COMPONENTS OF MODULI STACKS OF TWO-DIMENSIONAL GALOIS REPRESENTATIONS

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Abstract. In the article [CEGS20b] we introduced various moduli stacks of two-dimensional tamely potentially Barsotti–Tate representations of the absolute Galois group of a p-adic local field, as well as related moduli stacks of Breuil–Kisin modules with descent data. We study the irreducible components of these stacks, establishing in particular that the components of the former are naturally indexed by certain Serre weights.

Contents

1. Introduction

Fix a prime number p, and let $K/\mathbf{Q}_p$ be a finite extension with residue field k and absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$. In the paper [CEGS20b], inspired by a construction of Kisin [Kis09] in the setting of formal deformations, we constructed and began to study the geometry of certain moduli stacks $\mathcal{Z}^{\mathrm{dd}}$. The stacks $\mathcal{Z}^{\mathrm{dd}}$ can be thought of as moduli of two-dimensional tamely potentially Barsotti–Tate representations of G_K; they are in fact moduli stacks of étale φ-modules with descent data, and by construction are equipped with a partial resolution

$$\mathcal{C}^{\mathrm{dd,BT}} \to \mathcal{Z}^{\mathrm{dd}}$$

where $\mathcal{C}^{\mathrm{dd,BT}}$ is a moduli stack of rank two Breuil–Kisin modules with tame descent data and height one.

The first author was supported in part by NSF grant DMS-1501064, by a Royal Society University Research Fellowship, and by ERC Starting Grant 804176. The second author was supported in part by the NSF grants DMS-1303450, DMS-1601871, DMS-1902307, DMS-1952705, and DMS-2201242. The third author was supported in part by a Leverhulme Prize, EPSRC grant EP/L025485/1, Marie Curie Career Integration Grant 303605, ERC Starting Grant 306326, and a Royal Society Wolfson Research Merit Award. The fourth author was supported in part by NSF CAREER grant DMS-1564367 and NSF grant DMS-1702161.

The purpose of this paper is to make an explicit study of the morphism $\mathcal{C}^{\mathrm{dd,BT}} \to \mathcal{Z}^{\mathrm{dd}}$ at the level of irreducible components. To be precise, for each two-dimensional tame inertial type τ there are closed substacks $\mathcal{C}^{\tau,\mathrm{BT}} \subset \mathcal{C}^{\mathrm{dd,BT}}$ and $\mathcal{Z}^{\tau} \subset \mathcal{Z}^{\mathrm{dd}}$ corresponding to representations having inertial type τ, and a morphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{Z}^{\tau}$. These are p-adic formal algebraic stacks; let $\mathcal{C}^{\tau,\mathrm{BT},1}$ be the special fibre of $\mathcal{C}^{\tau,\mathrm{BT}}$, and $\mathcal{Z}^{\tau,1}$ its scheme-theoretic image in $\mathcal{Z}^{\tau}$ (in the sense of [EG21]). These were proved in [CEGS20b] to be equidimensional of dimension $[K : \mathbf{Q}_p]$. Moreover the finite type points $\mathrm{Spec}(\mathbf{F}) \to \mathcal{Z}^{\tau,1}$ are in bijection with Galois representations $G_K \to \mathrm{GL}_2(\mathbf{F})$ admitting a potentially Barsotti–Tate lift of type τ.

(In fact $\mathcal{C}^{\tau,\mathrm{BT},1}$ is shown in [CEGS20b] to be reduced, from which it follows that $\mathcal{Z}^{\tau,1}$ is also reduced. The special fibre of $\mathcal{Z}^{\tau}$ need not be reduced, so it need not equal $\mathcal{Z}^{\tau,1}$, but it will be proved in the sequel [CEGS20a] that it is *generically reduced*, using the results of this paper as input.)

Much of the work in our study of the irreducible components of $\mathcal{Z}^{\tau,1}$ involves an explicit construction of families of extensions of characters. Intuitively, a natural source of "families" of representations $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is given by the extensions of two fixed characters. Indeed, given two characters $\chi_1, \chi_2 : G_K \to \overline{\mathbf{F}}_p^{\times}$, the $\overline{\mathbf{F}}_p$-vector space $\mathrm{Ext}^1_{G_K}(\chi_2, \chi_1)$ is usually $[K : \mathbf{Q}_p]$-dimensional, and a back of the envelope calculation suggests that as a stack the collection of these representations should have dimension $[K : \mathbf{Q}_p] - 2$: the difference between an extension and a representation counts for a -1, as does the $\mathbf{G}_m$ of endomorphisms. Twisting χ_1, χ_2 independently by unramified characters gives a candidate for a $[K : \mathbf{Q}_p]$-dimensional family; if contained in $\mathcal{Z}^{\tau}$, then since $\mathcal{Z}^{\tau}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$, the closure of such a family should be an irreducible component of $\mathcal{Z}^{\tau}$.

Since there are only finitely many possibilities for the restrictions of the χ_i to the inertia subgroup I_K, this gives a finite list of maximal-dimensional families. On the other hand, there are up to unramified twist only finitely many irreducible two-dimensional representations of G_K, which suggests that the irreducible representations should correspond to 0-dimensional substacks. Together these considerations suggest that the irreducible components of our moduli stack should be given by the closures of the families of extensions considered in the previous paragraph, and in particular that the irreducible representations should arise as limits of reducible representations. This could not literally be the case for families of Galois representations, rather than families of étale φ-modules, and may seem surprising at first glance, but it is indeed what happens.

In the body of the paper we make this analysis rigorous, and we show that the different families that we have constructed exhaust the irreducible components. We can therefore label the irreducible components of $\mathcal{Z}^{\tau,1}$ as follows. A component is specified by an ordered pair of characters $I_K \to \overline{\mathbf{F}}_p^{\times}$, which via local class field theory corresponds to a pair of characters $k^{\times} \to \overline{\mathbf{F}}_p^{\times}$. Such a pair can be thought of as the highest weight of a *Serre weight*: an irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(k)$. To each irreducible component we have thus associated a Serre weight. (In fact, we need to make a shift in this dictionary, corresponding to half the sum of the positive roots of $\mathrm{GL}_2(k)$, but we ignore this for the purposes of this introduction.)

This might seem artificial, but in fact it is completely natural, for the following reason. Following the pioneering work of Serre [Ser87] and Buzzard–Diamond–Jarvis [BDJ10] (as extended in [Sch08] and [Gee11]), we now know how to associate a set $W(\overline{r})$ of Serre weights to each continuous representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with the property that if F is a totally real field and $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is an irreducible representation coming from a Hilbert modular form, then the possible weights of Hilbert modular forms giving rise to $\overline{\rho}$ are precisely determined by the sets $W(\overline{\rho}|_{G_{F_v}})$ for places $v|p$ of F (see for example [BLGG13, GK14, GLS15]).

Going back to our labelling of irreducible components above, we have associated a Serre weight $\overline{\sigma}$ to each irreducible component of $\mathcal{Z}^{\tau,1}$. The inertial local Langlands correspondence assigns a finite set of Serre weights $\mathrm{JH}(\overline{\sigma}(\tau))$ to τ, the Jordan–Hölder factors of the reduction mod p of the representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ corresponding to τ. One of our main theorems is that the components of $\mathcal{Z}^{\tau,1}$ are labeled precisely by the Serre weights $\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau))$. Furthermore the component labeled by $\overline{\sigma}$ has a dense set of finite type points $\overline{r}$ with $\overline{\sigma} \in W(\overline{r})$. In the sequel [CEGS20a] this will be strengthened to the statement that the representations $\overline{r}$ on the irreducible component labelled by $\overline{\sigma}$ are precisely the representations with $\overline{\sigma} \in W(\overline{r})$,

We also study the irreducible components of the stack $\mathcal{C}^{\tau,\mathrm{BT},1}$. If τ is a non-scalar principal series type then the set $\mathrm{JH}(\overline{\sigma}(\tau))$ can be identified with a subset of the power set $\mathcal{S}$ of the set of embeddings $k \hookrightarrow \overline{\mathbf{F}}_p$ (hence, after fixing one such embedding, with a subset $\mathcal{P}_\tau$ of $\mathbf{Z}/f\mathbf{Z}$). For generic choices of τ, this subset is the whole of $\mathcal{S}$. We are able to show, using the theory of Dieudonné modules, that for any non-scalar principal series type τ the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ can be identified with $\mathcal{S}$, and those irreducible components not corresponding to elements of $\mathrm{JH}(\overline{\sigma}(\tau))$ have image in $\mathcal{Z}^{\tau}$ of positive codimension. There is an analogous statement for cuspidal types, while for scalar types, both $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$ are irreducible.

To state our main results precisely we must first introduce a bit more notation. Fix a tame inertial type τ and a uniformiser π of K. Let L be the unramified quadratic extension of K, and write f for the inertial degree of $K/\mathbf{Q}_p$. We set $K' = K(\pi^{1/p^f-1})$ if τ is principal series, and set $K' = L(\pi^{1/(p^{2f}-1)})$ if τ is cuspidal. Our moduli stacks of p-adic Hodge theoretic objects with descent data will have descent data from K' to K. Let f' be the inertial degree of $K'/\mathbf{Q}_p$, so that $f' = f$ if the type τ is principal series, while $f' = 2f$ if the type τ is cuspidal.

We say that a subset $J \subset \mathbf{Z}/f'\mathbf{Z}$ is a *shape* if:

— τ is scalar and $J = \varnothing$,

— τ is a non-scalar principal series type and J is arbitrary, or

— τ is cuspidal and J has the property that $i \in J$ if and only if $i + f \notin J$.

If τ is non-scalar then there are exactly 2^f shapes.

As above, write $\sigma(\tau)$ for the representation of $\mathrm{GL}_2(\mathcal{O}_K)$ corresponding to τ under the inertial local Langlands correspondence of Henniart. The Jordan–Hölder factors of the reduction mod p of $\sigma(\tau)$ are parameterized by an explicit set of shapes $\mathcal{P}_\tau$, and we write $\overline{\sigma}(\tau)_J$ for the factor corresponding to J.

To each shape J, we will associate a closed substack $\overline{\mathcal{C}}(J)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$. The stack $\overline{\mathcal{Z}}(J)$ is then defined to be the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ under the map $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$, in the sense of [EG21]. Then the following is our main result.

Theorem 1.1. *The irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$ are as follows.*

(1) The irreducible components of $\mathcal{C}^{\tau,1}$ are precisely the $\overline{\mathcal{C}}(J)$ for shapes J, and if $J \neq J'$ then $\overline{\mathcal{C}}(J) \neq \overline{\mathcal{C}}(J')$.

(2) The irreducible components of $\mathcal{Z}^{\tau,1}$ are precisely the $\overline{\mathcal{Z}}(J)$ for shapes $J \in \mathcal{P}_\tau$, and if $J \neq J'$ then $\overline{\mathcal{Z}}(J) \neq \overline{\mathcal{Z}}(J')$.

(3) *For each $J \in \mathcal{P}_\tau$, there is a dense open substack $\mathcal{U}$ of $\overline{\mathcal{C}}(J)$ such that the map $\overline{\mathcal{C}}(J) \to \overline{\mathcal{Z}}(J)$ restricts to an open immersion on $\mathcal{U}$.*

(4) *For each $J \in \mathcal{P}_\tau$, there is a dense set of finite type points of $\overline{\mathcal{Z}}(J)$ with the property that the corresponding Galois representations have $\overline{\sigma}(\tau)_J$ as a Serre weight, and which furthermore admit a unique Breuil–Kisin model of type τ.*

Remark 1.2. We emphasize in Theorem 1.1 that the components of $\mathcal{Z}^{\tau,1}$ are indexed by shapes $J \in \mathcal{P}_\tau$, *not* by all shapes. If $J \notin \mathcal{P}_\tau$, then the stack $\overline{\mathcal{Z}}(J)$ has dimension strictly smaller than $[K : \mathbf{Q}_p]$, and so is properly contained in some component of $\mathcal{Z}^{\tau,1}$. We anticipate that the loci $\overline{\mathcal{Z}}(J)$ will nevertheless be of interest when $J \notin \mathcal{P}_\tau$: we expect that they will correspond to "phantom" (partial weight one) Serre weights of relevance to the geometric variant of the weight part of Serre's conjecture proposed by Diamond–Sasaki [DS17]. This will be the subject of future work.

We assume that $p > 2$ in much of the paper; while we expect that our results should also hold if $p = 2$, there are several reasons to exclude this case. We are frequently able to considerably simplify our arguments by assuming that the extension K'/K is not just tamely ramified, but in fact of degree prime to p; this is problematic when $p = 2$, as the consideration of cuspidal types involves a quadratic unramified extension. Furthermore, in the sequel [CEGS20a] we will use results on the Breuil–Mézard conjecture which ultimately depend on automorphy lifting theorems that are not available in the case $p = 2$ at present (although it is plausible that the methods of [Tho17] could be used to prove them).

We conclude this introduction by discussing the relationship between our results and those of [EG22]. Two of us (M.E. and T.G.) have constructed moduli stacks $\mathcal{X}_d$ of rank d étale (φ, Γ)-modules for K, as well as substacks $\mathcal{X}_d^{\lambda,\tau}$ which may be regarded as stacks of potentially crystalline representations of G_K with inertial type τ and Hodge type λ. When $d = 2$ and λ is the trivial Hodge type, these are stacks $\mathcal{X}_2^{\tau,\mathrm{BT}}$ of potentially Barsotti–Tate representations of G_K of inertial type τ, and we anticipate that $\mathcal{X}_2^{\tau,\mathrm{BT}}$ is isomorphic to $\mathcal{Z}^{\tau,\mathrm{BT}}$ (but since we do not need this fact, we have not proved it).

One of the main results of the book [EG22] is that the irreducible components of the underlying reduced stacks $\mathcal{X}_{d,\mathrm{red}}$ are in bijection with the irreducible representations of $\mathrm{GL}_d(k)$. This bijection is characterised in essentially exactly the same way as our description of the components of $\mathcal{Z}^{\tau,1}$ in this paper: a Serre weight has a highest weight, which corresponds to a tuple of inertial characters, which gives rise to a family of successive extensions of 1-dimensionals representations. Then the closure of this family is a component of $\mathcal{X}_{d,\mathrm{red}}$.

The crucial difference between our setting and that of [EG22] is that we could prove in [CEGS20b] that the stacks $\mathcal{Z}^{\tau,1}$ are reduced. The proof makes use of the resolution $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$ and the fact that we are able to relate the stack $\mathcal{C}^{\tau,\mathrm{BT}}$ to a local model at Iwahori level, whose special fibre is known to be reduced. In the sequel [CEGS20a] we combine the characterisation of the components of $\mathcal{Z}^{\tau,1}$ from this paper with the reducedness of $\mathcal{Z}^{\tau,1}$ from [CEGS20b] to prove that the special fibre of $\mathcal{Z}^\tau$ is *generically* reduced. This will then allow us to completely characterise *all* of the finite type points on each component of $\mathcal{Z}^{\tau,1}$ (not just a dense set of points), and to prove geometrisations of the Breuil–Mézard conjecture and of the weight part of Serre's conjecture for the stacks $\mathcal{Z}^{\mathrm{dd},1}$. Furthermore, by means of a comparison of versal rings, these results can be transported to the stacks $\mathcal{X}_2^{\tau,\mathrm{BT}}$ of [EG22] as well.

1.3. **Acknowledgements.** We would like to thank Kalyani Kansal for helpful comments.

1.4. **Notation and conventions.**

Topological groups. If M is an abelian topological group with a linear topology, then as in [Sta13, Tag 07E7] we say that M is *complete* if the natural morphism $M \to \varinjlim_i M/U_i$ is an isomorphism, where $\{U_i\}_{i\in I}$ is some (equivalently any) fundamental system of neighbourhoods of 0 consisting of subgroups. Note that in some other references this would be referred to as being *complete and separated*. In particular, any p-adically complete ring A is by definition p-adically separated.

Galois theory and local class field theory. If M is a field, we let G_M denote its absolute Galois group. If M is a global field and v is a place of M, let M_v denote the completion of M at v. If M is a local field, we write I_M for the inertia subgroup of G_M.

Let p be a prime number. Fix a finite extension $K/\mathbf{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Let e and f be the ramification and inertial degrees of K, respectively, and write $\#k = p^f$ for the cardinality of k. Let K'/K be a finite tamely ramified Galois extension. Let k' be the residue field of K', and let e', f' be the ramification and inertial degrees of K' respectively.

Our representations of G_K will have coefficients in $\overline{\mathbf{Q}}_p$, a fixed algebraic closure of $\mathbf{Q}_p$ whose residue field we denote by $\overline{\mathbf{F}}_p$. Let E be a finite extension of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$ and containing the image of every embedding of K' into $\overline{\mathbf{Q}}_p$. Let $\mathcal{O}$ be the ring of integers in E, with uniformiser ϖ and residue field $\mathbf{F} \subset \overline{\mathbf{F}}_p$.

Fix an embedding $\sigma_0 : k' \hookrightarrow \mathbf{F}$, and recursively define $\sigma_i : k' \hookrightarrow \mathbf{F}$ for all $i \in \mathbf{Z}$ so that $\sigma_{i+1}^p = \sigma_i$; of course, we have $\sigma_{i+f'} = \sigma_i$ for all i. We let $e_i \in k' \otimes_{\mathbf{F}_p} \mathbf{F}$ denote the idempotent satisfying $(x \otimes 1)e_i = (1 \otimes \sigma_i(x))e_i$ for all $x \in k'$; note that $\varphi(e_i) = e_{i+1}$. We also denote by e_i the natural lift of e_i to an idempotent in $W(k') \otimes_{\mathbf{Z}_p} \mathcal{O}$. If M is an $W(k') \otimes_{\mathbf{Z}_p} \mathcal{O}$-module, then we write M_i for e_iM.

We write $\mathrm{Art}_K \colon K^\times \to W_K^{\mathrm{ab}}$ for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements.

Lemma 1.4.1. *Let π be any uniformiser of $\mathcal{O}_K$. The composite $I_K \to \mathcal{O}_K^\times \to k^\times$, where the map $I_K \to \mathcal{O}_K^\times$ is induced by the restriction of Art_K^{-1}, sends an element $g \in I_K$ to the image in $k^\times$ of $g(\pi^{1/(p^f-1)})/\pi^{1/(p^f-1)}$.*

Proof. This follows (for example) from the construction in [Yos08, Prop. 4.4(iii), Prop. 4.7(ii), Cor. 4.9, Def. 4.10]. □

For each $\sigma \in \mathrm{Hom}(k, \overline{\mathbf{F}}_p)$ we define the fundamental character ω_σ to σ to be the composite

$$I_K \longrightarrow \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma} \overline{\mathbf{F}}_p^\times,$$

where the map $I_K \to \mathcal{O}_K^\times$ is induced by the restriction of Art_K^{-1}. Let ε denote the p-adic cyclotomic character and $\overline{\varepsilon}$ the mod p cyclotomic character, so that $\prod_{\sigma\in\mathrm{Hom}(k,\overline{\mathbf{F}}_p)} \omega_\sigma^e = \overline{\varepsilon}$. We will often identify characters of I_K with characters of $k^\times$ via the Artin map.

