le s - s(({a_i}^2+1)s-a_i)/2 \le s - s({a_i}^2+1-a_i)/2 \le s - 3s/2 < 0$. Meanwhile if $a_i = 1$, then for $s=0$ we have 0, for $s=1$ we have $1 - \lceil 1/2 \rceil = 1$, while for $s > 1$ we have $s - \lceil s(2s-1)/2\rceil \le s - s(2s-1)/2 \le s - 3s/2 < 0$.) It follows that as a function of a_i the bound is maximised at $a_i = 1$ and $s = 0$ or 1, when it is equal to $[K:\mathbf{Q}_p]d(d-1)/2-1$, and it is otherwise strictly smaller.

By Lemma 5.3.2, it follows that the locus in $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ of representations of the form $0 \to \overline{\rho}_{d-a_i} \to \overline{\rho} \to \overline{\alpha}' \to 0$, with $\overline{\alpha}'$ an unramified twist of $\overline{\alpha}_i$ for which $\dim_{\overline{\mathbf{F}}_p} \mathrm{Ext}^2_{G_K}(\overline{\alpha}', \overline{\rho}_{d-a_i}) = s$, is of dimension at most $[K:\mathbf{Q}_p]d(d-1)/2$, with equality holding only if $a_i = 1$ and $s = 0$ or 1.

Putting this together, we claim that (2) holds in dimension d if we take $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ to be the union of the twists by unramified characters of the substacks $\mathcal{X}_{s,\overline{\alpha}_i}$ for which $\dim \overline{\alpha}_i > 1$ or $s > 1$, together with the union of the twists by unramified characters of the substacks of the $\mathcal{X}_{s,\overline{\alpha}_i,\overline{\mathbf{F}}_p}$ for which $\dim \overline{\alpha}_i = 1$, $s = 0$ or 1, and $\overline{\rho}_{d-1}$ is an $\overline{\mathbf{F}}_p$-point of $\mathcal{X}^{\mathrm{small}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$. Indeed, by construction, $\mathcal{X}^{\mathrm{small}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is certainly a finite union of closed substacks of dimension less than $[K:\mathbf{Q}_p]d(d-1)/2$, and in order to see that it exhausts the remaining points of $\mathcal{X}_{d,\mathrm{red},\overline{\mathbf{F}}_p}(\overline{\mathbf{F}}_p)$, we only need to verify that if $\underline{k}$ is in the très ramifiée case, and $\overline{\rho}$ corresponds to a point of $\mathcal{U}^{\underline{k}_{d-1}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$ with $\dim_{\overline{\mathbf{F}}_p} \mathrm{Ext}^2_{G_K}(\overline{\alpha}, \overline{\rho}) = 1$, then every extension of $\overline{\alpha}$ by $\overline{\rho}$ is contained in $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. By construction, it is enough to check that such a $\overline{\rho}$ is automatically a point of $\mathcal{U}^{\underline{k}_{d-1},\mathrm{fixed}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$. To see this, note that $\mathrm{Ext}^2_{G_K}(\overline{\alpha}, \overline{\rho}) \neq 0$ and that $\overline{\rho}$ is maximally nonsplit force $\overline{\rho}$ to be a point of $\mathcal{U}^{\underline{k}_{d-1},\nu_1=1}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$, which is equal to $\mathcal{U}^{\underline{k}_{d-1},\mathrm{fixed}}_{d-1,\mathrm{red},\overline{\mathbf{F}}_p}$ by the inductive hypothesis. This completes the proof of (2).

We now prove (3) in dimension d. In the case $r = 0$, there is nothing to prove. If $\dim \mathrm{Hom}_{G_K}(\overline{\rho}, \overline{\alpha}) \geq r \geq 1$, then we may place $\overline{\rho}$ in a short exact sequence

$$0 \to \overline{\theta} \to \overline{\rho} \to \overline{\alpha}^{\oplus r} \to 0,$$

where $\overline{\theta}$ is of dimension $d - ra < d$. We may apply part (2) so as to find that $\mathcal{X}_{d-ar,\mathrm{red},\overline{\mathbf{F}}_p}$ has dimension at most $[K:\mathbf{Q}_p](d-ar)(d-ar-1)/2$. Let $\mathcal{U}_s$ be the locally closed substack of $\mathcal{X}_{d-ar,\mathrm{red},\overline{\mathbf{F}}_p}$ over which $\dim H^2(G_K, \overline{\theta} \otimes \overline{\alpha}^\vee) = s$; by the inductive hypothesis, this locus has dimension at most $[K:\mathbf{Q}_p](d-ar)(d-ar-1)/2 - s\big((a^2+1)s - a\big)/2$, and over this locus we may construct a universal family of extensions

$$0 \to \overline{\theta} \to \overline{\rho}_{\mathcal{U}_s} \to \overline{\alpha}^{\oplus r} \to 0.$$

