

PRESCRIBED LIFTS OF 2-DIMENSIONAL REPRESENTATIONS

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ABSTRACT. Let F be a totally real field, and let p be prime. Under standard Taylor–Wiles hypotheses, we show that an irreducible totally odd Galois representation $\bar{\rho} : \text{Gal}_F \rightarrow \text{GL}_2(\overline{\mathbf{F}}_p)$ admits lifts lying on arbitrary prescribed components of local deformation rings, allowing potentially semistable conditions with arbitrary regular Hodge–Tate weights.

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1. INTRODUCTION

Let F be a totally real field, and let Σ be a finite set of finite places of F containing all places above a fixed prime p . We write $\text{Gal}_F := \text{Gal}(\overline{F}/F)$ for the absolute Galois group of F , and we denote by $\text{Gal}_{F,\Sigma}$ the Galois group of the maximal extension of F unramified at all finite places outside Σ .

We consider the following local-to-global lifting problem: let \mathcal{O} be the ring of integers in a finite extension of \mathbf{Q}_p , with residue field \mathbf{F} , and let

$$\bar{\rho} : \text{Gal}_{F,\Sigma} \rightarrow \text{GL}_2(\mathbf{F})$$

be a continuous representation. Suppose that we are given a de Rham character $\psi : \text{Gal}_{F,\Sigma} \rightarrow \mathcal{O}^\times$ lifting $\det \bar{\rho}$, and local conditions on lifts of $\bar{\rho}$ at the places in Σ . At places away from p these conditions are allowed to be any non-empty union of irreducible components of the appropriate fixed-determinant local deformation ring, while at places above p they are allowed to be any non-empty union of irreducible components of a potentially semistable deformation ring with prescribed regular Hodge–Tate weights. We write \mathcal{S} for this collection of data, and $R_{\mathcal{S}}$

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for the corresponding global deformation ring; the precise definition is given in Definition 2.1.2.

Our main theorem is the following.

Theorem 1.1.1 (Theorem 4.2.1). *Suppose that $\bar{\rho}$ is totally odd, and that $\bar{\rho}|_{\mathrm{Gal}_F(\zeta_p)}$ is absolutely irreducible. If $p = 2$, assume that $\bar{\rho}$ has non-solvable image, and if $p = 5$, assume that $\bar{\rho}$ is not exceptional in the sense of Definition 4.1.1.*

Then, for every global deformation problem \mathcal{S} , the ring $R_{\mathcal{S}}$ is a finite \mathcal{O} -algebra of positive \mathcal{O} -rank. In particular, the set of characteristic-zero lifts of $\bar{\rho}$ to representations of $\mathrm{Gal}_{F,\Sigma}$ of type \mathcal{S} is finite and non-empty.

In short, subject to the hypotheses of the theorem, arbitrary prescribed non-empty sets of local components occur globally.

Remark 1.1.2. Although the theorem as we have phrased it (allowing our local conditions to be unions of irreducible components of local deformation rings) is equivalent to the analogous version in which we prescribe a single choice of irreducible component at each prime, we find our phrasing convenient. For example, imposing a bound on the prime-to- p conductor of lifts of $\bar{\rho}$ amounts to considering a certain union of local components. Thus the theorem immediately implies that the set of lifts of $\bar{\rho}$ satisfying any given bound on the prime-to- p conductor (compatible with the prime-to- p conductor of $\bar{\rho}$), as well as any prescribed p -adic Hodge-theoretic conditions at p , is finite and non-empty.

In particular, the theorem gives a Galois-theoretic analogue of theorems on level raising and lowering, and on change of weight, for Hilbert modular forms; see [CEG22, §1.2] for a historical overview of this connection, an explanation of the basic strategy for proving such results (which goes back to Khare–Wintenberger), and an account of the subsequent generalisations to Hilbert modular forms (see for example [KW09a; KW09b; Kis07; Gee11]). In brief: the global deformation ring with the prescribed local conditions has a Galois-cohomological lower bound on its dimension, which in the present setting says that $\dim R_{\mathcal{S}} \geq 1$; this is Proposition 2.1.4. Thus, if one also knows that $R_{\mathcal{S}}$ is finite over \mathcal{O} , this lower bound forces $R_{\mathcal{S}}$ to have positive \mathcal{O} -rank, so that $R_{\mathcal{S}} \otimes_{\mathcal{O}} \mathbf{Q}_p$ is a non-zero finite \mathbf{Q}_p -algebra. The points of its Spec are exactly the desired lifts.