Inertial local Langlands. A two-dimensional *tame inertial type* is (the isomorphism class of) a tamely ramified representation $\tau : I_K \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ that extends to a representation of G_K and whose kernel is open. Such a representation is of the form $\tau \simeq \eta \oplus \eta'$, and we say that τ is a *tame principal series type* if η, η' both extend to characters of G_K. Otherwise, $\eta' = \eta^q$, and η extends to a character of G_L, where L/K is a quadratic unramified extension. In this case we say that τ is a *tame cuspidal type*.

Henniart's appendix to [BM02] associates a finite dimensional irreducible E-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ to each inertial type τ; we refer to this association as the *inertial local Langlands correspondence*. Since we are only working with tame inertial types, this correspondence can be made very explicit as follows.

If $\tau \simeq \eta \oplus \eta'$ is a tame principal series type, then we also write $\eta, \eta' : k^\times \to \mathcal{O}^\times$ for the multiplicative characters determined by $\eta \circ \mathrm{Art}_K|_{\mathcal{O}_K^\times}, \eta' \circ \mathrm{Art}_K|_{\mathcal{O}_K^\times}$ respectively. If $\eta = \eta'$, then we set $\sigma(\tau) = \eta \circ \det$. Otherwise, we write I for the Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$ consisting of matrices which are upper triangular modulo a uniformiser ϖ_K of K, and write $\chi = \eta' \otimes \eta : I \to \mathcal{O}^\times$ for the character

$$\begin{pmatrix} a & b \\ \varpi_K c & d \end{pmatrix} \mapsto \eta'(\overline{a})\eta(\overline{d}).$$

Then $\sigma(\tau) := \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$.

If $\tau = \eta \oplus \eta^q$ is a tame cuspidal type, then as above we write L/K for a quadratic unramified extension, and l for the residue field of $\mathcal{O}_L$. We write $\eta : l^\times \to \mathcal{O}^\times$ for the multiplicative character determined by $\eta \circ \mathrm{Art}_L|_{\mathcal{O}_L^\times}$; then $\sigma(\tau)$ is the inflation to $\mathrm{GL}_2(\mathcal{O}_K)$ of the cuspidal representation of $\mathrm{GL}_2(k)$ denoted by $\Theta(\eta)$ in [Dia07].

p-adic Hodge theory. We normalise Hodge–Tate weights so that all Hodge–Tate weights of the cyclotomic character are equal to -1. We say that a potentially crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has *Hodge type* 0, or is *potentially Barsotti–Tate*, if for each $\varsigma : K \hookrightarrow \overline{\mathbf{Q}}_p$, the Hodge–Tate weights of ρ with respect to ς are 0 and 1. (Note that this is a more restrictive definition of potentially Barsotti–Tate than is sometimes used; however, we will have no reason to deal with representations with non-regular Hodge-Tate weights, and so we exclude them from consideration. Note also that it is more usual in the literature to say that ρ is potentially Barsotti–Tate if it is potentially crystalline, and $\rho^\vee$ has Hodge type 0.)

We say that a potentially crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has *inertial type* τ if the traces of elements of I_K acting on τ and on

$$\mathrm{D}_{\mathrm{pcris}}(\rho) = \varinjlim_{K'/K} (\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V_\rho)^{G_{K'}}$$

are equal (here V_ρ is the underlying vector space of V_ρ). A representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *has a potentially Barsotti–Tate lift of type* τ if and only if $\overline{r}$ admits a lift to a representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ of Hodge type 0 and inertial type τ.

Serre weights. By definition, a *Serre weight* is an irreducible $\mathbf{F}$-representation of $\mathrm{GL}_2(k)$. Concretely, such a representation is of the form

$$\overline{\sigma}_{\vec{t},\vec{s}} := \otimes_{j=0}^{f-1} (\det^{t_j} \mathrm{Sym}^{s_j} k^2) \otimes_{k,\sigma_j} \mathbf{F},$$

where $0 \le s_j, t_j \le p-1$ and not all t_j are equal to $p-1$. We say that a Serre weight is *Steinberg* if $s_j = p-1$ for all j, and *non-Steinberg* otherwise.

A remark on normalisations. Given a continuous representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, there is an associated (nonempty) set of Serre weights $W(\overline{r})$ whose precise definition we will recall in Appendix A. There are in fact several different definitions of $W(\overline{r})$ in the literature; as a result of the papers [BLGG13, GK14, GLS15], these definitions are known to be equivalent up to normalisation.

However, the normalisations of Hodge–Tate weights and of inertial local Langlands used in [GK14, GLS15, EGS15] are not all the same, and so for clarity we lay out how they differ, and how they compare to the normalisations of this paper.

Our conventions for Hodge–Tate weights and inertial types agree with those of [GK14, EGS15], but our representation $\sigma(\tau)$ is the representation $\sigma(\tau^\vee)$ of [GK14, EGS15] (where $\tau^\vee = \eta^{-1} \oplus (\eta')^{-1}$); to see this, note the dual in the definition of $\sigma(\tau)$ in [GK14, Thm. 2.1.3] and the discussion in §1.9 of [EGS15]. [1]

In all cases one chooses to normalise the set of Serre weights so that the condition of Lemma A.5(1) holds. Consequently, our set of weights $W(\overline{r})$ is the set of duals of the weights $W(\overline{r})$ considered in [GK14]. In turn, the paper [GLS15] has the opposite convention for the signs of Hodge–Tate weights to our convention (and to the convention of [GK14]), so we find that our set of weights $W(\overline{r})$ is the set of duals of the weights $W(\overline{r}^\vee)$ considered in [GLS15].

Stacks. We follow the terminology of [Sta13]; in particular, we write "algebraic stack" rather than "Artin stack". More precisely, an algebraic stack is a stack in groupoids in the *fppf* topology, whose diagonal is representable by algebraic spaces, which admits a smooth surjection from a scheme. See [Sta13, Tag 026N] for a discussion of how this definition relates to others in the literature, and [Sta13, Tag 04XB] for key properties of morphisms representable by algebraic spaces.

For a commutative ring A, an *fppf stack over* A (or *fppf* A-stack) is a stack fibred in groupoids over the big *fppf* site of $\operatorname{Spec} A$.

2. Preliminaries

We begin by reviewing the various constructions and results that we will need from [CEGS20b]. Section 2.1 recalls the definition and a few basic algebraic properties of Breuil–Kisin modules with coefficients and descent data, while Section 2.2 does the same for étale φ-modules. In Section 2.3 we define the stacks $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$ (as well as various other related stacks) and state the main results of [CEGS20b]. Finally, in Section 2.4 we introduce and study stacks of Dieudonné modules that will be used at the end of the paper to determine the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$.

2.1. **Breuil–Kisin modules with descent data.**

Recall that we have a finite tamely ramified Galois extension K'/K. Suppose further that there exists a uniformiser π' of $\mathcal{O}_{K'}$ such that $\pi := (\pi')^{e(K'/K)}$ is an element of K, where $e(K'/K)$ is the ramification index of K'/K. Recall that k' is the residue field of K', while e', f' are the ramification and inertial degrees of K' respectively. Let $E(u)$ be the minimal polynomial of π' over $W(k')[1/p]$.

Let φ denote the arithmetic Frobenius automorphism of k', which lifts uniquely to an automorphism of $W(k')$ that we also denote by φ. Define $\mathfrak{S} := W(k')[[u]]$,

1. However, this dual is erroneously omitted when the inertial local Langlands correspondence is made explicit at the end of [EGS15, §3.1]. See Remark A.1.

and extend φ to $\mathfrak{S}$ by

$$\varphi\left(\sum a_i u^i\right) = \sum \varphi(a_i) u^{pi}.$$

By our assumptions that $(\pi')^{e(K'/K)} \in K$ and that K'/K is Galois, for each $g \in \mathrm{Gal}(K'/K)$ we can write $g(\pi')/\pi' = h(g)$ with $h(g) \in \mu_{e(K'/K)}(K') \subset W(k')$, and we let $\mathrm{Gal}(K'/K)$ act on $\mathfrak{S}$ via

$$g\left(\sum a_i u^i\right) = \sum g(a_i) h(g)^i u^i.$$

Let A be a p-adically complete $\mathbf{Z}_p$-algebra, set $\mathfrak{S}_A := (W(k') \otimes_{\mathbf{Z}_p} A)[[u]]$, and extend the actions of φ and $\mathrm{Gal}(K'/K)$ on $\mathfrak{S}$ to actions on $\mathfrak{S}_A$ in the obvious (A-linear) fashion.

Lemma 2.1.1. *An $\mathfrak{S}_A$-module is projective if and only if it is projective as an $A[[u]]$-module.*

Proof. Suppose that $\mathfrak{M}$ is an $\mathfrak{S}_A$-module that is projective as an $A[[u]]$-module. Certainly $W(k') \otimes_{\mathbf{Z}_p} \mathfrak{M}$ is projective over $\mathfrak{S}_A$, and we claim that it has $\mathfrak{M}$ as an $\mathfrak{S}_A$-module direct summand. Indeed, this follows by rewriting $\mathfrak{M}$ as $W(k') \otimes_{W(k')} \mathfrak{M}$ and noting that $W(k')$ is a $W(k')$-module direct summand of $W(k') \otimes_{\mathbf{Z}_p} W(k')$. ☐

The actions of φ and $\mathrm{Gal}(K'/K)$ on $\mathfrak{S}_A$ extend to actions on $\mathfrak{S}_A[1/u] = (W(k') \otimes_{\mathbf{Z}_p} A)((u))$ in the obvious way. It will sometimes be necessary to consider the subring $\mathfrak{S}_A^0 := (W(k) \otimes_{\mathbf{Z}_p} A)[[v]]$ of $\mathfrak{S}_A$ consisting of power series in $v := u^{e(K'/K)}$, on which $\mathrm{Gal}(K'/K)$ acts trivially.

Definition 2.1.2. Fix a p-adically complete $\mathbf{Z}_p$-algebra A. A *Breuil–Kisin module with A-coefficients and descent data from K' to K* (or often simply a *Breuil–Kisin module*) is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \{\hat{g}\}_{g \in \mathrm{Gal}(K'/K)})$ consisting of a $\mathfrak{S}_A$-module $\mathfrak{M}$ and a φ-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that:

- — the $\mathfrak{S}_A$-module $\mathfrak{M}$ is finitely generated and projective, and
- — the induced map $\Phi_{\mathfrak{M}} = 1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is an isomorphism after inverting $E(u)$ (here as usual we write $\varphi^*\mathfrak{M} := \mathfrak{S}_A \otimes_{\varphi, \mathfrak{S}_A} \mathfrak{M}$),

together with additive bijections $\hat{g} : \mathfrak{M} \to \mathfrak{M}$, satisfying the further properties that the maps $\hat{g}$ commute with $\varphi_{\mathfrak{M}}$, satisfy $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$, and have $\hat{g}(sm) = g(s)\hat{g}(m)$ for all $s \in \mathfrak{S}_A$, $m \in \mathfrak{M}$. We say that $\mathfrak{M}$ is has *height at most h* if the cokernel of $\Phi_{\mathfrak{M}}$ is killed by $E(u)^h$.

The Breuil–Kisin module $\mathfrak{M}$ is said to be of rank d if the underlying finitely generated projective $\mathfrak{S}_A$-module has constant rank d. It is said to be free if the underlying $\mathfrak{S}_A$-module is free.

A morphism of Breuil–Kisin modules with descent data is a morphism of $\mathfrak{S}_A$-modules that commutes with φ and with the $\hat{g}$. In the case that $K' = K$ the data of the $\hat{g}$ is trivial, so it can be forgotten, giving the category of *Breuil–Kisin modules with A-coefficients.* In this case it will sometimes be convenient to elide the difference between a Breuil–Kisin module with trivial descent data, and a Breuil–Kisin module without descent data, in order to avoid making separate definitions in the case of Breuil–Kisin modules without descent data.

Remark 2.1.3. We refer the reader to [EG21, §5.1] for a discussion of foundational results concerning finitely generated modules over the power series ring $A[[u]]$. In particular (using Lemma 2.1.1) we note the following.

(1) An $\mathfrak{S}_A$-module $\mathfrak{M}$ is finitely generated and projective if and only if it is u-torsion free and u-adically complete, and $\mathfrak{M}/u\mathfrak{M}$ is a finitely generated projective A-module ([EG21, Prop. 5.1.8]).

(2) If the $\mathfrak{S}_A$-module $\mathfrak{M}$ is projective of rank d, then it is Zariski locally free of rank d in the sense that there is a cover of $\operatorname{Spec} A$ by affine opens $\operatorname{Spec} B_i$ such that each of the base-changed modules $\mathfrak{M} \otimes_{\mathfrak{S}_A} \mathfrak{S}_{B_i}$ is free of rank d ([EG21, Prop. 5.1.9]).

(3) If A is coherent (so in particular, if A is Noetherian), then $A[[u]]$ is faithfully flat over A, and so $\mathfrak{S}_A$ is faithfully flat over A, but this need not hold if A is not coherent.

Definition 2.1.4. If Q is any (not necessarily finitely generated) A-module, and $\mathfrak{M}$ is an $A[[u]]$-module, then we let $\mathfrak{M} \widehat{\otimes}_A Q$ denote the u-adic completion of $\mathfrak{M} \otimes_A Q$.

Lemma 2.1.5. *If $\mathfrak{M}$ is a Breuil–Kisin module and B is an A-algebra, then the base change $\mathfrak{M} \widehat{\otimes}_A B$ is a Breuil–Kisin module.*

Proof. This is [CEGS20b, Lem. 2.1.4]. □

We make the following two further remarks concerning base change.

Remark 2.1.6. (1) If A is Noetherian, if Q is finitely generated over A, and if $\mathfrak{N}$ is finitely generated over $A[[u]]$, then $\mathfrak{N} \otimes_A Q$ is finitely generated over $A[[u]]$, and hence (by the Artin–Rees lemma) is automatically u-adically complete. Thus in this case the natural morphism $\mathfrak{N} \otimes_A Q \to \mathfrak{N} \widehat{\otimes}_A Q$ is an isomorphism.

(2) Note that $A[[u]] \widehat{\otimes}_A Q = Q[[u]]$ (the $A[[u]]$-module consisting of power series with coefficients in the A-module Q), and so if $\mathfrak{N}$ is Zariski locally free on $\operatorname{Spec} A$, then $\mathfrak{N} \widehat{\otimes}_A Q$ is Zariski locally isomorphic to a direct sum of copies of $Q[[u]]$, and hence is u-torsion free (as well as being u-adically complete). In particular, by Remark 2.1.3(2), this holds if $\mathfrak{N}$ is projective.

Let A be a $\mathbf{Z}_p$-algebra. We define a *Dieudonné module of rank d with A-coefficients and descent data from K' to K* to be a finitely generated projective $W(k') \otimes_{\mathbf{Z}_p} A$-module D of constant rank d on $\operatorname{Spec} A$, together with:

— A-linear endomorphisms F, V satisfying $FV = VF = p$ such that F is φ-semilinear and V is φ^{-1}-semilinear for the action of $W(k')$, and

— a $W(k') \otimes_{\mathbf{Z}_p} A$-semilinear action of $\operatorname{Gal}(K'/K)$ which commutes with F and V.

Definition 2.1.7. If $\mathfrak{M}$ is a Breuil–Kisin module of height at most 1 and rank d with descent data, then there is a corresponding Dieudonné module $D = D(\mathfrak{M})$ of rank d defined as follows. We set $D := \mathfrak{M}/u\mathfrak{M}$ with the induced action of $\operatorname{Gal}(K'/K)$, and F given by the induced action of φ. The endomorphism V is determined as follows. Write $E(0) = cp$, so that we have $p \equiv c^{-1}E(u) \pmod{u}$. The condition that the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)$ allows us to factor the multiplication-by-$E(u)$ map on $\mathfrak{M}$ uniquely as $\mathfrak{V} \circ \varphi$, and V is defined to be $c^{-1}\mathfrak{V}$ modulo u.

2.2. **Étale φ-modules and Galois representations.**

Definition 2.2.1. Let A be a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$. A *weak étale φ-module* with A-coefficients and descent data from K' to K is a triple $(M, \varphi_M, \{\hat{g}\})$ consisting of:

— a finitely generated $\mathfrak{S}_A[1/u]$-module M;

— a φ-semilinear map $\varphi_M : M \to M$ with the property that the induced map

$$\Phi_M = 1 \otimes \varphi_M : \varphi^* M := \mathfrak{S}_A[1/u] \otimes_{\varphi, \mathfrak{S}_A[1/u]} M \to M$$

is an isomorphism,

together with additive bijections $\hat{g} : M \to M$ for $g \in \mathrm{Gal}(K'/K)$, satisfying the further properties that the maps $\hat{g}$ commute with φ_M, satisfy $\hat{g_1} \circ \hat{g_2} = \widehat{g_1 \circ g_2}$, and have $\hat{g}(sm) = g(s)\hat{g}(m)$ for all $s \in \mathfrak{S}_A[1/u]$, $m \in M$.

If M as above is projective as an $\mathfrak{S}_A[1/u]$-module then we say simply that M is an étale φ-module. The étale φ-module M is said to be of rank d if the underlying finitely generated projective $\mathfrak{S}_A[1/u]$-module has constant rank d.

Remark 2.2.2. We could also consider étale φ-modules for general p-adically complete $\mathbf{Z}_p$-algebras A, but we would need to replace $\mathfrak{S}_A[1/u]$ by its p-adic completion. As we will not need to consider these modules in this paper, we do not do so here, but we refer the interested reader to [EG22].

A morphism of weak étale φ-modules with A-coefficients and descent data from K' to K is a morphism of $\mathfrak{S}_A[1/u]$-modules that commutes with φ and with the $\hat{g}$. Again, in the case $K' = K$ the descent data is trivial, and we obtain the usual category of étale φ-modules with A-coefficients.

Note that if A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra, and $\mathfrak{M}$ is a Breuil–Kisin module with descent data, then $\mathfrak{M}[1/u]$ naturally has the structure of an étale φ-module with descent data.