The locus of $\overline{\rho}$ we are interested in is contained in the scheme-theoretic image of this family in $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$, and Proposition 5.4.4 shows that this scheme-theoretic image has dimension bounded above by

$$\begin{aligned}
&[K:\mathbf{Q}_p](d-ar)(d-ar-1)/2 - s((a^2+1)s-a)/2 + r([K:\mathbf{Q}_p]a(d-ar)+s) - r^2 \\
=&[K:\mathbf{Q}_p]d(d-1)/2 - r((a^2+1)r-a)/2 - (r-s)^2/2 - as(as-1)/2 + (1-[K:\mathbf{Q}_p])(ar(ar-1))/2 \\
\leq&[K:\mathbf{Q}_p]d(d-1)/2 - r((a^2+1)r-a)/2.
\end{aligned}$$

Since this conclusion holds for each of the finitely many values of s (and since the dimension is an integer, allowing us to take the floor of this upper bound), we have proved (3).

Finally, we prove (4') in dimension d. We may assume that $\mathcal{X}_{\overline{\alpha}}$ has dimension $[K:\mathbf{Q}_p]d(d-1)/2 - 1$, or there is nothing to prove. By Tate local duality, the condition $\mathrm{Ext}^2_{G_K}(\overline{\alpha}, \overline{\rho}) \neq 0$ is equivalent to $\mathrm{Hom}_{G_K}(\overline{\rho}, \overline{\alpha} \otimes \overline{\epsilon}) \neq 0$; so $\overline{\rho}$ is a point of $\mathcal{X}_{\overline{\alpha}}$ if and only if $\overline{\rho}$ admits $\overline{\alpha} \otimes \overline{\epsilon}$ as a quotient character, and in particular $\mathcal{X}_{\overline{\alpha}}$

is essentially twistable. Furthermore, after replacing $\overline{\alpha}$ by an unramified twist, $\mathcal{X}_{\overline{\alpha}}$ contains $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ (i.e. we take $\overline{\alpha}$ to be the unique quotient character of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$).

Write $\mathcal{X}'_{\overline{\alpha}} := \mathcal{X}_{\overline{\alpha}} \setminus (\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p} \cap \mathcal{X}_{\overline{\alpha}})$, and suppose for the sake of contradiction that this has dimension $[K:\mathbf{Q}_p]d(d-1)/2-1$. Since $\mathcal{X}_{\overline{\alpha}}$ is essentially twistable, so is $\mathcal{X}'_{\overline{\alpha}}$, so the closed substack of $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ obtained by taking unramified twists of $\mathcal{X}_{\overline{\alpha}}$ has dimension $[K:\mathbf{Q}_p]d(d-1)/2$, so is equal to $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. In particular we see that there is a dense open substack of $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ whose $\overline{\mathbf{F}}_p$-points are unramified twists of $\overline{\mathbf{F}}_p$-points of $\mathcal{X}'_{\overline{\alpha}}$.

Since there is also a dense open substack of $\mathcal{U}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ whose $\overline{\mathbf{F}}_p$-points are unramified twists of $\overline{\mathbf{F}}_p$-points of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, we conclude that there is an $\overline{\mathbf{F}}_p$-point of $\mathcal{X}'_{\overline{\alpha}}$ which is an unramified twist of an $\overline{\mathbf{F}}_p$-point of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$. But the only unramified twist of an $\overline{\mathbf{F}}_p$-point of $\mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ which admits $\overline{\alpha}$ as a quotient is the trivial twist, so we conclude that $\mathcal{X}'_{\overline{\alpha}} \cap \mathcal{U}^{\underline{k},\mathrm{fixed}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ is nonempty, a contradiction. This completes the proof of the theorem. □

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