The crux of the argument is therefore to prove the finiteness of the global deformation ring. In previous work, this finiteness is ultimately obtained from automorphy: after a suitable potential modularity argument, the deformation ring is compared with a Hecke algebra, which is finite over \mathcal{O} . For two-dimensional totally odd representations over totally real fields, the required finiteness in “parallel weight 2” is already known. Here this means that the labelled Hodge–Tate weights are $\{0, 1\}$ at every embedding; the terminology comes from the fact that the Galois representations associated to Hilbert modular forms of parallel weight 2 have these Hodge–Tate weights. In the form we need, this is a theorem of Snowden [Sno09] (building on Kisin’s modularity lifting theorems [Kis09b]), which is recalled below as Proposition 4.1.2.

The main new point of this paper is that we can deduce the general case of arbitrary regular Hodge–Tate weights from this weight 2 case. We do not use any new modularity lifting theorems to do this; indeed, we do not establish the potential modularity of our lifts. Instead, we make use of the geometry of the stack \mathcal{X} of étale (φ, Γ) -modules [EG23] (or, more precisely, its fixed-determinant variant). A

“topological” form of the Breuil–Mézard philosophy, established in [EG23], states that the underlying reduced substacks of the relevant potentially semistable substacks are equal to suitable unions of components of the underlying reduced substack \mathcal{X}_{red} . Furthermore, each of these components already occurs in the underlying reduced substack of a potentially semistable substack of parallel weight 2. (Indeed, Proposition 3.1.2 shows that every residual two-dimensional local representation with the required determinant is contained in the special fibre of a tamely potentially semistable weight 2 substack.)

Passing from stacks to local deformation rings via versal maps gives Corollary 3.2.1: modulo ϖ and up to nilpotent thickening, any potentially semistable local deformation ring is a quotient of a weight 2 local deformation ring. Tensoring these local comparisons together and using the standard description of global deformation rings as global-over-local tensor products then gives a surjection

$$R_{S'}/\varpi \rightarrow R_S/(\varpi, I),$$

where S' is a weight 2 deformation problem and I is nilpotent. Since $R_{S'}$ is finite over \mathcal{O} , it follows that R_S/ϖ is finite over \mathbf{F} , and hence that R_S is finite over \mathcal{O} . As sketched above, the Galois-cohomological lower bound then completes the proof of the theorem.

Remark 1.1.3. In the particular case when p splits completely in F , it is well-known that Theorem 1.1.1 can be deduced from the Breuil–Mézard conjecture for GL_2/\mathbf{Q}_p , following the methods of Kisin’s paper [Kis09c]. In particular [Cal12, Prop. 3.7] proves a very closely related result (under a residual modularity hypothesis, which is easily removed via standard potential modularity arguments).

After finishing writing this paper, we learned that a different proof (again in the case that p splits completely in F , and under an easily removed residual modularity hypothesis) was given in [HP19, Prop. 8.4]. The proof of this result in [HP19] is similar to ours; rather than using the general “topological Breuil–Mézard” statement that we establish using the stack \mathcal{X} , Hu and Paškūnas prove an analogous statement for deformation rings, in the framework of the geometric Breuil–Mézard conjecture of [EG14] (which is known in the case that p splits completely, *cf.* the various citations in the proof of *loc. cit.*).

It should also be noted that the idea of proving the finiteness of a global deformation ring via a comparison of the special fibres of local deformation rings also appears in Allen and Calegari’s paper [AC14], where they compare unramified and ordinary deformation rings.

Remark 1.1.4. The hypotheses in Theorem 1.1.1 are those that are required to implement the usual Taylor–Wiles argument in this two-dimensional setting. Some minor improvements are probably within reach. In particular, it is possible that the exceptional case when $p = 5$ could be handled (for example, using the methods of [KT17]), but we have not seriously investigated this.