Suppose that A is an $\mathcal{O}$-algebra (where $\mathcal{O}$ is as in Section 1.4). In making calculations, it is often convenient to use the idempotents e_i (again as in Section 1.4). In particular if $\mathfrak{M}$ is a Breuil–Kisin module, then writing as usual $\mathfrak{M}_i := e_i\mathfrak{M}$, we write $\Phi_{\mathfrak{M},i} : \varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$ for the morphism induced by $\Phi_{\mathfrak{M}}$. Similarly if M is an étale φ-module then we write $M_i := e_iM$, and we write $\Phi_{M,i} : \varphi^*(M_{i-1}) \to M_i$ for the morphism induced by Φ_M.

To connect étale φ-modules to G_{K_∞}-representations we begin by recalling from [Kis09] some constructions arising in p-adic Hodge theory and the theory of fields of norms, which go back to [Fon90]. Following Fontaine, we write $R := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\bar{K}}/p$. Fix a compatible system $(\sqrt[p^n]{\pi})_{n \geq 0}$ of p^nth roots of π in $\bar{K}$ (compatible in the obvious sense that $\left(\sqrt[p^{n+1}]{\pi}\right)^p = \sqrt[p^n]{\pi}$), and let $K_\infty := \cup_n K(\sqrt[p^n]{\pi})$, and also $K'_\infty := \cup_n K'(\sqrt[p^n]{\pi})$. Since $(e(K'/K), p) = 1$, the compatible system $(\sqrt[p^n]{\pi})_{n \geq 0}$ determines a unique compatible system $(\sqrt[p^n]{\pi'})_{n \geq 0}$ of p^nth roots of π' such that $(\sqrt[p^n]{\pi'})^{e(K'/K)} = \sqrt[p^n]{\pi}$. Write $\underline{\pi}' = (\sqrt[p^n]{\pi'})_{n \geq 0} \in R$, and $[\underline{\pi}'] \in W(R)$ for its image under the natural multiplicative map $R \to W(R)$. We have a Frobenius-equivariant inclusion $\mathfrak{S} \hookrightarrow W(R)$ by sending $u \mapsto [\underline{\pi}']$. We can naturally identify $\mathrm{Gal}(K'_\infty/K_\infty)$ with $\mathrm{Gal}(K'/K)$, and doing this we see that the action of $g \in G_{K_\infty}$ on u is via $g(u) = h(g)u$.

We let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}[1/u]$, and let $\mathcal{E}$ be the field of fractions of $\mathcal{O}_{\mathcal{E}}$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{E} \hookrightarrow W(\mathrm{Frac}(R))[1/p]$. Let $\mathcal{E}^{\mathrm{nr}}$ be the maximal unramified extension of $\mathcal{E}$ in $W(\mathrm{Frac}(R))[1/p]$, and let $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}} \subset W(\mathrm{Frac}(R))$ denote its ring of integers. Let $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ be the p-adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$. Note that $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ is stable under the action of G_{K_∞}.

Definition 2.2.3. Suppose that A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$. If M is a weak étale φ-module with A-coefficients and descent data, set $T_A(M) :=$

$\big(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} M\big)^{\varphi=1}$, an A-module with a G_{K_∞}-action (via the diagonal action on $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ and M, the latter given by the $\hat{g}$). If $\mathfrak{M}$ is a Breuil–Kisin module with A-coefficients and descent data, set $T_A(\mathfrak{M}) := T_A(\mathfrak{M}[1/u])$.

Lemma 2.2.4. *Suppose that A is a local $\mathbf{Z}_p$-algebra and that $|A| < \infty$. Then T_A induces an equivalence of categories from the category of weak étale φ-modules with A-coefficients and descent data to the category of continuous representations of G_{K_∞} on finite A-modules. If $A \to A'$ is finite, then there is a natural isomorphism $T_A(M) \otimes_A A' \xrightarrow{\sim} T_{A'}(M \otimes_A A')$. A weak étale φ-module with A-coefficients and descent data M is free of rank d if and only if $T_A(M)$ is a free A-module of rank d.*

Proof. This is due to Fontaine [Fon90], and can be proved in exactly the same way as [Kis09, Lem. 1.2.7]. ☐

We will frequently simply write T for T_A. Note that if we let M' be the étale φ-module obtained from M by forgetting the descent data, then by definition we have $T(M') = T(M)|_{G_{K'_\infty}}$.

Remark 2.2.5. Although étale φ-modules naturally give rise to representations of G_{K_∞}, those coming from Breuil–Kisin modules of height at most 1 admit canonical extensions to G_K by [Kis09, Prop. 1.1.13].

Lemma 2.2.6. *If $\overline{r}, \overline{r}' : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ are continuous representations, both of which arise as the reduction mod p of potentially Barsotti–Tate representations of tame inertial type, and there is an isomorphism $\overline{r}|_{G_{K_\infty}} \cong \overline{r}'|_{G_{K_\infty}}$, then $\overline{r} \cong \overline{r}'$.*

Proof. The extension K_∞/K is totally wildly ramified. Since the irreducible $\overline{\mathbf{F}}_p$-representations of G_K are induced from tamely ramified characters, we see that $\overline{r}|_{G_{K_\infty}}$ is irreducible if and only if $\overline{r}$ is irreducible, and if $\overline{r}$ or $\overline{r}'$ is irreducible then we are done. In the reducible case, we see that $\overline{r}$ and $\overline{r}'$ are extensions of the same characters, and the result then follows from [GLS15, Lem. 5.4.2] and Lemma A.5 (2). ☐

2.3. Recollections from [CEGS20b].

The main objects of study in this paper are certain algebraic stacks $\mathcal{C}^{\tau,\mathrm{BT}}$ and $\mathcal{Z}^{\tau,1}$, of rank two Breuil–Kisin modules and étale φ-modules respectively, that were introduced and studied in [CEGS20b]. We review their definitions now, and recall the main properties of these stacks that were established in [CEGS20b].

To define $\mathcal{C}^{\tau,\mathrm{BT},1}$ we first introduce stacks of Breuil–Kisin modules with descent data; then we impose two conditions on them, corresponding (in the analogy with Galois representations) to fixing an inertial type τ and requiring all pairs Hodge–Tate weights to be $\{0,1\}$.

Take K'/K to be any Galois extension such that $[K' : K]$ is prime to p, and write $N = K \cdot W(k')[1/p]$.

Definition 2.3.1. For each integer $a \geq 1$, we let $\mathcal{C}^{\mathrm{dd},a}_{d,h}$ be the *fppf* stack over $\mathcal{O}/\varpi^a$ which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{C}^{\mathrm{dd},a}_{d,h}(A)$ of rank d Breuil–Kisin modules of height at most h with A-coefficients and descent data from K' to K.

By [Sta13, Tag 04WV], we may also regard each of the stacks $\mathcal{C}^{\mathrm{dd},a}_{d,h}$ as an *fppf* stack over $\mathcal{O}$, and we then write $\mathcal{C}^{\mathrm{dd}}_{d,h} := \varinjlim_a \mathcal{C}^{\mathrm{dd},a}_{d,h}$; this is again an *fppf* stack over $\mathcal{O}$.

We will omit the subscripts d, h from this notation when doing so will not cause confusion.

Definition 2.3.2. Let τ be a d-dimensional E-representation of $I(K'/K)$. We say that an object $\mathfrak{M}$ of $\mathcal{C}^{\mathrm{dd},a}$ is has *type* τ if Zariski locally on $\operatorname{Spec} A$ there is an $I(K'/K)$-equivariant isomorphism $\mathfrak{M}_i/u\mathfrak{M}_i \cong A \otimes_{\mathcal{O}} \tau^\circ$ for each i. (Here we recall that $\mathfrak{M}_i := e_i\mathfrak{M}$, and τ° denotes an $\mathcal{O}$-lattice in τ.)

Definition 2.3.3. Let $\mathcal{C}^\tau$ be the étale substack of $\mathcal{C}^{\mathrm{dd}}$ consisting of the objects of type τ. This is an open and closed substack of $\mathcal{C}^{\mathrm{dd}}$ (see [CEGS20b, Prop. 3.3.5]).

For the remainder of this section we fix $d = 2$ and $h = 1$. Suppose that A is an $\mathcal{O}/\varpi^a$-algebra and consider a pair $(\mathfrak{L}, \mathfrak{L}^+)$, where:

— $\mathfrak{L}$ is a rank 2 projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module, with a $\mathrm{Gal}(K'/K)$-semilinear, A-linear action of $\mathrm{Gal}(K'/K)$;

— $\mathfrak{L}^+$ is an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$, which is locally on $\operatorname{Spec} A$ a direct summand of $\mathfrak{L}$ as an A-module (or equivalently, for which $\mathfrak{L}/\mathfrak{L}^+$ is projective as an A-module), and is preserved by $\mathrm{Gal}(K'/K)$.

For each character $\xi : I(K'/K) \to \mathcal{O}^\times$, let $\mathfrak{L}_\xi$ (resp. $\mathfrak{L}^+_\xi$) be the $\mathcal{O}_N \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$ (resp. $\mathfrak{L}^+$) on which $I(K'/K)$ acts through ξ. We say that the pair $(\mathfrak{L}, \mathfrak{L}^+)$ *satisfies the strong determinant condition* if Zariski locally on $\operatorname{Spec} A$ the following condition holds: for all $\alpha \in \mathcal{O}_N$ and all ξ, we have

$$\det_A(\alpha|\mathfrak{L}^+_\xi) = \prod_{\psi : N \hookrightarrow E} \psi(\alpha) \tag{2.3.4}$$

as polynomial functions on $\mathcal{O}_N$ in the sense of [Kot92, §5].

Definition 2.3.5. An object $\mathfrak{M}$ of $\mathcal{C}^{\mathrm{dd},a}$ *satisfies the strong determinant condition* if the pair $(\mathfrak{M}/E(u)\mathfrak{M}, \operatorname{im}\Phi_{\mathfrak{M}}/E(u)\mathfrak{M})$ satisfies the strong determinant condition as in the previous paragraph.

We define $\mathcal{C}^{\tau,\mathrm{BT}}$ to be the substack of $\mathcal{C}^\tau$ of objects satisfying the strong determinant condition. This is a ϖ-adic formal algebraic stack of finite presentation over $\mathcal{O}$ by [CEGS20b, Prop. 4.2.7], and so its special fibre $\mathcal{C}^{\tau,\mathrm{BT},1}$ is an algebraic stack, locally of finite type over $\mathbf{F}$. The $\mathrm{Spf}(\mathcal{O}_{E'})$-points of $\mathcal{C}^{\tau,\mathrm{BT}}$, for any finite extension E'/E, correspond to potentially Barsotti–Tate Galois representations $G_K \to \mathrm{GL}_2(\mathcal{O}_{E'})$ of inertial type τ ([CEGS20b, Lem. 4.2.16]).

The following result combines [CEGS20b, Cor. 4.5.3, Prop. 5.2.21].

Theorem 2.3.6. *We have:*

(1) $\mathcal{C}^{\tau,\mathrm{BT}}$ is analytically normal, and Cohen–Macaulay.

(2) The special fibre $\mathcal{C}^{\tau,\mathrm{BT},1}$ is reduced and equidimensional of dimension equal to $[K : \mathbf{Q}_p]$.

(3) $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}$.

We now introduce our stacks of étale φ-modules.

Definition 2.3.7. Let $\mathcal{R}^{\mathrm{dd},1}$ be the *fppf* $\mathbf{F}$-stack which associates to any $\mathbf{F}$-algebra A the groupoid $\mathcal{R}^{\mathrm{dd},1}(A)$ of rank 2 étale φ-modules with A-coefficients and descent data from K' to K.

Inverting u gives a proper morphism $\mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1}$, which then restricts to a proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{R}^{\mathrm{dd},1}$ for each τ.

We now briefly remind the reader of some definitions from [EG21, §3.2]. Let $\mathcal{X} \to \mathcal{F}$ be a proper morphism of stacks over a locally Noetherian base-scheme S, where $\mathcal{X}$ is an algebraic stack which is locally of finite presentation over S, and the diagonal of $\mathcal{F}$ is representable by algebraic spaces and locally of finite presentation.

We refer to [EG21, Defn. 3.2.8] for the definition of the *scheme-theoretic image* $\mathcal{Z}$ of the proper morphism $\mathcal{X} \to \mathcal{F}$. By definition, it is a full subcategory in groupoids of $\mathcal{F}$, and in fact by [EG21, Lem. 3.2.9] it is a Zariski substack of $\mathcal{F}$. By [EG21, Lem. 3.2.14], the finite type points of $\mathcal{Z}$ are precisely the finite type points of $\mathcal{F}$ for which the corresponding fibre of $\mathcal{X}$ is nonzero.

The results of [EG21, §3.2] give criteria for $\mathcal{Z}$ to be an algebraic stack, and prove a number of associated results (such as universal properties of the morphism $\mathcal{Z} \to \mathcal{F}$, and a description of versal deformation rings for $\mathcal{Z}$). This formalism applies in particular to the proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{R}^{\mathrm{dd},1}$, and so we make the following definition.

Definition 2.3.8. We define $\mathcal{Z}^{\tau,1}$ to be the scheme-theoretic image (in the sense of [EG21, Defn. 3.2.8]) of the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{R}^{\mathrm{dd},1}$.

In [CEGS20b, Thm. 5.1.2, Prop. 5.2.20] we established the following properties of this construction.

Proposition 2.3.9.

(1) $\mathcal{Z}^{\tau,1}$ is an algebraic stack of finite presentation over $\mathbf{F}$, and is a closed substack of $\mathcal{R}^{\mathrm{dd},1}$.

(2) The morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{R}^{\mathrm{dd},1}$ factors through a morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\mathrm{dd},1}$ which is representable by algebraic spaces, scheme-theoretically dominant, and proper.

(3) The $\overline{\mathbf{F}}_p$-points of $\mathcal{Z}^{\tau,1}$ are naturally in bijection with the continuous representations $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which have a potentially Barsotti–Tate lift of type τ.

Theorem 2.3.10. *The algebraic stacks $\mathcal{Z}^{\tau,1}$ are equidimensional of dimension equal to $[K : \mathbf{Q}_p]$.*

2.4. **Dieudonné and gauge stacks.** We now specialise the choice of K' in the following way. Choose a tame inertial type $\tau = \eta \oplus \eta'$. Fix a uniformiser π of K. If τ is a tame principal series type, we take $K' = K(\pi^{1/(p^f-1)})$, while if τ is a tame cuspidal type, we let L be an unramified quadratic extension of K, and set $K' = L(\pi^{1/(p^{2f}-1)})$. Let N be the maximal unramified extension of K in K'. In either case K'/K is a Galois extension; in the principal series case, we have $e' = (p^f - 1)e$, $f' = f$, and in the cuspidal case we have $e' = (p^{2f} - 1)e$, $f' = 2f$. We refer to this choice of extension as the *standard choice* (for the fixed type τ and uniformiser π).

For the rest of this section we assume that $\eta \neq \eta'$ (we will not need to consider Dieudonné modules for scalar types). Let $\mathfrak{M}$ be an object of $\mathcal{C}^{\tau,\mathrm{BT}}(A)$, and let $D := \mathfrak{M}/u\mathfrak{M}$ be its corresponding Dieudonné module as in Definition 2.1.7. The group $I(K'/K)$ is abelian of order prime to p, and so we can write $D = D_\eta \oplus D_{\eta'}$, where D_η is the submodule on which $I(K'/K)$ acts via η. Setting $D_{\eta,j} := e_j D_\eta$, it follows from the projectivity of $\mathfrak{M}$ that each $D_{\eta,j}$ is an invertible A-module. The maps F, V induce linear maps $F : D_{\eta,j} \to D_{\eta,j+1}$ and $V : D_{\eta,j+1} \to D_{\eta,j}$ such that $FV = VF = p$.

Definition 2.4.1. If τ is a principal series type we define a stack

$$\mathcal{D}_\eta := \Big[\big(\operatorname{Spec} W(k)[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]/(X_jY_j - p)_{j=0,\dots,f-1}\big)/\mathbf{G}_m^f\Big],$$

where the f copies of $\mathbf{G}_m$ act as $(u_0, \dots, u_{f-1}) \cdot (X_j, Y_j) \mapsto (u_j u_{j+1}^{-1} X_j, u_{j+1} u_j^{-1} Y_j)$.

If instead τ is a cuspidal type we define

$$\mathcal{D}_\eta := \Big[\big(\operatorname{Spec} W(k)[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]/(X_jY_j - p)_{j=0,\dots,f-1}) \times \mathbf{G}_m\big)/\mathbf{G}_m^{f+1}\Big],$$

where the $f+1$ copies of $\mathbf{G}_m$ act as

$$(u_0, \dots, u_{f-1}, u_f) \cdot ((X_j, Y_j), \alpha) \mapsto ((u_j u_{j+1}^{-1} X_j, u_{j+1} u_j^{-1} Y_j), \alpha).$$

In [CEGS20b, Sec. 4.6] we explained how the stack $\mathcal{D}_\eta$ classifies the line bundles $D_{\eta,j}$ together with the maps F, V, so that in either case (principal series or cuspidal) there is a natural map $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{D}_\eta$.

It will be helpful to introduce another stack, the stack $\mathcal{G}_\eta$ of η-gauges. This classifies f-tuples of line bundles $\mathcal{D}_j$ $(j = 0, \dots, f-1)$ equipped with sections $X_j \in \mathcal{D}_j$ and $Y_j \in \mathcal{D}_j^{-1}$. Explicitly, it can be written as the quotient stack

$$\mathcal{G}_\eta := \Big[\big(\operatorname{Spec} W(k)[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]/(X_jY_j - p)_{j=0,\dots,f-1}\big)/\mathbf{G}_m^f\Big],$$

where the f copies of $\mathbf{G}_m$ act as follows:

$$(v_0, \dots, v_{f-1}) \cdot (X_j, Y_j) \mapsto (v_j X_j, v_j^{-1} Y_j).$$

There is a morphism of stacks $\mathcal{D}_\eta \to \mathcal{G}_\eta$ which we can define explicitly using their descriptions as quotient stacks. Indeed, in the principal series case we have a morphism $\mathbf{G}_m^f \to \mathbf{G}_m^f$ given by $(u_j)_{j=0,\dots,f-1} \mapsto (u_j u_{j+1}^{-1})_{j=0,\dots,f-1}$, which is compatible with the actions of these two groups on $\operatorname{Spec} W(k)[(X_j, Y_j)_{j=0,\dots,f-1}]/(X_jY_j - p)_{j=0,\dots,f-1}$, and we are just considering the map from the quotient by the first $\mathbf{G}_m^f$ to the quotient by the second $\mathbf{G}_m^f$. In the cuspidal case we have a morphism $\mathbf{G}_m^{f+1} \to \mathbf{G}_m^f$ given by $(u_j)_{j=0,\dots,f} \mapsto (u_j u_{j+1}^{-1})_{j=0,\dots,f-1}$, and the morphism $\mathcal{D}_\eta \to \mathcal{G}_\eta$ is the obvious one which forgets the factor of $\mathbf{G}_m$ coming from α.