In an ideal formulation, one would replace the Taylor–Wiles condition by the precise cohomological condition that guarantees the expected lower bound on the global deformation ring, for instance the appropriate vanishing of the obstruction group or dual Selmer group. (Without such a condition, the result need not hold; for example, see [All19, §6.4] for a discussion from the point of view of Galois deformations of an example due to Serre for $F = \mathbf{Q}$ and $p = 3$, with $\bar{\rho}$ induced from $\mathbf{Q}(\sqrt{-3})$.) We restrict throughout, though, to the residually irreducible case and,

indeed, to the Taylor–Wiles case; in forthcoming work, we will relate Theorem 1.1.1 to the local-to-global morphisms of moduli stacks of Galois representations discussed in [EGH25, §9.2], and discuss the residually reducible case.

Remark 1.1.5. One could hope to generalise Theorem 4.2.1 to other number fields, and to representations valued in reductive groups other than GL_2 . As alluded to in the previous remark, at least if $\bar{\rho}$ is irreducible, we expect that such generalisations should hold if and only if the Galois-cohomological lower bound on the dimension of the global deformation ring is positive (equivalently, is equal to 1). This requires the usual “numerical coincidence” in the Taylor–Wiles method to hold, as described at the beginning of the introduction to [CHT08]; in particular, it forces the number field to be totally real, and the representation $\bar{\rho}$ to be totally odd, which in turn forces an appropriate self-duality to hold. The difficulty in actually proving results in these settings is the absence of sufficiently powerful automorphy lifting theorems. (See Remark 3.1.3 below for a brief explanation of how our local results extend to arbitrary dimension.)

The main higher-dimensional setting in which automorphy lifting theorems are available is that of automorphic forms on unitary groups. The corresponding Galois representations are valued in the L -groups (or C -groups) of these unitary groups, which are of the form $\mathrm{GL}_n \rtimes \{\pm 1\}$, with -1 acting by inverse-transpose; these Galois representations can be interpreted as essentially conjugate-self-dual representations of Gal_E , where E is a totally imaginary CM field.

The automorphy lifting theorems of [BLGGT] could be applied to extend our results to this setting *if* one knew that (for example) all crystalline representations with Hodge–Tate weights $0, 1, \dots, n-1$ are potentially diagonalizable. This is known for $n = 2$ by [GK14], and for $n = 3$, it was recently proved (for $p \geq 5$) in [BHL26]. It should then be straightforward to adapt the methods of this paper to prove the corresponding lifting results for $n = 2$ or 3 , but we have not attempted to do so.

Remark 1.1.6. There is another, purely Galois-theoretic approach to producing lifts with prescribed local properties at a fixed finite set of places, going back to Ramakrishna [Ram02]. It requires one to allow additional ramification at auxiliary places to kill elements of Selmer groups, and so does not let one prescribe the local behaviour at every place at once. However, it is considerably more flexible, and Fakhruddin–Khare–Patrikis [FKP21] have extended it so as to produce p -adic lifts of mod- p representations into quite general reductive groups (provided, roughly, that p is sufficiently large and that the “numerical coincidence” mentioned in the previous paragraph holds). Their results in particular apply to essentially self-dual n -dimensional representations (for p sufficiently large, depending on the totally real field F).

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2. DEFORMATION RINGS

2.1. Definitions. Let F be a totally real field. Let \mathcal{O} be the ring of integers in a finite extension E/\mathbf{Q}_p , with uniformizer ϖ and residue field $\mathcal{O}/\varpi = \mathbf{F}$. We assume without further comment that E is chosen to be sufficiently large, so that

for example the image of each embedding $F \hookrightarrow \overline{\mathbf{Q}}_p$ is contained in E , and the irreducible components of the deformation rings considered below are geometrically irreducible.

Let $\overline{\rho} : \text{Gal}_F \rightarrow \text{GL}_2(\mathbf{F})$ be absolutely irreducible and totally odd, in the sense that $\det \overline{\rho}(c) = -1$ for each complex conjugation $c \in \text{Gal}_F$. Let $\psi : \text{Gal}_F \rightarrow \mathcal{O}^\times$ denote a character whose reduction modulo ϖ — which we denote by $\overline{\psi}$ — is equal to $\det \overline{\rho}$, and which is de Rham at all places lying over p . For each finite place v of F , let R_v denote the universal lifting \mathcal{O} -algebra for liftings of $\overline{\rho}|_{\text{Gal}_{F_v}}$, and let R_v^ψ denote the quotient of R_v corresponding to liftings of determinant $\psi|_{\text{Gal}_{F_v}}$.

Definition 2.1.1. If $v|p$ is a place of F , then a *regular Hodge type* at v is a tuple $\lambda_v = \{\lambda_{\sigma,1}, \lambda_{\sigma,2}\}_{\sigma: F_v \hookrightarrow \overline{\mathbf{Q}}_p}$, where $\lambda_{\sigma,1} > \lambda_{\sigma,2}$ are distinct integers. We say that λ_v has *weight* \mathcal{Q} if $\{\lambda_{\sigma,1}, \lambda_{\sigma,2}\} = \{1, 0\}$ for all σ , in which case we write $[0, 1]_v$ for λ_v .

Given a finite set Σ of finite places of F , containing all places at which either $\overline{\rho}$ or ψ is ramified and all places lying above p , we let R_Σ^{univ} denote the universal deformation \mathcal{O} -algebra for deformations of $\overline{\rho}$ which are unramified at all finite places of F not contained in Σ (recall that if R is a complete local \mathcal{O} -algebra with residue field \mathbf{F} , a *deformation* of $\overline{\rho}$ to R is an equivalence class of liftings under conjugation by $\ker(\text{GL}_2(R) \rightarrow \text{GL}_2(\mathbf{F}))$). We let $R_\Sigma^{\text{univ}, \psi}$ denote the quotient of R_Σ^{univ} corresponding to deformations with determinant ψ .

Definition 2.1.2. A *global deformation problem* \mathcal{S} is a tuple

$$\mathcal{S} = (\psi, \Sigma, \{\overline{R}_v^\psi\}_{v \in \Sigma})$$

consisting of the following data:

- As above, $\psi : \text{Gal}_F \rightarrow \mathcal{O}^\times$ is a de Rham character such that $\overline{\psi} = \det \overline{\rho}$.
- Σ , as above, is a finite set of places of F , containing all the places above p , and all the places at which $\overline{\rho}$ or ψ is ramified.
- For each $v \nmid p$ with $v \in \Sigma$, we let \overline{R}_v^ψ be a quotient of R_v^ψ corresponding to a (non-empty) union of irreducible components of $R_v^\psi[1/p]$.
- For each $v|p$, we let λ_v be a regular Hodge–Tate type, we let E_v/F_v be a finite extension, and we write $R_v^{\lambda_v, E_v, \psi}$ for the \mathcal{O} -flat quotient of R_v^ψ corresponding to potentially semistable lifts of $\overline{\rho}|_{\text{Gal}_{F_v}}$ which become semistable over E_v . We assume that $R_v^{\lambda_v, E_v, \psi}$ is non-zero, and we let \overline{R}_v^ψ be a quotient of $R_v^{\lambda_v, E_v, \psi}$ corresponding to a non-empty union of irreducible components of $R_v^{\lambda_v, E_v, \psi}[1/p]$.

We let $R_\mathcal{S}$ be the quotient of $R_\Sigma^{\text{univ}, \psi}$ given by the universal deformation \mathcal{O} -algebra for deformations of $\overline{\rho}$ of determinant ψ , which are unramified at all finite places of F not contained in Σ , and whose restrictions to each Gal_{F_v} , $v \in \Sigma$ correspond to points of \overline{R}_v^ψ .