Composing our morphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{D}_\eta$ with the forgetful morphism $\mathcal{D}_\eta \to \mathcal{G}_\eta$, we obtain a morphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{G}_\eta$.

For our analysis of the irreducible components of the stacks $\mathcal{C}^{\tau,\mathrm{BT},1}$ at the end of Section 3, it will be useful to have a more directly geometric interpretation of a morphism $S \to \mathcal{G}_\eta$, in the case that the source is a *flat* $W(k)$-scheme, or, more generally, a flat p-adic formal algebraic stack over $\operatorname{Spf} W(k)$. In order to do this we will need some basic material on effective Cartier divisors for (formal) algebraic stacks; while it is presumably possible to develop this theory in considerable generality, we only need a very special case, and we limit ourselves to this setting.

The property of a closed subscheme being an effective Cartier divisor is not preserved under arbitrary pull-back, but it is preserved under flat pull-back. More precisely, we have the following result.

Lemma 2.4.2. *If X is a scheme, and Z is a closed subscheme of X, then the following are equivalent:*

(1) Z is an effective Cartier divisor on X.

(2) *For any flat morphism of schemes $U \to X$, the pull-back $Z \times_X U$ is an effective Cartier divisor on U.*

(3) *For some fpqc covering $\{X_i \to X\}$ of X, each of the pull-backs $Z \times_X X_i$ is an effective Cartier divisor on X_i.*

Proof. Since Z is an effective Cartier divisor if and only if its ideal sheaf $\mathcal{I}_Z$ is an invertible sheaf on X, this follows from the fact that the invertibility of a quasi-coherent sheaf is a local property in the *fpqc* topology. □

Lemma 2.4.3. *If A is a Noetherian adic topological ring, then pull-back under the natural morphism $\operatorname{Spf} A \to \operatorname{Spec} A$ induces a bijection between the closed subschemes of $\operatorname{Spec} A$ and the closed subspaces of $\operatorname{Spf} A$.*

Proof. It follows from [Sta13, Tag 0ANQ] that closed immersions $Z \to \operatorname{Spf} A$ are necessarily of the form $\operatorname{Spf} B \to \operatorname{Spf} A$, and correspond to continuous morphisms $A \to B$, for some complete linearly topologized ring B, which are taut (in the sense of [Sta13, Tag 0AMX]), have closed kernel, and dense image. Since A is adic, it admits a countable basis of neighbourhoods of the origin, and so it follows from [Sta13, Tag 0APT] (recalling also [Sta13, Tag 0AMV]) that $A \to B$ is surjective. Because any ideal of definition I of A is finitely generated, it follows from [Sta13, Tag 0APU] that B is endowed with the I-adic topology. Finally, since A is Noetherian, any ideal in A is I-adically closed. Thus closed immersions $\operatorname{Spf} B \to \operatorname{Spf} A$ are determined by giving the kernel of the corresponding morphism $A \to B$, which can be arbitrary. The same is true of closed immersions $\operatorname{Spec} B \to \operatorname{Spec} A$, and so the lemma follows. □

Definition 2.4.4. If A is a Noetherian adic topological ring, then we say that a closed subspace of $\operatorname{Spf} A$ is an *effective Cartier divisor* on $\operatorname{Spf} A$ if the corresponding closed subscheme of $\operatorname{Spec} A$ is an effective Cartier divisor on $\operatorname{Spec} A$.

Lemma 2.4.5. *Let $\operatorname{Spf} B \to \operatorname{Spf} A$ be a flat adic morphism of Noetherian affine formal algebraic spaces. If $Z \hookrightarrow \operatorname{Spf} A$ is a Cartier divisor, then $Z \times_{\operatorname{Spf} A} \operatorname{Spf} B \hookrightarrow \operatorname{Spf} B$ is a Cartier divisor. Conversely, if $\operatorname{Spf} B \to \operatorname{Spf} A$ is furthermore surjective, and if $Z \hookrightarrow \operatorname{Spf} A$ is a closed subspace for which the base-change $Z \times_{\operatorname{Spf} A} \operatorname{Spf} B \hookrightarrow \operatorname{Spf} B$ is a Cartier divisor, then Z is a Cartier divisor on $\operatorname{Spf} A$.*

Proof. The morphism $\operatorname{Spf} B \to \operatorname{Spf} A$ corresponds to an adic flat morphism $A \to B$ ([Sta13, Tag 0AN0] and [Eme, Lem. 8.18]) and hence is induced by a flat morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, which is furthermore faithfully flat if and only if $\operatorname{Spf} B \to \operatorname{Spf} A$ is surjective (again by [Eme, Lem. 8.18]). The present lemma thus follows from Lemma 2.4.2. □

The preceding lemma justifies the following definition.

Definition 2.4.6. We say that a closed substack $\mathcal{Z}$ of a locally Noetherian formal algebraic stack $\mathcal{X}$ is an *effective Cartier divisor* on $\mathcal{X}$ if for any morphism $U \to \mathcal{X}$ whose source is a Noetherian affine formal algebraic space, and which is representable by algebraic spaces and flat, the pull-back $\mathcal{Z} \times_{\mathcal{X}} U$ is an effective Cartier divisor on U.

We consider the $W(k)$-scheme $\operatorname{Spec} W(k)[X,Y]/(XY-p)$, which we endow with a $\mathbf{G}_m$-action via $u \cdot (X,Y) := (uX, u^{-1}Y)$. There is an obvious morphism

$$\operatorname{Spec} W(k)[X,Y]/(XY-p) \to \operatorname{Spec} W(k)[X] = \mathbf{A}^1$$

given by $(X,Y) \to X$, which is $\mathbf{G}_m$-equivariant (for the action of $\mathbf{G}_m$ on $\mathbf{A}^1$ given by $u \cdot X := uX$), and so induces a morphism

$$[\big(\operatorname{Spec} W(k)[X,Y]/(XY-p)\big)/\mathbf{G}_m] \to [\mathbf{A}^1/\mathbf{G}_m]. \tag{2.4.7}$$

Lemma 2.4.8. *If $\mathcal{X}$ is a locally Noetherian p-adic formal algebraic stack which is furthermore flat over* $\operatorname{Spf} W(k)$*, then the groupoid of morphisms*

$$\mathcal{X} \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m]$$

is in fact a setoid, and is equivalent to the set of effective Cartier divisors on $\mathcal{X}$ that are contained in the effective Cartier divisor $(\operatorname{Spec} k) \times_{\operatorname{Spf} W(k)} \mathcal{X}$ *on $\mathcal{X}$.*

Proof. Essentially by definition (and taking into account [Eme, Lem. 8.18]), it suffices to prove this in the case when $\mathcal{X} = \operatorname{Spf} B$, where B is a flat Noetherian adic $W(k)$-algebra admitting (p) as an ideal of definition. In this case, the restriction map

$$[\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m](\operatorname{Spec} B) \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m](\operatorname{Spf} B)$$

is an equivalence of groupoids. Indeed, the essential surjectivity follows from the (standard and easily verified) fact that if $\{M_i\}$ is a compatible family of locally free B/p^iB-modules of rank one, then $M := \varprojlim M_i$ is a locally free B-module of rank one, for which each of the natural morphisms $M/p^iM \to M_i$ is an isomorphism. The full faithfulness follows from the fact that a locally free B-module of rank one is p-adically complete, and so is recovered as the inverse limit of its compatible family of quotients $\{M/p^iM\}$.

We are therefore reduced to the same statement with $\mathcal{X} = \operatorname{Spec} B$. The composite morphism $\operatorname{Spec} B \to [\mathbf{A}^1/\mathbf{G}_m]$ induced by (2.4.7) corresponds to giving a pair $(\mathcal{D}, X)$ where $\mathcal{D}$ is a line bundle on $\operatorname{Spec} B$, and X is a global section of $\mathcal{D}^{-1}$. Indeed, giving a morphism $\operatorname{Spec} B \to [\mathbf{A}^1/\mathbf{G}_m]$ is equivalent to giving a $\mathbf{G}_m$-torsor $P \to \operatorname{Spec} B$, together with a $\mathbf{G}_m$-equivariant morphism $P \to \mathbf{A}^1$. Giving a $\mathbf{G}_m$-torsor P over $\operatorname{Spec} B$ is equivalent to giving an invertible sheaf $\mathcal{D}$ on $\operatorname{Spec} B$ (the associated $\mathbf{G}_m$-torsor is then obtained by deleting the zero section from the line bundle $D \to X$ corresponding to $\mathcal{D}$), and giving a $\mathbf{G}_m$-equivariant morphism $P \to \mathbf{A}^1$ is equivalent to giving a global section of $\mathcal{D}^{-1}$.

It follows that giving a morphism $\operatorname{Spec} B \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m]$ corresponds to giving a line bundle $\mathcal{D}$ and sections $X \in \mathcal{D}^{-1}$, $Y \in \mathcal{D}$ satisfying $XY = p$. To say that B is flat over $W(k)$ is just to say that p is a regular element on B, and so we see that X (resp. Y) is a regular section of $\mathcal{D}^{-1}$ (resp. $\mathcal{D}$). Again, since p is a regular element on B, we see that Y is uniquely determined by X and the equation $XY = p$, and so giving a morphism $\operatorname{Spec} B \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m]$ is equivalent to giving a line bundle $\mathcal{D}$ and a regular section X of $\mathcal{D}^{-1}$, such that $pB \subset X \otimes_B \mathcal{D} \subset \mathcal{D}^{-1} \otimes_B \mathcal{D} \xrightarrow{\sim} B$; this last condition guarantees the existence of the (then uniquely determined) Y.

Now giving a line bundle $\mathcal{D}$ on $\operatorname{Spec} B$ and a regular section $X \in \mathcal{D}^{-1}$ is the same as giving the zero locus D of X, which is a Cartier divisor on $\operatorname{Spec} B$. (There is a canonical isomorphism $(\mathcal{D}, X) \cong \big(\mathcal{I}_D, 1\big)$, where $\mathcal{I}_D$ denotes the ideal sheaf of D.) The condition that $pB \subset X \otimes_B \mathcal{D}$ is equivalent to the condition that $p \in \mathcal{I}_D$, i.e. that D be contained in $\operatorname{Spec} B/pB$, and we are done. $\square$

Lemma 2.4.9. *If $\mathcal{S}$ is a locally Noetherian p-adic formal algebraic stack which is flat over $W(k)$, then giving a morphism $\mathcal{S} \to \mathcal{G}_\eta$ over $W(k)$ is equivalent to giving* *a collection of effective Cartier divisors $\mathcal{D}_j$ on $\mathcal{S}$ $(j = 0, \dots, f-1)$, with each $\mathcal{D}_j$ contained in the Cartier divisor $\overline{\mathcal{S}}$ cut out by the equation $p = 0$ on $\mathcal{S}$ (i.e. the* special fibre *of $\mathcal{S}$).*

Proof. This follows immediately from Lemma 2.4.8, by the definition of $\mathcal{G}_\eta$. $\square$

3. Families of extensions of Breuil–Kisin modules

The goal of the next two sections is to construct certain universal families of extensions of rank one Breuil–Kisin modules over $\mathbf{F}$ with descent data; these families will be used in Section 5 to describe the generic behaviour of the various irreducible components of the special fibres of $\mathcal{C}^{\tau,\mathrm{BT}}$ and $\mathcal{Z}^\tau$.

In Subsections 3.1 and 3.2 we present some generalities on extensions of Breuil–Kisin modules and on families of these extensions, respectively. In Subsection 3.3 we specialize the discussion of Subsection 3.2 to the case of extensions of two rank one Breuil–Kisin modules, and thus explain how to construct our desired families of extensions. In Section 4 we recall the fundamental computations related to extensions of rank one Breuil–Kisin modules from [DS15], to which the results of Subsection 3.3 will be applied at the end of Subsection 4.2 to construct the components $\overline{\mathcal{C}}(J)$ and $\overline{\mathcal{Z}}(J)$ of Theorem 1.1.

We assume throughout this section that $[K' : K]$ is not divisible by p; since we are assuming throughout the paper that K'/K is tamely ramified, this is equivalent to assuming that K' does not contain an unramified extension of K of degree p. In our final applications K'/K will contain unramified extensions of degree at most 2, and p will be odd, so this assumption will be satisfied. (In fact, we specialize to such a context begining in Subsection 5.2.)

3.1. Extensions of Breuil–Kisin modules with descent data.

When discussing the general theory of extensions of Breuil–Kisin modules, it is convenient to embed the category of Breuil–Kisin modules in a larger category which is abelian, contains enough injectives and projectives, and is closed under passing to arbitrary limits and colimits. The simplest way to obtain such a category is as the category of modules over some ring, and so we briefly recall how a Breuil–Kisin module with A-coefficients and descent data can be interpreted as a module over a certain A-algebra.

Let $\mathfrak{S}_A[F]$ denote the twisted polynomial ring over $\mathfrak{S}_A$, in which the variable F obeys the following commutation relation with respect to elements $s \in \mathfrak{S}_A$:

$$F \cdot s = \varphi(s) \cdot F.$$

Let $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$ denote the twisted group ring over $\mathfrak{S}_A[F]$, in which the elements $g \in \mathrm{Gal}(K'/K)$ commute with F, and obey the following commutation relations with elements $s \in \mathfrak{S}_A$:

$$g \cdot s = g(s) \cdot g.$$

One immediately confirms that giving a left $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$-module $\mathfrak{M}$ is equivalent to equipping the underlying $\mathfrak{S}_A$-module $\mathfrak{M}$ with a φ-linear morphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ and a semi-linear action of $\mathrm{Gal}(K'/K)$ which commutes with φ.

In particular, if we let $\mathcal{K}(A)$ denote the category of left $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$-modules, then a Breuil–Kisin module with descent data from K' to K may naturally be regarded as an object of $\mathcal{K}(A)$. In the following lemma, we record the fact that extensions of Breuil–Kisin modules with descent data may be computed as extensions in the category $\mathcal{K}(A)$.

Lemma 3.1.1. *If $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is a short exact sequence in $\mathcal{K}(A)$, such that $\mathfrak{M}'$ (resp. $\mathfrak{M}''$) is a Breuil–Kisin module with descent data of rank d' and height at most h' (resp. of rank d'' and height at most h''), then $\mathfrak{M}$ is a Breuil–Kisin module with descent data of rank $d' + d''$ and height at most $h' + h''$.*

More generally, if $E(u)^h \in \operatorname{Ann}_{\mathfrak{S}_A}(\operatorname{coker}\Phi_{\mathfrak{M}'})\operatorname{Ann}_{\mathfrak{S}_A}(\operatorname{coker}\Phi_{\mathfrak{M}''})$, then $\mathfrak{M}$ is a Breuil–Kisin module with descent data of height at most h.

Proof. Note that since $\Phi_{\mathfrak{M}'}[1/E(u)]$ and $\Phi_{\mathfrak{M}''}[1/E(u)]$ are both isomorphisms by assumption, it follows from the snake lemma that $\Phi_{\mathfrak{M}}[1/E(u)]$ is isomorphism. Similarly we have a short exact sequence of $\mathfrak{S}_A$-modules

$$0 \to \operatorname{coker}\Phi_{\mathfrak{M}'} \to \operatorname{coker}\Phi_{\mathfrak{M}} \to \operatorname{coker}\Phi_{\mathfrak{M}''} \to 0.$$

The claims about the height and rank of $\mathfrak{M}$ follow immediately. □

We now turn to giving an explicit description of the functors $\operatorname{Ext}^i(\mathfrak{M}, -)$ for a Breuil–Kisin module with descent data $\mathfrak{M}$.

Definition 3.1.2. Let $\mathfrak{M}$ be a Breuil–Kisin module with A-coefficients and descent data (of some height). If $\mathfrak{N}$ is any object of $\mathcal{K}(A)$, then we let $C^\bullet_{\mathfrak{M}}(\mathfrak{N})$ denote the complex

$$\operatorname{Hom}_{\mathfrak{S}_A[\operatorname{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{N}) \to \operatorname{Hom}_{\mathfrak{S}_A[\operatorname{Gal}(K'/K)]}(\varphi^*\mathfrak{M}, \mathfrak{N}),$$

with differential being given by

$$\alpha \mapsto \Phi_{\mathfrak{N}} \circ \varphi^*\alpha - \alpha \circ \Phi_{\mathfrak{M}}.$$

Also let $\Phi^*_{\mathfrak{M}}$ denote the map $C^0_{\mathfrak{M}}(\mathfrak{N}) \to C^1_{\mathfrak{M}}(\mathfrak{N})$ given by $\alpha \mapsto \alpha \circ \Phi_{\mathfrak{M}}$. When $\mathfrak{M}$ is clear from the context we will usually suppress it from the notation and write simply $C^\bullet(\mathfrak{N})$.

Each $C^i(\mathfrak{N})$ is naturally an $\mathfrak{S}^0_A$-module. The formation of $C^\bullet(\mathfrak{N})$ is evidently functorial in $\mathfrak{N}$, and is also exact in $\mathfrak{N}$, since $\mathfrak{M}$, and hence also $\varphi^*\mathfrak{M}$, is projective over $\mathfrak{S}_A$, and since $\operatorname{Gal}(K'/K)$ has prime-to-p order. Thus the cohomology functors $H^0\big(C^\bullet(-)\big)$ and $H^1\big(C^\bullet(-)\big)$ form a δ-functor on $\mathcal{K}(A)$.

Lemma 3.1.3. *There is a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, -) \cong H^0\big(C^\bullet(-)\big).$$

Proof. This is immediate. □

It follows from this lemma and a standard dimension shifting argument (or, equivalently, the theory of δ-functors) that there is an embedding of functors

$$\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, -) \hookrightarrow H^1\big(C^\bullet(-)\big). \tag{3.1.4}$$

Lemma 3.1.5. *The embedding of functors (3.1.4) is an isomorphism.*

Proof. We first describe the embedding (3.1.4) explicitly. Suppose that

$$0 \to \mathfrak{N} \to \mathfrak{E} \to \mathfrak{M} \to 0$$

is an extension in $\mathcal{K}(A)$. Since $\mathfrak{M}$ is projective over $\mathfrak{S}_A$, and since $\operatorname{Gal}(K'/K)$ is of prime-to-p order, we split this short exact sequence over the twisted group ring $\mathfrak{S}_A[\operatorname{Gal}(K'/K)]$, say via some element $\sigma \in \operatorname{Hom}_{\mathfrak{S}_A[\operatorname{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{E})$. This splitting is well-defined up to the addition of an element $\alpha \in \operatorname{Hom}_{\mathfrak{S}_A[\operatorname{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{N})$.