Set $R^{\text{loc}} := \widehat{\otimes}_{v \in \Sigma} R_v$, and $R_\mathcal{S}^{\text{loc}} := \widehat{\otimes}_{v \in \Sigma} \overline{R}_v^\psi$, a quotient of R^{loc} . If we choose a representative for the universal deformation of $\overline{\rho}$ to $\text{GL}_2(R_\Sigma^{\text{univ}})$, we may regard R_Σ^{univ} as an R^{loc} -algebra, and $R_\mathcal{S}$ as an $R_\mathcal{S}^{\text{loc}}$ -algebra. From the definitions we see that

$$(2.1.3) \quad R_\mathcal{S} = R_\Sigma^{\text{univ}, \psi} \otimes_{R^{\text{loc}}} R_\mathcal{S}^{\text{loc}}.$$

The following dimension bound follows from a standard tangent-obstruction calculation using Galois cohomology, which goes back to Mazur [Maz89], and in this form is essentially due to Kisin [Kis07].

Proposition 2.1.4. *Suppose that $\bar{\rho}|_{\mathrm{Gal}_{F(\zeta_p)}}$ is absolutely irreducible, and if $p = 2$, assume that $\bar{\rho}$ has non-solvable image. Then $\dim R_{\mathcal{S}} \geq 1$.*

Proof. If $p > 2$ and λ_v has weight 2, this is [Sno09, Prop. 6.2.1], and the general case follows from an identical argument, using [Kis08, Thm. 3.3.4] for the dimensions of the rings $\bar{R}_v^\psi[1/p]$ for $v|p$. Note that when $p = 2$, the assumption that $\bar{\rho}$ has non-solvable image guarantees the vanishing of the group $H^0(\mathrm{Gal}_{F,S}, (\mathrm{ad}^0 V)^*(1))$ considered in [Kis07, Prop. 4.1.5]; see [KW09b, Lem. 4.3 2(ii)]. \square

Definition 2.1.5. We say that a lift $\rho : \mathrm{Gal}_F \rightarrow \mathrm{GL}_2(\bar{\mathbf{Z}}_p)$ of $\bar{\rho} \otimes_{\mathbf{F}} \bar{\mathbf{F}}_p$ is of type \mathcal{S} if it has determinant ψ , and if for each place $v \in \Sigma$, the corresponding homomorphism $R_v^\psi \rightarrow \bar{\mathbf{Z}}_p$ factors through \bar{R}_v^ψ . By definition, the isomorphism classes of lifts of type \mathcal{S} are in bijection with the set of \mathcal{O} -algebra homomorphisms $R_{\mathcal{S}} \rightarrow \bar{\mathbf{Q}}_p$.

3. COMPARING THE SPECIAL FIBRES OF POTENTIALLY SEMISTABLE DEFORMATION RINGS

3.1. Moduli stacks of étale (φ, Γ) -modules. We now make use of the moduli stacks of étale (φ, Γ) -modules constructed in [EG23], or more precisely of the fixed-determinant variant of [DEG26, App. C]. Fix a place $v|p$ of F , and let $\psi : \mathrm{Gal}_{F_v} \rightarrow \mathcal{O}^\times$ be a de Rham character. We write \mathcal{X}^ψ for the moduli stack of étale (φ, Γ) -modules defined in [DEG26, Defn. C.1.1], taking the field K to be F_v and the positive integer d to be 2; this is a formal algebraic stack over $\mathrm{Spf} \mathcal{O}$, with the property that the groupoid $\mathcal{X}^\psi(\mathrm{Spf} \mathcal{O})$ is naturally equivalent to the groupoid of continuous representations $\mathrm{Gal}_{F_v} \rightarrow \mathrm{GL}_2(\mathcal{O})$ with determinant ψ . The special fibre \mathcal{X}^ψ/ϖ only depends on $\bar{\psi}$, and if \mathbf{F}'/\mathbf{F} is a finite extension, $\mathcal{X}^\psi(\mathrm{Spec} \mathbf{F}')$ corresponds to continuous representations $\mathrm{Gal}_{F_v} \rightarrow \mathrm{GL}_2(\mathbf{F}')$ with determinant $\bar{\psi}$.

For each regular Hodge type λ_v and finite extension E_v/F_v , we write $\mathcal{X}^{\lambda_v, E_v, \psi}$ for the closed substack of \mathcal{X}^ψ corresponding to potentially semistable representations which become semistable over E_v . By [DEG26, Cor. C.1.4, Thm. C.1.8] (and the obvious variants for potentially semistable stacks, whose proofs are identical), the underlying reduced substack $\mathcal{X}_{\mathrm{red}}^\psi$ of \mathcal{X}^ψ is a finite type algebraic stack over \mathbf{F} , and is equidimensional of dimension $[F_v : \mathbf{Q}_p]$. Furthermore, the special fibre $\mathcal{X}^{\lambda_v, E_v, \psi}/\varpi$ of each $\mathcal{X}^{\lambda_v, E_v, \psi}$ is also a finite type algebraic stack over \mathbf{F} , and is again equidimensional of dimension $[F_v : \mathbf{Q}_p]$. In particular, the underlying reduced substack of $\mathcal{X}^{\lambda_v, E_v, \psi}$ is a union of irreducible components of $\mathcal{X}_{\mathrm{red}}^\psi$.

Definition 3.1.1. We let ε denote the p -adic cyclotomic character; our convention is that it has Hodge–Tate weight -1 . Let $\psi_1 := \varepsilon^{-1}(\widetilde{\varepsilon \det \bar{\rho}})$, where the tilde denotes the Teichmüller lift; in local notation we also write ψ_1 for its restriction to Gal_{F_v} . Write $K_v := F_v(\varpi_v^{1/(q_v-1)})$, where ϖ_v is a uniformizer of F_v and q_v denotes the cardinality of the residue field of \mathcal{O}_{F_v} .

By definition, $\mathcal{X}^{[0,1]_v, K_v, \psi_1}$ is the closed substack of \mathcal{X}^{ψ_1} corresponding to representations which become semistable over K_v and are of weight 2. The reason for our choice of K_v is the following proposition.

Proposition 3.1.2. *The special fibre $\mathcal{X}^{[0,1]_v, K_v, \psi_1}/\varpi$ contains $\mathcal{X}_{\mathrm{red}}^{\psi_1}$.*

Proof. Since $\mathcal{X}^{[0,1]_v, K_v, \psi_1}/\varpi$ and $\mathcal{X}_{\mathrm{red}}^{\psi_1}$ are of finite type over \mathbf{F} , it suffices to show that for each finite extension \mathbf{F}'/\mathbf{F} , each $\bar{r} : \mathrm{Gal}_{F_v} \rightarrow \mathrm{GL}_2(\mathbf{F}')$ of determinant $\bar{\psi}_1$

admits a lift to a weight 2 representation which becomes semistable over K_v . At least when $p > 2$, this is an immediate consequence of the Breuil–Mézard conjecture; see e.g. [EG23, §8.6]. We can also prove it (for all p) more directly as follows. Since both substacks are closed in $\mathcal{X}^{\psi_1}/\varpi$, it suffices to show that $\mathcal{X}^{[0,1]_v, K_v, \psi_1}/\varpi$ contains a dense set of closed points of $\mathcal{X}_{\text{red}}^{\psi_1}$. Any sufficiently generic closed point of $\mathcal{X}_{\text{red}}^{\psi_1}$ corresponds to a reducible \bar{r} . It therefore suffices to show that if $\bar{r} \cong \begin{pmatrix} \bar{\chi}_1 & * \\ 0 & \bar{\chi}_2 \bar{\varepsilon}^{-1} \end{pmatrix}$

then \bar{r} admits a lift of the form $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \varepsilon^{-1} \end{pmatrix}$ where χ_i is an unramified twist of the Teichmüller lift $\tilde{\chi}_i$ of $\bar{\chi}_i$, and $\chi_1 \chi_2 = \widetilde{\bar{\chi}_1 \bar{\chi}_2}$ (such a lift is automatically semistable over K_v). This follows from a straightforward calculation in Galois cohomology, cf. [BGG12, Lem. 6.1.6]. \square

Remark 3.1.3. In fact, by a straightforward induction on n one can prove an analogue of Proposition 3.1.2 for GL_n , replacing $[0, 1]_v$ with any regular Hodge type for GL_n .