This splitting is a homomorphism in $\mathcal{K}(A)$ if and only if the element

$$\Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}} \in \operatorname{Hom}_{\mathfrak{S}_A[\operatorname{Gal}(K'/K)]}(\varphi^*\mathfrak{M}, \mathfrak{N})$$

vanishes. If we replace σ by $\sigma + \alpha$, then this element is replaced by

$$(\Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}}) + (\Phi_{\mathfrak{N}} \circ \varphi^*\alpha - \alpha \circ \Phi_{\mathfrak{M}}).$$

Thus the coset of $\Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}}$ in $H^1\bigl(C^\bullet(\mathfrak{N})\bigr)$ is well-defined, independent of the choice of σ, and this coset is the image of the class of the extension $\mathfrak{E}$ under the embedding

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \hookrightarrow H^1\bigl(C^\bullet(\mathfrak{N})\bigr) \tag{3.1.6}$$

(up to a possible overall sign, which we ignore, since it doesn't affect the claim of the lemma).

Now, given any element $\nu \in \mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\varphi^*\mathfrak{M}, \mathfrak{N})$, we may give the $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module $\mathfrak{E} := \mathfrak{N} \oplus \mathfrak{M}$ the structure of a $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$-module as follows: we need to define a φ-linear morphism $\mathfrak{E} \to \mathfrak{E}$, or equivalently a linear morphism $\Phi_{\mathfrak{E}} : \varphi^*\mathfrak{E} \to \mathfrak{E}$. We do this by setting

$$\Phi_{\mathfrak{E}} := \begin{pmatrix} \Phi_{\mathfrak{N}} & \nu \\ 0 & \Phi_{\mathfrak{M}} \end{pmatrix}.$$

Then $\mathfrak{E}$ is an extension of $\mathfrak{M}$ by $\mathfrak{N}$, and if we let σ denote the obvious embedding of $\mathfrak{M}$ into $\mathfrak{E}$, then one computes that

$$\nu = \Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}}.$$

This shows that (3.1.6) is an isomorphism, as claimed. □

Another dimension shifting argument, taking into account the preceding lemma, shows that $\mathrm{Ext}^2_{\mathcal{K}(A)}(\mathfrak{M}, -)$ embeds into $H^2\bigl(C^\bullet(-)\bigr)$. Since the target of this embedding vanishes, we find that the same is true of the source. This yields the following corollary.

Corollary 3.1.7. *If $\mathfrak{M}$ is a Breuil–Kisin module with A-coefficients and descent data, then* $\mathrm{Ext}^2_{\mathcal{K}(A)}(\mathfrak{M}, -) = 0$.

We summarise the above discussion in the following corollary.

Corollary 3.1.8. *If $\mathfrak{M}$ is a Breuil–Kisin module with A-coefficients and descent data, and $\mathfrak{N}$ is an object of $\mathcal{K}(A)$, then we have a natural short exact sequence*

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to C^0(\mathfrak{N}) \to C^1(\mathfrak{N}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to 0.$$

The following lemma records the behaviour of these complexes with respect to base change.

Lemma 3.1.9. *Suppose that $\mathfrak{M}$, $\mathfrak{N}$ are Breuil–Kisin modules with descent data and A-coefficients, that B is an A-algebra, and that Q is a B-module. Then the complexes $C^\bullet_{\mathfrak{M}}(\mathfrak{N} \widehat{\otimes}_A Q)$ and $C^\bullet_{\mathfrak{M} \widehat{\otimes}_A B}(\mathfrak{N} \widehat{\otimes}_A Q)$ coincide, the former complex formed with respect to $\mathcal{K}(A)$ and the latter with respect to $\mathcal{K}(B)$.*

Proof. Indeed, there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{N} \widehat{\otimes}_A Q) \cong \mathrm{Hom}_{\mathfrak{S}_B[\mathrm{Gal}(K'/K)]}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A Q),$$

and similarly with $\varphi^*\mathfrak{M}$ in place of $\mathfrak{M}$. □

The following slightly technical lemma is crucial for establishing finiteness properties, and also base-change properties, of Exts of Breuil–Kisin modules.

Lemma 3.1.10. *Let A be a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, suppose that $\mathfrak{M}$ is a Breuil–Kisin module with descent data and A-coefficients, of height at most h, and suppose that $\mathfrak{N}$ is a u-adically complete, u-torsion free object of $\mathcal{K}(A)$.*

Let $C^\bullet$ be the complex defined in Definition 3.1.2*, and write δ for its differential. Suppose that Q is an A-module with the property that $C^i \otimes_A Q$ is v-torsion free for $i = 0, 1$ and v-adically separated for $i = 0$.*

Then:

1. *For any integer $M \geq (eah+1)/(p-1)$, $\ker(\delta \otimes \mathrm{id}_Q) \cap v^M C^0 \otimes_A Q = 0$.*
2. *For any integer $N \geq (peah+1)/(p-1)$, $\delta \otimes \mathrm{id}_Q$ induces an isomorphism*
$$(\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1 \otimes_A Q) \xrightarrow{\sim} v^N(C^1 \otimes_A Q).$$

Consequently, for N as in (2) *the natural morphism of complexes of A-modules*

$$[C^0 \otimes_A Q \xrightarrow{\delta \otimes \mathrm{id}_Q} C^1 \otimes_A Q] \to [C^0 \otimes_A Q / \big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1 \otimes_A Q)\big) \xrightarrow{\delta \otimes \mathrm{id}_Q} C^1 \otimes_A Q / v^N C^1 \otimes_A Q]$$

is a quasi-isomorphism.

Since we are assuming that the $C^i \otimes_A Q$ are v-torsion free, the expression $v^r C^i(\mathfrak{N}) \otimes_A Q$ may be interpreted as denoting either $v^r\big(C^i(\mathfrak{N}) \otimes_A Q\big)$ or $\big(v^r C^i(\mathfrak{N})\big) \otimes_A Q$, the two being naturally isomorphic.

Remark 3.1.11. Before giving the proof of Lemma 3.1.10, we observe that the hypotheses on the $C^i \otimes_A Q$ are satisfied if either $Q = A$, or else $\mathfrak{N}$ is a projective $\mathfrak{S}_A$-module and Q is a finitely generated B-module for some finitely generated A-algebra B. (Indeed $C^1 \otimes_A Q$ is v-adically separated as well in these cases.)

(1) Since $\mathfrak{M}$ is projective of finite rank over $A[[u]]$, and since $\mathfrak{N}$ is u-adically complete and u-torsion free, each C^i is v-adically separated and v-torsion free. In particular the hypothesis on Q is always satisfied by $Q = A$. (In fact since $\mathfrak{N}$ is u-adically complete it also follows that the C^i are v-adically complete. Here we use that $\mathrm{Gal}(K'/K)$ has order prime to p to see that C^0 is an $\mathfrak{S}^0_A$-module direct summand of $\mathrm{Hom}_{\mathfrak{S}_A}(\mathfrak{M}, \mathfrak{N})$, and similarly for C^1.)

(2) Suppose $\mathfrak{N}$ is a projective $\mathfrak{S}_A$-module. Then the C^i are projective $\mathfrak{S}^0_A$-modules, again using that $\mathrm{Gal}(K'/K)$ has order prime to p. Since each $C^i(\mathfrak{N})/vC^i(\mathfrak{N})$ is A-flat, it follows that $C^i(\mathfrak{N}) \otimes_A Q$ is v-torsion free. If furthermore B is a finitely generated A-algebra, and Q is a finitely generated B-module, then the $C^i(\mathfrak{N}) \otimes_A Q$ are v-adically separated (being finitely generated modules over the ring $A[[v]] \otimes_A B$, which is a finitely generated algebra over the Noetherian ring $A[[v]]$, and hence is itself Noetherian).

Proof of Lemma 3.1.10. Since $p^a = 0$ in A, there exists $H(u) \in \mathfrak{S}_A$ with $u^{e'ah} = E(u)^h H(u)$ in $\mathfrak{S}_A$. Thus the image of $\Phi_{\mathfrak{M}}$ contains $u^{e'ah}\mathfrak{M} = v^{eah}\mathfrak{M}$, and there exists a map $\Upsilon : \mathfrak{M} \to \varphi^*\mathfrak{M}$ such that $\Phi_{\mathfrak{M}} \circ \Upsilon$ is multiplication by v^{eah}.

We begin with (1). Suppose that $f \in \ker(\delta \otimes \mathrm{id}_Q) \cap v^M C^0 \otimes_A Q$. Since $C^0 \otimes_A Q$ is v-adically separated, it is enough, applying induction on M, to show that $f \in v^{M+1} C^0 \otimes_A Q$. Since $f \in \ker(\delta \otimes \mathrm{id}_Q)$, we have $f \circ \Phi_{\mathfrak{M}} = \Phi_{\mathfrak{N}} \circ \varphi^* f$. Since $f \in v^M C^0 \otimes_A Q$, we have $f \circ \Phi_{\mathfrak{M}} = \Phi_{\mathfrak{N}} \circ \varphi^* f \in v^{pM} C^1 \otimes_A Q$. Precomposing with Υ gives $v^{eah} f \in v^{pM} C^0 \otimes_A Q$. Since $C^0 \otimes_A Q$ is v-torsion free, it follows that $f \in v^{pM-eah} C^0 \otimes_A Q \subseteq u^{M+1} C^0 \otimes_A Q$, as required.

We now move on to (2). Set $M = N - eah$. By precomposing with Υ we see that $\alpha \circ \Phi_{\mathfrak{M}} \in v^N C^1 \otimes_A Q$ implies $\alpha \in v^M C^0 \otimes_A Q$; from this, together with the inequality $pM \geq N$, it is straightforward to check that

$$(\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1 \otimes_A Q) = (\delta \otimes \mathrm{id}_Q)^{-1}(v^N C^1 \otimes_A Q) \cap v^M C^0 \otimes_A Q.$$

Note that M satisfies the condition in (1). To complete the proof we will show that for any M as in (1) and any $N \geq M + eah$ the map δ induces an isomorphism

$$(\delta \otimes \mathrm{id}_Q)^{-1}(v^N C^1 \otimes_A Q) \cap v^M C^0 \otimes_A Q \xrightarrow{\sim} v^N C^1 \otimes_A Q.$$

By (1), $\delta \otimes \mathrm{id}_Q$ induces an injection $(\delta \otimes \mathrm{id}_Q)^{-1}(v^N C^1 \otimes_A Q) \cap v^M C^0 \otimes_A Q \hookrightarrow v^N C^1 \otimes_A Q$, so it is enough to show that $(\delta \otimes \mathrm{id}_Q)(v^M C^0 \otimes_A Q) \supseteq v^N C^1 \otimes_A Q$. Equivalently, we need to show that

$$v^N C^1 \otimes_A Q \to (C^1 \otimes_A Q)/(\delta \otimes \mathrm{id}_Q)\big(v^M C^0 \otimes_A Q)$$

is identically zero. Since the formation of cokernels is compatible with tensor products, we see that this morphism is obtained by tensoring the corresponding morphism

$$v^N C^1 \to C^1/\delta\big(v^M C^0\big)$$

with Q over A, so we are reduced to the case $Q = A$. (Recall from Remark 3.1.11(1) that the hypotheses of the Lemma are satisfied in this case, and that C^1 is v-adically separated.)

We claim that for any $g \in v^N C^1$, we can find an $f \in v^{N-eah} C^0$ such that $\delta(f) - g \in v^{p(N-eah)} C^1$. Admitting the claim, given any $g \in v^N C^1$, we may find $h \in v^M C^0$ with $\delta(h) = g$ by successive approximation in the following way: Set $h_0 = f$ for f as in the claim; then $h_0 \in v^{N-eah} C^0 \subseteq v^M C^0$, and $\delta(h_0) - g \in v^{p(N-eah)} C^1 \subseteq v^{N+1} C^1$. Applying the claim again with N replaced by $N+1$, and g replaced by $g - \delta(h_0)$, we find $f \in v^{N+1-eah} C^0 \subseteq v^{M+1} C^0$ with $\delta(f) - g + \delta(h_0) \in v^{p(N+1-eah)} C^1 \subseteq v^{N+1} C^1$. Setting $h_1 = h_0 + f$, and proceeding inductively, we obtain a Cauchy sequence converging (in the v-adically complete $A[[v]]$-module C^0) to the required element h.

It remains to prove the claim. Since $\delta(f) = \Phi_{\mathfrak{N}} \circ \varphi^* f - f \circ \Phi_{\mathfrak{M}}$, and since if $f \in v^{N-eah} C^0$ then $\Phi_{\mathfrak{N}} \circ \varphi^* f \in v^{p(N-eah)} C^1$, it is enough to show that we can find an $f \in v^{N-eah} C^0$ with $f \circ \Phi_{\mathfrak{M}} = -g$. Since $\Phi_{\mathfrak{M}}$ is injective, the map $\Upsilon \circ \Phi_{\mathfrak{M}}$ is also multiplication by v^{eah}, and so it suffices to take f with $v^{eah} f = -g \circ \Upsilon \in v^N C^0$. $\square$

Corollary 3.1.12. *Let A be a Noetherian $\mathcal{O}/\varpi^a$-algebra, and let $\mathfrak{M}$, $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients. If B is a finitely generated A-algebra, and Q is a finitely generated B-module, then the natural morphism of complexes of B-modules*

$$[C^0(\mathfrak{N}) \otimes_A Q \xrightarrow{\delta \otimes \mathrm{id}_Q} C^1(\mathfrak{N}) \otimes_A Q] \to [C^0(\mathfrak{N} \widehat{\otimes}_A Q) \xrightarrow{\delta} C^1(\mathfrak{N} \widehat{\otimes}_A Q)]$$

is a quasi-isomorphism.

Proof. By Remarks 3.1.11 and 2.1.6(2) we can apply Lemma 3.1.10 to both $C^i(\mathfrak{N} \widehat{\otimes}_A Q)$ and $C^i(\mathfrak{N}) \otimes_A Q$, and we see that it is enough to show that the natural morphism of complexes

$$\begin{array}{c} [\big(C^0(\mathfrak{N}) \otimes_A Q\big)/(\Phi^*_{\mathfrak{M}} \otimes \mathrm{id}_Q)^{-1}\big(v^N C^1(\mathfrak{N}) \otimes_A Q\big) \xrightarrow{\delta} \big(C^1(\mathfrak{N}) \otimes_A Q\big)/\big(v^N C^1(\mathfrak{N}) \otimes_A Q\big)] \\ \big\downarrow \\ [C^0(\mathfrak{N} \widehat{\otimes}_A Q)/\big(\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1(\mathfrak{N} \widehat{\otimes}_A Q)\big) \xrightarrow{\delta} C^1(\mathfrak{N} \widehat{\otimes}_A Q)/v^N C^1(\mathfrak{N} \widehat{\otimes}_A Q)] \end{array}$$

is a quasi-isomorphism. In fact, it is even an isomorphism. $\square$

Proposition 3.1.13. *Let A be a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let $\mathfrak{M}$, $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients. Then $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ and $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N})$ for $i \geq 1$ are finitely presented A-modules.*

If furthermore A is Noetherian, then $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ and $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N})$ for $i \geq 1$ are also finitely presented (equivalently, finitely generated) A-modules.

Proof. The statements for $\mathfrak{N}/u^i\mathfrak{N}$ follow easily from those for $\mathfrak{N}$, by considering the short exact sequence $0 \to u^i\mathfrak{N} \to \mathfrak{N} \to \mathfrak{N}/u^i\mathfrak{N} \to 0$ in $\mathcal{K}(A)$ and applying Corollary 3.1.7. By Corollary 3.1.8, it is enough to consider the cohomology of the complex $C^\bullet$. By Lemma 3.1.10 with $Q = A$, the cohomology of $C^\bullet$ agrees with the cohomology of the induced complex

$$C^0/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1)\big) \to C^1/v^N C^1,$$

for an appropriately chosen value of N. It follows that for an appropriately chosen value of N, $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ can be computed as the cokernel of the induced morphism $C^0/v^N C^0 \to C^1/v^N C^1$.

Under our hypothesis on $\mathfrak{N}$, $C^0/v^N C^0$ and $C^1/v^N C^1$ are finitely generated projective A-modules, and thus finitely presented. It follows that $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is finitely presented.

In the case that A is furthermore assumed to be Noetherian, it is enough to note that since $v^N C^0 \subseteq (\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1)$, the quotient $C^0/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1)\big)$ is a finitely generated A-module. ☐

Proposition 3.1.14. *Let A be a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let $\mathfrak{M}$ and $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients. Let B be an A-algebra, and let $f_B : \mathfrak{M}\widehat{\otimes}_A B \to \mathfrak{N}\widehat{\otimes}_A B$ be a morphism of Breuil–Kisin modules with B-coefficients.*

Then there is a finite type A-subalgebra B' of B and a morphism of Breuil–Kisin modules $f_{B'} : \mathfrak{M}\widehat{\otimes}_A B' \to \mathfrak{N}\widehat{\otimes}_A B'$ such that f_B is the base change of $f_{B'}$.

Proof. By Lemmas 3.1.3 and 3.1.9 (the latter applied with $Q = B$) we can and do think of f_B as being an element of the kernel of $\delta : C^0(\mathfrak{N}\widehat{\otimes}_A B) \to C^1(\mathfrak{N}\widehat{\otimes}_A B)$, the complex $C^\bullet$ here and throughout this proof denoting $C^\bullet_{\mathfrak{M}}$ as usual.

Fix N as in Lemma 3.1.10, and write $\overline{f}_B$ for the corresponding element of $C^0(\mathfrak{N}\widehat{\otimes}_A B)/v^N = (C^0(\mathfrak{N})/v^N) \otimes_A B$ (this equality following easily from the assumption that $\mathfrak{M}$ and $\mathfrak{N}$ are projective $\mathfrak{S}_A$-modules of finite rank). Since $C^0(\mathfrak{N})/v^N$ is a projective A-module of finite rank, it follows that for some finite type A-subalgebra B' of B, there is an element $\overline{f}_{B'} \in (C^0(\mathfrak{N})/v^N) \otimes_A B' = C^0(\mathfrak{N}\widehat{\otimes}_A B')/v^N$ such that $\overline{f}_{B'} \otimes_{B'} B = \overline{f}_B$. Denote also by $\overline{f}_{B'}$ the induced element of

$$C^0(\mathfrak{N}\widehat{\otimes}_A B')/\big(\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1(\mathfrak{N}\widehat{\otimes}_A B')).$$

By Lemma 3.1.10 (and Lemma 3.1.3) we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0(C^\bullet(\mathfrak{N}\widehat{\otimes}_A B')) \arrow[r] \arrow[d] & C^0(\mathfrak{N}\widehat{\otimes}_A B')/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1(\mathfrak{N}\widehat{\otimes}_A B'))\big) \arrow[r, "\delta"] \arrow[d] & C^1(\mathfrak{N}\widehat{\otimes}_A B')/v^N \arrow[d] \\
0 \arrow[r] & H^0(C^\bullet(\mathfrak{N}\widehat{\otimes}_A B)) \arrow[r] & C^0(\mathfrak{N}\widehat{\otimes}_A B)/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1(\mathfrak{N}\widehat{\otimes}_A B))\big) \arrow[r, "\delta"] & C^1(\mathfrak{N}\widehat{\otimes}_A B)/v^N
\end{tikzcd}$$

in which the vertical arrows are induced by $\widehat{\otimes}_{B'}B$. By a diagram chase we only need to show that $\delta(\overline{f}_{B'}) = 0$. Since $\delta(f_B) = 0$, it is enough to show that the right hand vertical arrow is an injection. This arrow can be rewritten as the tensor product of the injection of A-algebras $B' \hookrightarrow B$ with the flat (even projective of finite rank) A-module $C^1(\mathfrak{N})/v^N$, so the result follows. $\square$

We have the following key base-change result for Ext^1's of Breuil–Kisin modules with descent data.

Proposition 3.1.15. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a $\mathcal{O}/\varpi^a$-algebra A. Then for any A-algebra B, and for any B-module Q, there are natural isomorphisms* $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A B) \otimes_B Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A Q)$.

Proof. We first prove the lemma in the case of an A-module Q. It follows from Lemmas 3.1.5 and 3.1.10 that we may compute $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})$ as the cokernel of the morphism

$$C^0(\mathfrak{N})/v^N C^0(\mathfrak{N}) \xrightarrow{\delta} C^1(\mathfrak{N})/v^N C^1(\mathfrak{N}),$$

for some sufficiently large value of N (not depending on $\mathfrak{N}$), and hence that we may compute $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q$ as the cokernel of the morphism

$$\big(C^0(\mathfrak{N})/v^N C^0(\mathfrak{N})\big) \otimes_A Q \xrightarrow{\delta} \big(C^1(\mathfrak{N})/v^N C^1(\mathfrak{N})\big) \otimes_A Q.$$

We may similarly compute $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}\widehat{\otimes}_A Q)$ as the cokernel of the morphism

$$C^0(\mathfrak{N}\widehat{\otimes}_A Q)/v^N C^0(\mathfrak{N}\widehat{\otimes}_A Q) \xrightarrow{\delta} C^1(\mathfrak{N}\widehat{\otimes}_A Q)/v^N C^1(\mathfrak{N}\widehat{\otimes}_A Q).$$

(Remark 2.1.6 (2) shows that $\mathfrak{N}\widehat{\otimes}_A Q$ satisfies the necessary hypotheses for Lemma 3.1.10 to apply.) Once we note that the natural morphism

$$\big(C^i(\mathfrak{N})/v^N C^i(\mathfrak{N})\big) \otimes_A Q \to C^i(\mathfrak{N}\widehat{\otimes}_A Q)/v^N C^i(\mathfrak{N}\widehat{\otimes}_A Q)$$

is an isomorphism for $i = 0$ and 1 (because $\mathfrak{M}$ is a finitely generated projective $\mathfrak{S}_A$-module), we obtain the desired isomorphism

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}\widehat{\otimes}_A Q).$$

If B is an A-algebra, and Q is a B-module, then by Lemma 3.1.9 there is a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}\widehat{\otimes}_A Q) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A Q);$$

combined with the preceding base-change result, this yields one of our claimed isomorphisms, namely

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A Q).$$

Taking Q to be B itself, we then obtain an isomorphism

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A B \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A B).$$

This allows us to identify $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q$, which is naturally isomorphic to $\big(\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A B\big) \otimes_B Q$, with $\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A B) \otimes_B Q$, yielding the second claimed isomorphism. $\square$

In contrast to the situation for extensions (*cf.* Proposition 3.1.15), the formation of homomorphisms between Breuil–Kisin modules is in general not compatible with arbitrary base-change, as the following example shows.

Example 3.1.16. Take $A = (\mathbf{Z}/p\mathbf{Z})[x^{\pm 1}, y^{\pm 1}]$, and let $\mathfrak{M}_x$ be the free Breuil–Kisin module of rank one and A-coefficients with $\varphi(e) = xe$ for some generator e of $\mathfrak{M}_x$. Similarly define $\mathfrak{M}_y$ with $\varphi(e') = ye'$ for some generator e' of $\mathfrak{M}_y$. Then $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}_x, \mathfrak{M}_y) = 0$. On the other hand, if $B = A/(x-y)$ then $\mathfrak{M}_x \widehat{\otimes}_A B$ and $\mathfrak{M}_y \widehat{\otimes}_A B$ are isomorphic, so that $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_x \widehat{\otimes} B, \mathfrak{M}_y \widehat{\otimes} B) \not\cong \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}_x, \mathfrak{M}_y) \otimes_A B$.

However, it is possible to establish such a compatibility in some settings. Corollary 3.1.18, which gives a criterion for the vanishing of $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B)$ for any A-algebra B, is a first example of a result in this direction. Lemma 3.1.20 deals with flat base change, and Lemma 3.1.21, which will be important in Section 3.3, proves that formation of homomorphisms is compatible with base-change over a dense open subscheme of Spec A.

Proposition 3.1.17. *Suppose that A is a Noetherian $\mathcal{O}/\varpi^a$-algebra, and that $\mathfrak{M}$ and $\mathfrak{N}$ are objects of $\mathcal{K}(A)$ that are finitely generated over $\mathfrak{S}_A$ (or, equivalently, over $A[[u]]$). Suppose also that $\mathfrak{N}$ is a flat $\mathfrak{S}_A$-module. Consider the following conditions:*

(1) $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B) = 0$ *for any finite type A-algebra B.*

(2) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = 0$ *for each maximal ideal $\mathfrak{m}$ of A.*

(3) $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q) = 0$ *for any finitely generated A-module Q.*

Then we have (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). If A is furthermore Jacobson, then all three conditions are equivalent.

Proof. If $\mathfrak{m}$ is a maximal ideal of A, then $\kappa(\mathfrak{m})$ is certainly a finite type A-algebra, and so evidently (1) implies (2). It is even a finitely generated A-module, and so also (2) follows from (3).

We next prove that (2) implies (3). To this end, recall that if A is any ring, and M is any A-module, then M injects into the product of its localizations at all maximal ideals. If A is Noetherian, and M is finitely generated, then, by combining this fact with the Artin–Rees Lemma, we see that M embeds into the product of its completions at all maximal ideals. Another way to express this is that, if I runs over all cofinite length ideals in A (i.e. all ideals for which A/I is finite length), then M embeds into the projective limit of the quotients M/IM (the point being that this projective limit is the same as the product over all $\mathfrak{m}$-adic completions). We are going to apply this observation with A replaced by $\mathfrak{S}_A$, and with M taken to be $\mathfrak{N} \otimes_A Q$ for some finitely generated A-module Q.

In $A[[u]]$, one sees that u lies in the Jacobson radical (because $1 + fu$ is invertible in $A[[u]]$ for every $f \in A[[u]]$), and thus in every maximal ideal, and so the maximal ideals of $A[[u]]$ are of the form $(\mathfrak{m}, u)$, where $\mathfrak{m}$ runs over the maximal ideals of A. Thus the ideals of the form (I, u^n), where I is a cofinite length ideal in A, are cofinal in all cofinite length ideals in $A[[u]]$. Since $\mathfrak{S}_A$ is finite over $A[[u]]$, we see that the ideals (I, u^n) in $\mathfrak{S}_A$ are also cofinal in all cofinite length ideals in $A[[u]]$. Since $A[[u]]$, and hence $\mathfrak{S}_A$, is furthermore Noetherian when A is, we see that if Q is a finitely generated A-module, and $\mathfrak{N}$ is a finitely generated $\mathfrak{S}_A$-module, then $\mathfrak{N} \otimes_A (Q/IQ)$ is u-adically complete, for any cofinite length ideal I in A, and hence equal to the limit over n of $\mathfrak{N} \otimes_A Q/(I, u^n)$. Putting this together with the observation of the preceding paragraph, we see that the natural morphism

$$\mathfrak{N} \otimes_A Q \to \varprojlim_I \mathfrak{N} \otimes_A (Q/IQ)$$

(where I runs over all cofinite length ideals of A) is an embedding. The induced morphism

$$\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q) \to \varprojlim_I \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A (Q/IQ))$$

is then evidently also an embedding.

Thus, to conclude that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q)$ vanishes, it suffices to show that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A (Q/IQ))$ vanishes for each cofinite length ideal I in A. An easy induction (using the flatness of $\mathfrak{N}$) on the length of A/I reduces this to showing that $\mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M}, \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big)$, or, equivalently, $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big)$, vanishes for each maximal ideal $\mathfrak{m}$. Since this is the hypothesis of (2), we see that indeed (2) implies (3).

It remains to show that (3) implies (1) when A is Jacobson. Applying the result "(2) implies (3)" (with A replaced by B, and taking Q in (3) to be B itself as a B-module) to $\mathfrak{M} \widehat{\otimes}_A B$ and $\mathfrak{N} \widehat{\otimes}_A B$, we see that it suffices to prove the vanishing of

$$\mathrm{Hom}_{\mathcal{K}(B)}\big((\mathfrak{M} \widehat{\otimes}_A B) \otimes_B \kappa(\mathfrak{n}), (\mathfrak{N} \widehat{\otimes}_A B) \otimes_B \kappa(\mathfrak{n})\big) = \mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M}, \mathfrak{N} \widehat{\otimes}_A \kappa(\mathfrak{n})\big)$$

for each maximal ideal $\mathfrak{n}$ of B. Since A is Jacobson, the field $\kappa(\mathfrak{n})$ is in fact a finitely generated A-module, hence $\mathfrak{N} \widehat{\otimes} \kappa(\mathfrak{n}) = \mathfrak{N} \otimes_A \kappa(\mathfrak{n})$, and so the desired vanishing is a special case of (3). $\square$

Corollary 3.1.18. *If A is a Noetherian and Jacobson $\mathcal{O}/\varpi^a$-algebra, and if $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and A-coefficients, then the following three conditions are equivalent:*

(1) $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B) = 0$ *for any A-algebra B.*

(2) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = 0$ *for each maximal ideal $\mathfrak{m}$ of A.*

(3) $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q) = 0$ *for any finitely generated A-module Q.*

Proof. By Proposition 3.1.17, we need only prove that if $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B)$ vanishes for all finitely generated A-algebras B, then it vanishes for all A-algebras B. This is immediate from Proposition 3.1.14. $\square$

Corollary 3.1.19. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and that furthermore $\mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big)$ vanishes for each maximal ideal $\mathfrak{m}$ of A. Then the A-module $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is projective of finite rank.*

Proof. By Proposition 3.1.13, in order to prove that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is projective of finite rank over A, it suffices to prove that it is flat over A. For this, it suffices to show that $Q \mapsto \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A Q$ is exact when applied to finitely generated A-modules Q. Proposition 3.1.15 (together with Remark 2.1.6 (1)) allows us to identify this functor with the functor $Q \mapsto \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q)$. Note that the functor $Q \mapsto \mathfrak{N} \otimes_A Q$ is an exact functor of Q, since $\mathfrak{S}_A$ is a flat A-module (as A is Noetherian; see Remark 2.1.3(3)). Thus, taking into account Corollary 3.1.7, we see that it suffices to show that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}\otimes_A Q) = 0$ for each finitely generated A-module Q, under the hypothesis that $\mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M}\otimes_A \kappa(\mathfrak{m}), \mathfrak{N}\otimes_A \kappa(\mathfrak{m})\big) = 0$ for each maximal ideal $\mathfrak{m}$ of A. This is the implication (2) $\implies$ (3) of Proposition 3.1.17. ☐

Lemma 3.1.20. *Suppose that $\mathfrak{M}$ is a Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A. Suppose that $\mathfrak{N}$ is either a Breuil–Kisin module with A-coefficients, or that $\mathfrak{N} = \mathfrak{N}'/u^N\mathfrak{N}'$, where $\mathfrak{N}'$ a Breuil–Kisin module with A-coefficients and $N \geq 1$. Then, if B is a finitely generated flat A-algebra, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A B.$$

Proof. By Corollary 3.1.8 and the flatness of B, we have a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A B \to C^0(\mathfrak{N}) \otimes_A B \to C^1(\mathfrak{N}) \otimes_A B$$

and therefore (applying Corollary 3.1.12 to treat the case that $\mathfrak{N}$ is projective) a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A B \to C^0(\mathfrak{N}\widehat{\otimes}_A B) \to C^1(\mathfrak{N}\widehat{\otimes}_A B).$$

The result follows from Corollary 3.1.8 and Lemma 3.1.9. ☐

Lemma 3.1.21. *Suppose that $\mathfrak{M}$ is a Breuil–Kisin module with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A which is furthermore a domain. Suppose also that $\mathfrak{N}$ is either a Breuil–Kisin module with A-coefficients, or that $\mathfrak{N} = \mathfrak{N}'/u^N\mathfrak{N}'$, where $\mathfrak{N}'$ is a Breuil–Kisin module with A-coefficients and $N \geq 1$. Then there is some nonzero $f \in A$ with the following property: writing $\mathfrak{M}_{A_f} = \mathfrak{M}\widehat{\otimes}_A A_f$ and $\mathfrak{N}_{A_f} = \mathfrak{N}\widehat{\otimes}_A A_f$, then for any finitely generated A_f-algebra B, and any finitely generated B-module Q, there are natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} Q &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} B) \otimes_B Q \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} Q).\end{aligned}$$

Proof of Lemma 3.1.21. Note that since A is Noetherian, by Remark 2.1.3(3) we see that $\mathfrak{N}$ is A-flat. By Corollary 3.1.8 we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to C^0(\mathfrak{N}) \to C^1(\mathfrak{N}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to 0.$$

Since by assumption $\mathfrak{M}$ is a projective $\mathfrak{S}_A$-module, and $\mathfrak{N}$ is a flat A-module, the $C^i(\mathfrak{N})$ are also flat A-modules.

By Proposition 3.1.13, $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is a finitely generated A-module, so by the generic freeness theorem [Sta13, Tag 051R] there is some nonzero $f \in A$ such that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f$ is free over A_f.

Since localisation is exact, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}, \mathfrak{N})_f \to C^0(\mathfrak{N})_f \to C^1(\mathfrak{N})_f \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f \to 0$$

and therefore (applying Corollary 3.1.12 to treat the case that $\mathfrak{N}$ is a Breuil–Kisin module) an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \to C^0(\mathfrak{N}_{A_f}) \to C^1(\mathfrak{N}_{A_f}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f \to 0.$$

Since the last three terms are flat over A_f, this sequence remains exact upon tensoring over A_f with Q. Applying Corollary 3.1.12 again to treat the case that $\mathfrak{N}$ is a Breuil–Kisin module, we see that in particular we have a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} Q \to C^0(\mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} Q) \to C^1(\mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} Q),$$

and Corollary 3.1.8 together with Lemma 3.1.9 yield one of the desired isomorphisms, namely

$$\mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} Q \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} Q).$$

If we consider the case when $Q = B$, we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} B \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B).$$

Rewriting the tensor product $- \otimes_{A_f} Q$ as $- \otimes_{A_f} B \otimes_B Q$, we then find that

$$\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B) \otimes_B Q \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} Q),$$

which gives the second desired isomorphism. $\square$

Variants on the preceding result may be proved using other versions of the generic freeness theorem.

Example 3.1.22. Returning to the setting of Example 3.1.16, one can check using Corollary 3.1.18 that the conclusion of Lemma 3.1.21 (for $\mathfrak{M} = \mathfrak{M}_x$ and $\mathfrak{N} = \mathfrak{M}_y$) holds with $f = x - y$. In this case all of the resulting Hom groups vanish (*cf.* also the proof of Lemma 3.3.7). It then follows from Corollary 3.1.19 that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f$ is projective over A_f, so that the proof of Lemma 3.1.21 even goes through with this choice of f.

As well as considering homomorphisms and extensions of Breuil–Kisin modules, we need to consider the homomorphisms and extensions of their associated étale φ-modules; recall that the passage to associated étale φ-modules amounts to inverting u, and so we briefly discuss this process in the general context of the category $\mathcal{K}(A)$.

We let $\mathcal{K}(A)[1/u]$ denote the full subcategory of $\mathcal{K}(A)$ consisting of objects on which multiplication by u is invertible. We may equally well regard it as the category of left $\mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)]$-modules (this notation being interpreted in the evident manner). There are natural isomorphisms (of bi-modules)

$$\mathfrak{S}_A[1/u] \otimes_{\mathfrak{S}_A} \mathfrak{S}_A[F, \mathrm{Gal}(K'/K)] \xrightarrow{\sim} \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)] \tag{3.1.23}$$

and

$$\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)] \otimes_{\mathfrak{S}_A} \mathfrak{S}_A[1/u] \xrightarrow{\sim} \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)]. \tag{3.1.24}$$

Thus (since $\mathfrak{S}_A \to \mathfrak{S}_A[1/u]$ is a flat morphism of commutative rings) the morphism of rings $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)] \to \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)]$ is both left and right flat.

If $\mathfrak{M}$ is an object of $\mathcal{K}(A)$, then we see from (3.1.23) that $\mathfrak{M}[1/u] := \mathfrak{S}_A[1/u] \otimes_{\mathfrak{S}_A} \mathfrak{M} \xrightarrow{\sim} \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)] \otimes_{\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]} \mathfrak{M}$ is naturally an object of $\mathcal{K}(A)[1/u]$. Our preceding remarks about flatness show that $\mathfrak{M} \mapsto \mathfrak{M}[1/u]$ is an exact functor $\mathcal{K}(A) \to \mathcal{K}(A)[1/u]$.

Lemma 3.1.25. *(1) If M and N are objects of $\mathcal{K}(A)[1/u]$, then there is a natural isomorphism*

$$\mathrm{Ext}^i_{\mathcal{K}(A)[1/u]}(M, N) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{K}(A)}(M, N).$$

(2) If $\mathfrak{M}$ is an object of $\mathcal{K}(A)$ and N is an object of $\mathcal{K}(A)[1/u]$, then there is a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{K}(A)}(\mathfrak{M}, N) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{K}(A)}(\mathfrak{M}[1/u], N),$$

for all $i \geq 0$.