3.2. Versal rings. Let ψ be a de Rham lift of $\det \bar{\rho}$. Then $\bar{\rho}|_{\text{Gal}_{F_v}}$ corresponds to a point $x \in \mathcal{X}^{\psi}(\mathbf{F})$, and there is an induced morphism $\text{Spf } R_v^{\psi} \rightarrow \mathcal{X}^{\psi}$ which is versal to \mathcal{X}^{ψ} at x (in a precise sense which we do not need to recall here; see [EG23, Prop. 3.6.3]). For each pair λ_v, E_v , this morphism pulls back to a versal morphism $\text{Spf } R_v^{\lambda_v, E_v, \psi} \rightarrow \mathcal{X}_{\text{red}}^{\lambda_v, E_v, \psi}$. In particular, the closed substack $\mathcal{X}_{\text{red}}^{\lambda_v, E_v, \psi}$ pulls back under this morphism to a closed formal subscheme of $\text{Spf } R_v^{\lambda_v, E_v, \psi}/\varpi$. Since for any complete Noetherian local ring R , the closed formal subschemes of $\text{Spf } R$ are in bijection (via passing to the \mathfrak{m}_R -adic completion) with the closed subschemes of $\text{Spec } R$, this pullback corresponds to a quotient $R_v^{\lambda_v, E_v, \psi}/(\varpi, I)$ of $R_v^{\lambda_v, E_v, \psi}/\varpi$ by an ideal I . Since $\mathcal{X}^{\lambda_v, E_v, \psi}/\varpi$ is of finite type over \mathbf{F} , the ideal I is nilpotent.

In the statement of the following corollary, note that if $\bar{\psi} = \bar{\psi}_1 = \det \bar{\rho}$, we may canonically identify R_v^{ψ}/ϖ with $R_v^{\psi_1}/\varpi$.

Corollary 3.2.1. *For each λ_v, E_v, ψ with $\bar{\psi} = \det(\bar{\rho}|_{\text{Gal}_{F_v}})$, there is a nilpotent ideal I in $R_v^{\lambda_v, E_v, \psi}/\varpi$ with the property that the natural surjection $R_v^{\psi} \rightarrow R_v^{\lambda_v, E_v, \psi}/(\varpi, I)$ factors through $R_v^{[0,1]_v, K_v, \psi_1}/\varpi$.*

Proof. Letting I be as defined above, this is immediate from Proposition 3.1.2. \square

4. FINITENESS OF GLOBAL DEFORMATION RINGS

4.1. Finiteness of weight two deformation rings.

Definition 4.1.1. Following [KT17], we say that $\bar{\rho}$ is *exceptional* if $p = 5$, the projective image $\text{Proj}(\bar{\rho})$ of $\bar{\rho}$ is isomorphic to $\text{PGL}_2(\mathbf{F}_5)$, and $\bar{F}^{\ker \text{Proj}(\bar{\rho})}$ contains ζ_5 .

Proposition 4.1.2. *Suppose that $\bar{\rho}$ is totally odd, and that $\bar{\rho}|_{\text{Gal}_{F(\zeta_p)}}$ is absolutely irreducible. If $p = 2$, assume that $\bar{\rho}$ has non-solvable image, and if $p = 5$, assume that $\bar{\rho}$ is not exceptional in the sense of Definition 4.1.1.*

Assume furthermore that λ_v has weight 2 for each $v|p$. Then R_S is a finite \mathcal{O} -algebra.