Proof. The morphism of (1) can be understood in various ways; for example, by thinking in terms of Yoneda Exts, and recalling that $\mathcal{K}(A)[1/u]$ is a full subcategory of $\mathcal{K}(A)$. If instead we think in terms of projective resolutions, we can begin with a projective resolution $\mathfrak{P}^\bullet \to M$ in $\mathcal{K}(A)$, and then consider the induced projective resolution $\mathfrak{P}^\bullet[1/u]$ of $M[1/u]$. Noting that $M[1/u] \xrightarrow{\sim} M$ for any object M of $\mathcal{K}(A)[1/u]$, we then find (via tensor adjunction) that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{P}^\bullet, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)[1/u]}(\mathfrak{P}^\bullet[1/u], N)$, which induces the desired isomorphism of Ext's by passing to cohomology.

Taking into account the isomorphism of (1), the claim of (2) is a general fact about tensoring over a flat ring map (as can again be seen by considering projective resolutions). □

Remark 3.1.26. The preceding lemma is fact an automatic consequence of the abstract categorical properties of our situation: the functor $\mathfrak{M} \mapsto \mathfrak{M}[1/u]$ is left adjoint to the inclusion $\mathcal{K}(A)[1/u] \subset \mathcal{K}(A)$, and restricts to (a functor naturally equivalent to) the identity functor on $\mathcal{K}(A)[1/u]$.

The following lemma expresses the Hom between étale φ-modules arising from Breuil–Kisin modules in terms of a certain direct limit.

Lemma 3.1.27. *Suppose that $\mathfrak{M}$ is a Breuil–Kisin module with descent data in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and that $\mathfrak{N}$ is an object of $\mathcal{K}(A)$ which is finitely generated and u-torsion free as an $\mathfrak{S}_A$-module. Then there is a natural isomorphism*

$$\varinjlim_i \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)[1/u]}(\mathfrak{M}[1/u], \mathfrak{N}[1/u]),$$

where the transition maps are induced by the inclusions $u^{i+1}\mathfrak{M} \subset u^i\mathfrak{M}$.

Remark 3.1.28. Note that since $\mathfrak{N}$ is u-torsion free, the transition maps in the colimit are injections, so the colimit is just an increasing union.

Proof. There are compatible injections $\mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}) \to \mathrm{Hom}_{\mathcal{K}(A)[1/u]}(\mathfrak{M}[1/u], \mathfrak{N}[1/u])$, taking $f' \in \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N})$ to $f \in \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u])$ where $f(m) = u^{-i}f'(u^i m)$. Conversely, given $f \in \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u])$, there is some i such that $f(\mathfrak{M}) \subset u^{-i}\mathfrak{N}$, as required. □

We have the following analogue of Proposition 3.1.17.

Corollary 3.1.29. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data in a Noetherian $\mathcal{O}/\varpi^a$-algebra A. Consider the following conditions:*

(1) $\mathrm{Hom}_{\mathcal{K}(B)[1/u]}\big((\mathfrak{M}\widehat{\otimes}_A B)[1/u], (\mathfrak{N}\widehat{\otimes}_A B)[1/u]\big) = 0$ *for any finite type A-algebra B.*

(2) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))[1/u]}\big((\mathfrak{M}\otimes_A \kappa(\mathfrak{m}))[1/u], (\mathfrak{N}\otimes_A \kappa(\mathfrak{m}))[1/u]\big) = 0$ *for each maximal ideal $\mathfrak{m}$ of A.*

(3) $\mathrm{Hom}_{\mathcal{K}(A)[1/u]}\big(\mathfrak{M}[1/u], (\mathfrak{N}\otimes_A Q)[1/u]\big) = 0$ *for any finitely generated A-module Q.*

Then we have (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). If A is furthermore Jacobson, then all three conditions are equivalent.

Proof. By Lemma 3.1.27, the three conditions are respectively equivalent to the following conditions.

(1′) $\mathrm{Hom}_{\mathcal{K}(B)}\big(u^i(\mathfrak{M}\widehat{\otimes}_A B), \mathfrak{N}\widehat{\otimes}_A B\big) = 0$ for any finite type A-algebra B and all $i \geq 0$.

(2′) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(u^i(\mathfrak{M}\otimes_A \kappa(\mathfrak{m})), \mathfrak{N}\otimes_A \kappa(\mathfrak{m})\big) = 0$ for each maximal ideal $\mathfrak{m}$ of A and all $i \geq 0$.

(3′) $\mathrm{Hom}_{\mathcal{K}(A)}\big(u^i\mathfrak{M}, \mathfrak{N}\otimes_A Q\big) = 0$ for any finitely generated A-module Q and all $i \geq 0$.

Since $\mathfrak{M}$ is projective, the first two conditions are in turn equivalent to

(1″) $\mathrm{Hom}_{\mathcal{K}(B)}\big((u^i\mathfrak{M})\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A B\big) = 0$ for any finite type A-algebra B and all $i \geq 0$.

(2″) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big((u^i\mathfrak{M})\otimes_A \kappa(\mathfrak{m}), \mathfrak{N}\otimes_A \kappa(\mathfrak{m})\big) = 0$ for each maximal ideal $\mathfrak{m}$ of A and all $i \geq 0$.

The result then follows from Proposition 3.1.17. ☐

Definition 3.1.30. If $\mathfrak{M}$ and $\mathfrak{N}$ are objects of $\mathcal{K}(A)$, then we define

$$\ker\text{-}\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) := \ker\big(\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}[1/u],\mathfrak{N}[1/u])\big).$$

The point of this definition is to capture, in the setting of Lemma 2.2.4, the non-split extensions of Breuil–Kisin modules whose underlying extension of Galois representations is split.

Suppose now that $\mathfrak{M}$ is a Breuil–Kisin module. The exact sequence in $\mathcal{K}(A)$

$$0 \to \mathfrak{N} \to \mathfrak{N}[1/u] \to \mathfrak{N}[1/u]/\mathfrak{N} \to 0$$

gives an exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & C^0(\mathfrak{N}) \arrow[r] \arrow[d] & C^0(\mathfrak{N}[1/u]) \arrow[r] \arrow[d] & C^0(\mathfrak{N}[1/u]/\mathfrak{N}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^1(\mathfrak{N}) \arrow[r] & C^1(\mathfrak{N}[1/u]) \arrow[r] & C^1(\mathfrak{N}[1/u]/\mathfrak{N}) \arrow[r] & 0. \end{tikzcd}$$

It follows from Corollary 3.1.8, Lemma 3.1.25(2), and the snake lemma that we have an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}[1/u]) \\ &\to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N}) \to \ker\text{-}\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \to 0. \end{aligned} \tag{3.1.31}$$

Lemma 3.1.32. *If $\mathfrak{M}$, $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and $\mathfrak{N}$ has height at most h, then $f(\mathfrak{M})$ is killed by u^i for any $f \in \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$ and any $i \geq \lfloor e'ah/(p-1)\rfloor$.*

Proof. Suppose that f is an element of $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$. Then $f(\mathfrak{M})$ is a finitely generated submodule of $\mathfrak{N}[1/u]/\mathfrak{N}$, and it therefore killed by u^i for some $i \geq 0$. Choosing i to be the exponent of $f(\mathfrak{M})$ (that is, choosing i to be minimal), it follows that $(\varphi^*f)(\varphi^*\mathfrak{M})$ has exponent precisely ip. (From the choice of i, we see that $u^{i-1}f(\mathfrak{M})$ is nonzero but killed by u, i.e., it is just a $W(k')\otimes A$-module, and so its pullback by $\varphi : \mathfrak{S}_A \to \mathfrak{S}_A$ has exponent precisely p. Then by the flatness of $\varphi : \mathfrak{S}_A \to \mathfrak{S}_A$ we have $u^{ip-1}(\varphi^*f)(\varphi^*\mathfrak{M}) = u^{p-1}\varphi^*(u^{i-1}f(\mathfrak{M})) \neq 0$.)

We claim that $u^{i+e'ah}(\varphi^*f)(\varphi^*\mathfrak{M}) = 0$; admitting this, we deduce that $i+e'ah \geq ip$, as required. To see the claim, take $x \in \varphi^*\mathfrak{M}$, so that $\Phi_{\mathfrak{N}}((u^i\varphi^*f)(x)) = u^if(\Phi_{\mathfrak{M}}(x)) = 0$. It is therefore enough to show that the kernel of

$$\Phi_{\mathfrak{N}} : \varphi^*\mathfrak{N}[1/u]/\varphi^*\mathfrak{N} \to \mathfrak{N}[1/u]/\mathfrak{N}$$

is killed by $u^{e'ah}$; but this follows immediately from an application of the snake lemma to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \varphi^*\mathfrak{N} \arrow[r] \arrow[d, "\Phi_{\mathfrak{N}}"'] & \varphi^*\mathfrak{N}[1/u] \arrow[r] \arrow[d, "\Phi_{\mathfrak{N}}"'] & \varphi^*\mathfrak{N}[1/u]/\varphi^*\mathfrak{N} \arrow[r] \arrow[d, "\Phi_{\mathfrak{N}}"'] & 0 \\ 0 \arrow[r] & \mathfrak{N} \arrow[r] & \mathfrak{N}[1/u] \arrow[r] & \mathfrak{N}[1/u]/\mathfrak{N} \arrow[r] & 0 \end{tikzcd}$$

together with the assumption that $\mathfrak{N}$ has height at most h and an argument as in the first line of the proof of Lemma 3.1.10. $\square$

Lemma 3.1.33. *If $\mathfrak{M}$, $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and $\mathfrak{N}$ has height at most h, then for any $i \geq \lfloor e'ah/(p-1) \rfloor$ we have an exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, u^i\mathfrak{N}) &\to \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}) \\ &\to \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N}) \to \text{ker-Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to 0. \end{aligned}$$

Proof. Comparing Lemma 3.1.32 with the proof of Lemma 3.1.27, we see that the direct limit in that proof has stabilised at i, and we obtain an isomorphism $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N})$ sending a map f to $f' : u^i m \mapsto u^i f(m)$. The same formula evidently identifies $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ with $\mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, u^i\mathfrak{N})$ and $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N})$ with $\mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}[1/u]/u^i\mathfrak{N})$. But any map in the latter group has image contained in $\mathfrak{N}/u^i\mathfrak{N}$ (by Lemma 3.1.32 applied to $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N})$, together with the identification in the previous sentence), so that $\mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}[1/u]/u^i\mathfrak{N}) = \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N})$. $\square$

Proposition 3.1.34. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-domain A. Then there is some nonzero $f \in A$ with the following property: if we write $\mathfrak{M}_{A_f} = \mathfrak{M}\widehat{\otimes}_A A_f$ and $\mathfrak{N}_{A_f} = \mathfrak{N}\widehat{\otimes}_A A_f$, then if B is any finitely generated A_f-algebra, and if Q is any finitely generated B-module, we have natural isomorphisms*

$$\begin{aligned} \text{ker-Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}, \mathfrak{N}) \otimes_{A_f} Q &\xrightarrow{\sim} \text{ker-Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}\widehat{\otimes}_{A_f} B) \otimes_B Q \\ &\xrightarrow{\sim} \text{ker-Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}\widehat{\otimes}_{A_f} Q). \end{aligned}$$

Proof. In view of Lemma 3.1.33, this follows from Lemma 3.1.21, with $\mathfrak{M}$ there being our $u^i\mathfrak{M}$, and $\mathfrak{N}$ being each of $\mathfrak{N}$, $\mathfrak{N}/u^i\mathfrak{N}$ in turn. $\square$

The following result will be crucial in our investigation of the decomposition of $\mathcal{C}^{\mathrm{dd},1}$ and $\mathcal{R}^{\mathrm{dd},1}$ into irreducible components.

Proposition 3.1.35. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A which is furthermore a domain, and suppose that $\mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big)$ vanishes for each maximal ideal $\mathfrak{m}$ of A. Then there is some nonzero $f \in A$ with the following property: if we write $\mathfrak{M}_{A_f} = \mathfrak{M}\widehat{\otimes}_A A_f$ and $\mathfrak{N}_{A_f} = \mathfrak{N}\widehat{\otimes}_A A_f$, then for any finitely generated A_f-algebra B, each of $\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B)$, $\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B)$, and*

$$\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B)/\,\text{ker-Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B)$$

is a finitely generated projective B-module.

Proof. Choose f as in Proposition 3.1.34, let B be a finitely generated A_f-algebra, and let Q be a finitely generated B-module. By Propositions 3.1.15 and 3.1.34, the morphism

$$\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)\otimes_B Q \to \operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)\otimes_B Q$$

is naturally identified with the morphism

$$\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}Q) \to \operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}Q);$$

in particular, it is injective. By Proposition 3.1.15 and Corollary 3.1.19 we see that $\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$ is a finitely generated projective B-module; hence it is also flat. Combining this with the injectivity just proved, we find that

$$\operatorname{Tor}^1_B\big(Q, \operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)/\,\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)\big) = 0$$

for every finitely generated B-module Q, and thus that

$$\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)/\,\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$$

is a finitely generated flat, and therefore finitely generated projective, B-module. Thus $\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$ is a direct summand of the finitely generated projective B-module $\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$, and so is itself a finitely generated projective B-module. ☐

3.2. **Families of extensions.** Let $\mathfrak{M}$ and $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients, so that $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})$ is an A-module. Suppose that $\psi : V \to \operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})$ is a homomorphism of A-modules whose source is a projective A-module of finite rank. Then we may regard ψ as an element of

$$\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})\otimes_A V^\vee = \operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}\otimes_A V^\vee),$$

and in this way ψ corresponds to an extension

$$0 \to \mathfrak{N}\otimes_A V^\vee \to \mathfrak{E} \to \mathfrak{M} \to 0, \tag{3.2.1}$$

which we refer to as the *family of extensions* of $\mathfrak{M}$ by $\mathfrak{N}$ parametrised by V (or by ψ, if we want to emphasise our choice of homomorphism). We let $\mathfrak{E}_v$ denote the pushforward of $\mathfrak{E}$ under the morphism $\mathfrak{N}\otimes_A V^\vee \to \mathfrak{N}$ given by evaluation on $v \in V$. In the special case that $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})$ itself is a projective A-module of finite rank, we can let V be $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})$ and take ψ be the identity map; in this case we refer to (3.2.1) as the *universal extension* of $\mathfrak{M}$ by $\mathfrak{N}$. The reason for this terminology is as follows: if $v \in \operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})$, then $\mathfrak{E}_v$ is the extension of $\mathfrak{M}$ by $\mathfrak{N}$ corresponding to the element v.

Let $B := A[V^\vee]$ denote the symmetric algebra over A generated by $V^\vee$. The short exact sequence (3.2.1) is a short exact sequence of Breuil–Kisin modules with descent data, and so forming its u-adically completed tensor product with B over A, we obtain a short exact sequence

$$0 \to \mathfrak{N}\otimes_A V^\vee\,\widehat{\otimes}_A B \to \mathfrak{E}\,\widehat{\otimes}_A B \to \mathfrak{M}\,\widehat{\otimes}_A B \to 0$$

of Breuil–Kisin modules with descent data over B (see Lemma 2.1.5). Pushing this short exact sequence forward under the natural map

$$V^\vee\,\widehat{\otimes}_A B = V^\vee \otimes_A B \to B$$

induced by the inclusion of $V^\vee$ in B and the multiplication map $B \otimes_A B \to B$, we obtain a short exact sequence

$$0 \to \mathfrak{N} \widehat{\otimes}_A B \to \widetilde{\mathfrak{E}} \to \mathfrak{M} \widehat{\otimes}_A B \to 0 \tag{3.2.2}$$

of Breuil–Kisin modules with descent data over B, which we call the *family of extensions* of $\mathfrak{M}$ by $\mathfrak{N}$ parametrised by $\operatorname{Spec} B$ (which we note is (the total space of) the vector bundle over $\operatorname{Spec} A$ corresponding to the projective A-module V).

If $\alpha_v : B \to A$ is the morphism induced by the evaluation map $V^\vee \to A$ given by some element $v \in V$, then base-changing (3.2.2) by α_v, we recover the short exact sequence

$$0 \to \mathfrak{N} \to \mathfrak{E}_v \to \mathfrak{M} \to 0.$$

More generally, suppose that A is a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let C be any A-algebra. Suppose that $\alpha_{\tilde{v}} : B \to C$ is the morphism induced by the evaluation map $V^\vee \to C$ corresponding to some element $\tilde{v} \in C \otimes_A V$. Then base-changing (3.2.2) by $\alpha_{\tilde{v}}$ yields a short exact sequence

$$0 \to \mathfrak{N} \widehat{\otimes}_A C \to \widetilde{\mathfrak{E}} \widehat{\otimes}_B C \to \mathfrak{M} \widehat{\otimes}_A C \to 0,$$

whose associated extension class corresponds to the image of $\tilde{v}$ under the natural morphism $C \otimes_A V \to C \otimes_A \operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \cong \operatorname{Ext}^1_{\mathcal{K}(C)}(\mathfrak{M} \widehat{\otimes}_A C, \mathfrak{N} \otimes_A C)$, the first arrow being induced by ψ and the second arrow being the isomorphism of Proposition 3.1.15.

3.2.3. *The functor represented by a universal family.* We now suppose that the ring A and the Breuil–Kisin modules $\mathfrak{M}$ and $\mathfrak{N}$ have the following properties:

Assumption 3.2.4. Let A be a Noetherian and Jacobson $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and assume that for each maximal ideal $\mathfrak{m}$ of A, we have that

$$\operatorname{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = \operatorname{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{N} \otimes_A \kappa(\mathfrak{m}), \mathfrak{M} \otimes_A \kappa(\mathfrak{m})\big) = 0.$$

By Corollary 3.1.19, this assumption implies in particular that $V := \operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is projective of finite rank, and so we may form $\operatorname{Spec} B := \operatorname{Spec} A[V^\vee]$, which parametrised the universal family of extensions. We are then able to give the following precise description of the functor represented by $\operatorname{Spec} B$.