Proof. Since each $\bar{R}_v^{\psi}[1/p]$ has only finitely many irreducible components, we may assume that each $\bar{R}_v^{\psi}[1/p]$ is irreducible. If $p > 2$, then bearing in mind [Sno09, Prop. 7.3.1, 7.4.1], the result is then a special case of [Sno09, Thm. 6.1.1]. (Note

that Snowden’s condition (A2) is slightly stronger than our assumption that $\bar{\rho}$ is not exceptional if $p = 5$, but this is due to the corresponding condition in the version of [Kis09b] that is cited in [Sno09]; the published version of [Kis09b] assumes only that $\bar{\rho}$ is not exceptional.) If $p = 2$ then a very similar argument works, appealing to [Kis09a] instead of [Kis09b]; the analogue for $p = 2$ of [Sno09, Thm. 5.1.1] can be proved in exactly the same way as [Tho17, Prop. 6.7] (noting that in the proof there, for each place $v|2$, one can choose whether to make the N -HBAV A/L_v have good ordinary reduction or good non-ordinary reduction). \square

4.2. The main theorem.

Theorem 4.2.1. *Suppose that $\bar{\rho}$ is totally odd and that $\bar{\rho}|_{\mathrm{Gal}_F(\zeta_p)}$ is absolutely irreducible. If $p = 2$, assume that $\bar{\rho}$ has non-solvable image, and if $p = 5$, assume that $\bar{\rho}$ is not exceptional in the sense of Definition 4.1.1.*

Then for any global deformation problem \mathcal{S} , the universal deformation ring $R_{\mathcal{S}}$ is a finite \mathcal{O} -algebra of positive \mathcal{O} -rank. In particular, the set of isomorphism classes of lifts of $\bar{\rho}$ of type \mathcal{S} is finite and non-empty.

Proof. In order to see that $R_{\mathcal{S}}$ is a finite \mathcal{O} -algebra, it suffices to show that $R_{\mathcal{S}}/\varpi$ is a finite \mathbf{F} -algebra. We now consider the deformation problem

$$\mathcal{S}' = (\psi_1, \Sigma, \{\bar{R}_v^{\psi_1}\}_{v \in \Sigma}),$$

where

- ψ_1 is as in Definition 3.1.1.
- $\bar{R}_v^{\psi_1} = R_v^{\psi_1}$ for $v \nmid p$.
- $\bar{R}_v^{\psi_1} = R_v^{[0,1]_v, K_v, \psi_1}$ if $v|p$, where K_v is as in Definition 3.1.1.

For $v \nmid p$, the canonical identification $R_v^{\psi_1}/\varpi = R_v^{\psi}/\varpi$ and the quotient map $R_v^{\psi}/\varpi \rightarrow \bar{R}_v^{\psi}/\varpi$, together with Corollary 3.2.1 for $v|p$, give a nilpotent ideal I^{loc} of $R_{\mathcal{S}'}^{\mathrm{loc}}/\varpi$ and a surjection of R^{loc} -algebras $R_{\mathcal{S}'}^{\mathrm{loc}}/\varpi \rightarrow R_{\mathcal{S}}^{\mathrm{loc}}/(\varpi, I^{\mathrm{loc}})$. Let I be the ideal of $R_{\mathcal{S}}/\varpi$ generated by the image of I^{loc} under the map $R_{\mathcal{S}}^{\mathrm{loc}}/\varpi \rightarrow R_{\mathcal{S}}/\varpi$. By (2.1.3), this induces a surjection of \mathbf{F} -algebras $R_{\mathcal{S}'}/\varpi \rightarrow R_{\mathcal{S}}/(\varpi, I)$. By Proposition 4.1.2, $R_{\mathcal{S}'}/\varpi$ is a finite \mathbf{F} -algebra, so the same is true of $R_{\mathcal{S}}/(\varpi, I)$. Since $R_{\mathcal{S}}$ is Noetherian and I is nilpotent, we deduce that $R_{\mathcal{S}}/\varpi$ is a finite \mathbf{F} -algebra, as claimed.

By Proposition 2.1.4, $R_{\mathcal{S}}$ has dimension at least 1, so it has dimension precisely 1, and has positive \mathcal{O} -rank. Thus $R_{\mathcal{S}}[1/p] \otimes_E \bar{\mathbf{Q}}_p$ is a non-zero finite $\bar{\mathbf{Q}}_p$ -algebra, so the set of \mathcal{O} -homomorphisms $R_{\mathcal{S}} \rightarrow \bar{\mathbf{Q}}_p$ is finite and non-empty, and we are done (see Definition 2.1.5). \square

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