Proposition 3.2.5. *The scheme* $\operatorname{Spec} B$ *represents the functor which, to any $\mathcal{O}/\varpi^a$-algebra C, associates the set of isomorphism classes of tuples $(\alpha, \mathfrak{E}, \iota, \pi)$, where α is a morphism $\alpha : \operatorname{Spec} C \to \operatorname{Spec} A$, $\mathfrak{E}$ is a Breuil–Kisin module with descent data and coefficients in C, and ι and π are morphisms $\alpha^*\mathfrak{N} \to \mathfrak{E}$ and $\mathfrak{E} \to \alpha^*\mathfrak{M}$ respectively, with the property that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ is short exact.*

Proof. We have already seen that giving a morphism $\operatorname{Spec} C \to \operatorname{Spec} B$ is equivalent to giving the composite morphism $\alpha : \operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A$, together with an extension class $[\mathfrak{E}] \in \operatorname{Ext}^1_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}, \alpha^*\mathfrak{N})$. Thus to prove the proposition, we just have to show that any automorphism of $\mathfrak{E}$ which restricts to the identity on $\alpha^*\mathfrak{N}$ and induces the identity on $\alpha^*\mathfrak{M}$ is itself the identity on $\mathfrak{E}$. This follows from Corollary 3.1.18, together with Assumption 3.2.4. ∎

Fix an integer $h \geq 0$ so that $E(u)^h \in \operatorname{Ann}_{\mathfrak{S}_A}(\operatorname{coker} \Phi_{\mathfrak{M}}) \operatorname{Ann}_{\mathfrak{S}_A}(\operatorname{coker} \Phi_{\mathfrak{N}})$, so that by Lemma 3.1.1, every Breuil–Kisin module parametrised by $\operatorname{Spec} B$ has height at most h. There is a natural action of $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$ on $\operatorname{Spec} B$, given by rescaling each of ι and π. There is also an evident forgetful morphism $\operatorname{Spec} B \to \operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}$, given by forgetting ι and π, which is evidently invariant under the $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$-action. (Here and below, $\mathcal{C}^{\mathrm{dd},a}$ denotes the moduli stack defined in Subsection 2.3 for our fixed choice of h and for d equal to the sum of the ranks of $\mathfrak{M}$ and $\mathfrak{N}$.) We thus obtain a morphism

$$\operatorname{Spec} B \times_{\mathcal{O}} \mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m \to \operatorname{Spec} B \times_{\operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}} \operatorname{Spec} B. \tag{3.2.6}$$

Corollary 3.2.7. *Suppose that* $\operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}) = \operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{N}) = C^\times$ *for any morphism* $\alpha : \operatorname{Spec} C \to \operatorname{Spec} A$. *Then the morphism* (3.2.6) *is an isomorphism, and consequently the induced morphism*

$$[\operatorname{Spec} B/\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m] \to \operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}$$

is a finite type monomorphism.

Proof. By Proposition 3.2.5, a morphism

$$\operatorname{Spec} C \to \operatorname{Spec} B \times_{\operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}} \operatorname{Spec} B$$

corresponds to an isomorphism class of tuples $(\alpha, \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

- α is a morphism $\alpha : \operatorname{Spec} C \to \operatorname{Spec} A$,
- $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in C,
- $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$ and $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ are morphisms with the property that
$$0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$$
is short exact,
- $\iota' : \alpha^*\mathfrak{N} \to \mathfrak{E}'$ and $\pi' : \mathfrak{E}' \to \alpha^*\mathfrak{M}$ are morphisms with the property that
$$0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} \alpha^*\mathfrak{M} \to 0$$
is short exact.

Assumption 3.2.4 and Corollary 3.1.18 together show that $\operatorname{Hom}_{\mathcal{K}(C)}(\alpha^*\mathfrak{N}, \alpha^*\mathfrak{M}) = 0$. It follows that the composite $\alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\beta} \mathfrak{E}'$ factors through ι', and the induced endomorphism of $\alpha^*\mathfrak{N}$ is injective. Reversing the roles of $\mathfrak{E}$ and $\mathfrak{E}'$, we see that it is in fact an automorphism of $\alpha^*\mathfrak{N}$, and it follows easily that β also induces an automorphism of $\alpha^*\mathfrak{M}$. Again, Assumption 3.2.4 and Proposition 3.1.18 together show that $\operatorname{Hom}_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}, \alpha^*\mathfrak{N}) = 0$, from which it follows easily that β is determined by the automorphisms of $\alpha^*\mathfrak{M}$ and $\alpha^*\mathfrak{N}$ that it induces.

Since $\operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}) = \operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{N}) = C^\times$ by assumption, we see that $\beta \circ \iota, \iota'$ and $\pi, \pi' \circ \beta$ differ only by the action of $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$, so the first claim of the corollary follows. The claim regarding the monomorphism is immediate from Lemma 3.2.8 below. Finally, note that $[\operatorname{Spec} B/\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m]$ is of finite type over $\operatorname{Spec} A$, while $\mathcal{C}^{\mathrm{dd},a}$ has finite type diagonal. It follows that the morphism $[\operatorname{Spec} B/\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m] \to \operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}$ is of finite type, as required. $\square$

Lemma 3.2.8. *Let X be a scheme over a base scheme S, let G be a smooth affine group scheme over S, and let $\rho : X \times_S G \to X$ be a* (*right*) *action of G on X. Let $X \to \mathcal{Y}$ be a G-equivariant morphism, whose target is an algebraic stack over S on which G acts trivially. Then the induced morphism*

$$[X/G] \to \mathcal{Y}$$

is a monomorphism if and only if the natural morphism

$$X \times_S G \to X \times_{\mathcal{Y}} X$$

(*induced by the morphisms* $\mathrm{pr}_1, \rho : X \times_S G \to X$) *is an isomorphism.*

Proof. We have a Cartesian diagram as follows.

$$\begin{tikzcd} X \times_S G \arrow[r] \arrow[d] & X \times_{\mathcal{Y}} X \arrow[d] \\ {[X/G]} \arrow[r] & {[X/G] \times_{\mathcal{Y}} [X/G]} \end{tikzcd}$$

The morphism $[X/G] \to \mathcal{Y}$ is a monomorphism if and only if the bottom horizontal morphism of this square is an isomorphism; since the right hand vertical arrow is a smooth surjection, this is the case if and only if the top horizontal morphism is an isomorphism, as required. □

3.3. **Families of extensions of rank one Breuil–Kisin modules.** In this section we construct universal families of extensions of rank one Breuil–Kisin modules. We will use these rank two families to study our moduli spaces of Breuil–Kisin modules, and the corresponding spaces of étale φ-modules. We show how to compute the dimensions of these universal families; in the subsequent sections, we will combine these results with explicit calculations to determine the irreducible components of our moduli spaces. In particular, we will show that each irreducible component has a dense open substack given by a family of extensions.

3.3.1. *Universal unramified twists.* Fix a free Breuil–Kisin module with descent data $\mathfrak{M}$ over $\mathbf{F}$, and write Φ_i for $\Phi_{\mathfrak{M},i} : \varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$. (Here we are using the notation of Section 2.1, so that $\mathfrak{M}_i = e_i\mathfrak{M}$ is cut out by the idempotent e_i of Section 1.4.) We will construct the "universal unramified twist" of $\mathfrak{M}$.

Definition 3.3.2. If Λ is an $\mathbf{F}$-algebra, and if $\lambda \in \Lambda^\times$, then we define $\mathfrak{M}_{\Lambda,\lambda}$ to be the free Breuil–Kisin module with descent data and Λ-coefficients whose underlying $\mathfrak{S}_\Lambda[\mathrm{Gal}(K'/K)]$-module is equal to $\mathfrak{M}\widehat{\otimes}_{\mathbf{F}}\Lambda$ (so the usual base change of $\mathfrak{M}$ to Λ), and for which $\Phi_{\mathfrak{M}_{\Lambda,\lambda}} : \varphi^*\mathfrak{M}_{\Lambda,\lambda} \to \mathfrak{M}_{\Lambda,\lambda}$ is defined via the f'-tuple $(\lambda\Phi_0, \Phi_1, \dots, \Phi_{f'-1})$. We refer to $\mathfrak{M}_{\Lambda,\lambda}$ as the *unramified twist* of $\mathfrak{M}$ by λ over Λ.

If M is a free étale φ-module with descent data, then we define $M_{\Lambda,\lambda}$ in the analogous fashion. If we write $X = \operatorname{Spec}\Lambda$, then we will sometimes write $\mathfrak{M}_{X,\lambda}$ (resp. $M_{X,\lambda}$) for $\mathfrak{M}_{\Lambda,\lambda}$ (resp. $M_{\Lambda,\lambda}$).

As usual, we write $\mathbf{G}_m := \operatorname{Spec}\mathbf{F}[x, x^{-1}]$. We may then form the rank one Breuil–Kisin module with descent data $\mathfrak{M}_{\mathbf{G}_m,x}$, which is the universal instance of an unramified twist: given $\lambda \in \Lambda^\times$, there is a corresponding morphism $\operatorname{Spec}\Lambda \to \mathbf{G}_m$ determined by the requirement that $x \in \Gamma(\mathbf{G}_m, \mathcal{O}_{\mathbf{G}_m}^\times)$ pulls-back to λ, and $\mathfrak{M}_{X,\lambda}$ is obtained by pulling back $\mathfrak{M}_{\mathbf{G}_m,x}$ under this morphism (that is, by base changing under the corresponding ring homomorphism $\mathbf{F}[x, x^{-1}] \to \Lambda$).

Lemma 3.3.3. *If* $\mathfrak{M}_\Lambda$ *is a Breuil–Kisin module of rank one with* Λ*-coefficients, then* $\mathrm{End}_{\mathcal{K}(\Lambda)}(\mathfrak{M}) = \Lambda$*. Similarly, if* M_Λ *is a étale* φ*-module of rank one with* Λ*-coefficients, then* $\mathrm{End}_{\mathcal{K}(\Lambda)}(M_\Lambda) = \Lambda$.

Proof. We give the proof for M_Λ, the argument for $\mathfrak{M}_\Lambda$ being essentially identical. One reduces easily to the case where M_Λ is free. Since an endomorphism ψ of M_Λ is in particular an endomorphism of the underlying $\mathfrak{S}_\Lambda[1/u]$-module, we see that there is some $\lambda \in \mathfrak{S}_\Lambda[1/u]$ such that ψ is given by multiplication by λ. The commutation relation with Φ_{M_Λ} means that we must have $\varphi(\lambda) = \lambda$, so that certainly (considering the powers of u in λ of lowest negative and positive degrees) $\lambda \in W(k') \otimes_{\mathbf{Z}_p} \Lambda$, and in fact $\lambda \in \Lambda$. Conversely, multiplication by any element of Λ is evidently an endomorphism of M_Λ, as required. $\square$

Lemma 3.3.4. *Let κ be a field of characteristic p, and let M_κ, N_κ be étale φ-modules of rank one with κ-coefficients and descent data. Then any nonzero element of* $\mathrm{Hom}_{\mathcal{K}(\kappa)}(M_\kappa, N_\kappa)$ *is an isomorphism.*

Proof. Since $\kappa((u))$ is a field, it is enough to show that if one of the induced maps $M_{\kappa,i} \to N_{\kappa,i}$ is nonzero, then they all are; but this follows from the commutation relation with φ. $\square$

Lemma 3.3.5. *If $\lambda, \lambda' \in \Lambda^\times$ and $\mathfrak{M}_{\Lambda,\lambda} \cong \mathfrak{M}_{\Lambda,\lambda'}$* (as Breuil–Kisin modules with descent data over Λ)*, then $\lambda = \lambda'$. Similarly, if $M_{\Lambda,\lambda} \cong M_{\Lambda,\lambda'}$, then $\lambda = \lambda'$.*

Proof. Again, we give the proof for M, the argument for $\mathfrak{M}$ being essentially identical. Write $M_i = \mathbf{F}((u))m_i$, and write $\Phi_i(1 \otimes m_{i-1}) = \theta_i m_i$, where $\theta_i \neq 0$. There are $\mu_i \in \Lambda[[u]][1/u]$ such that the given isomorphism $M_{\Lambda,\lambda} \cong M_{\Lambda,\lambda'}$ takes m_i to $\mu_i m_i$. The commutation relation between the given isomorphism and Φ_M imposes the condition

$$\lambda_i \mu_i \theta_i m_i = \lambda_i' \varphi(\mu_{i-1}) \theta_i m_i$$

where λ_i (resp. λ_i') equals 1 unless $i = 0$, when it equals λ (resp. λ').

Thus we have $\mu_i = (\lambda_i'/\lambda_i)\varphi(\mu_{i-1})$, so that in particular $\mu_0 = (\lambda'/\lambda)\varphi^{f'}(\mu_0)$. Considering the powers of u in μ_0 of lowest negative and positive degrees we conclude that $\mu_0 \in W(k') \otimes \Lambda$; but then $\mu_0 = \varphi^{f'}(\mu_0)$, so that $\lambda' = \lambda$, as required. $\square$

Remark 3.3.6. If $\mathfrak{M}$ has height at most h, and we let $\mathcal{C}$ (temporarily) denote the moduli stack of rank one Breuil–Kisin modules of height at most h with $\mathbf{F}$-coefficients and descent data then Lemma 3.3.5 can be interpreted as saying that the morphism $\mathbf{G}_m \to \mathcal{C}$ that classifies $\mathfrak{M}_{\mathbf{G}_m,x}$ is a monomorphism, i.e. the diagonal morphism $\mathbf{G}_m \to \mathbf{G}_m \times_{\mathcal{C}} \mathbf{G}_m$ is an isomorphism. Similarly, the morphism $\mathbf{G}_m \to \mathcal{R}$ (where we temporarily let $\mathcal{R}$ denote the moduli stack of rank one étale φ-modules with $\mathbf{F}$-coefficients and descent data) that classifies $M_{\mathbf{G}_m,x}$ is a monomorphism.

Now choose another rank one Breuil–Kisin module with descent data $\mathfrak{N}$ over $\mathbf{F}$. Let (x, y) denote the standard coordinates on $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$, and consider the rank one Breuil–Kisin modules with descent data $\mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, x}$ and $\mathfrak{N}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, y}$ over $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$.

Lemma 3.3.7. *There is a non-empty irreducible affine open subset* $\mathrm{Spec}\, A^{\mathrm{dist}}$ *of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ whose finite type points are exactly the maximal ideals $\mathfrak{m}$ of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ such that*

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\big) = 0$$

(*where we have written $\bar{x}$ and $\bar{y}$ to denote the images of x and y in $\kappa(\mathfrak{m})^\times$*).

Furthermore, if R is any finite-type A^{dist}-algebra, and if $\mathfrak{m}$ is any maximal ideal of R, then

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = \mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\big) = 0,$$

and also

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}, \mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}\big) = \mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u], \mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u]\big) = 0.$$

In particular, Assumption 3.2.4 is satisfied by $\mathfrak{M}_{A^{\mathrm{dist}},x}$ *and* $\mathfrak{N}_{A^{\mathrm{dist}},y}$.

Proof. If $\mathrm{Hom}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\big) = 0$ for all maximal ideals $\mathfrak{m}$ of $\mathbf{F}[x,y,x^{-1},y^{-1}]$, then we are done: $\mathrm{Spec}\, A^{\mathrm{dist}} = \mathbf{G}_m \times \mathbf{G}_m$. Otherwise, we see that for some finite extension $\mathbf{F}'/\mathbf{F}$ and some $a, a' \in \mathbf{F}'$, we have a non-zero morphism $\mathfrak{M}_{\mathbf{F}',a}[1/u] \to \mathfrak{N}_{\mathbf{F}',a'}[1/u]$. By Lemma 3.3.4, this morphism must in fact be an isomorphism. Since $\mathfrak{M}$ and $\mathfrak{N}$ are both defined over $\mathbf{F}$, we furthermore see that the ratio a'/a lies in $\mathbf{F}$. We then let $\mathrm{Spec}\, A^{\mathrm{dist}}$ be the affine open subset of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ where $a'x \neq ay$; the claimed property of $\mathrm{Spec}\, A^{\mathrm{dist}}$ then follows easily from Lemma 3.3.5.

For the remaining statements of the lemma, note that if $\mathfrak{m}$ is a maximal ideal in a finite type A^{dist}-algebra, then its pull-back to A^{dist} is again a maximal ideal $\mathfrak{m}'$ of A^{dist} (since A^{dist} is Jacobson), and the vanishing of

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\big)$$

follows from the corresponding statement for $\kappa(\mathfrak{m}')$, together with Lemma 3.1.20.

Inverting u induces an embedding

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) \hookrightarrow \mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\big),$$

and so certainly the vanishing of the target implies the vanishing of the source.

The statements in which the roles of $\mathfrak{M}$ and $\mathfrak{N}$ are reversed follow from Lemma 3.3.4. $\square$

Define $T := \mathrm{Ext}^1_{\mathcal{K}(\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m)}\big(\mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, x}, \mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, y}\big)$; it follows from Proposition 3.1.13 that T is finitely generated over $\mathbf{F}[x,x^{-1},y,y^{-1}]$, while Proposition 3.1.15 shows that $T_{A^{\mathrm{dist}}} := T \otimes_{\mathbf{F}[x^{\pm 1},y^{\pm 1}]} A^{\mathrm{dist}}$ is naturally isomorphic to $\mathrm{Ext}^1_{\mathcal{K}(A^{\mathrm{dist}})}\big(\mathfrak{M}_{A^{\mathrm{dist}},x}, \mathfrak{N}_{A^{\mathrm{dist}},y}\big)$. (Here and elsewhere we abuse notation by writing x, y for $x|_{A^{\mathrm{dist}}}$, $y|_{A^{\mathrm{dist}}}$.) Corollary 3.1.19 and Lemma 3.3.7 show that $T_{A^{\mathrm{dist}}}$ is in fact a finitely generated projective A^{dist}-module. If, for any A^{dist}-algebra B, we write $T_B := T_{A^{\mathrm{dist}}} \otimes_{A^{\mathrm{dist}}} B \xrightarrow{\sim} T \otimes_{\mathbf{F}[x^{\pm 1},y^{\pm 1}]} B$, then Proposition 3.1.15 again shows that $T_B \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}\big(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y}\big)$.

By Propositions 3.1.34 and 3.1.35, together with Lemma 3.3.7, there is a nonempty (so dense) affine open subset $\mathrm{Spec}\, A^{\text{k-free}}$ of $\mathrm{Spec}\, A^{\mathrm{dist}}$ with the properties that

$$U_{A^{\text{k-free}}} := \text{ker-Ext}^1_{\mathcal{K}(A^{\text{k-free}})}\big(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y}\big)$$

and

$$T_{A^{\text{k-free}}}/U_{A^{\text{k-free}}} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(A^{\text{k-free}})}\big(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y}\big)/\,\text{ker-Ext}^1_{\mathcal{K}(A^{\text{k-free}})}\big(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y}\big)$$

are finitely generated and projective over $A^{\text{k-free}}$, and furthermore so that for all finitely generated $A^{\text{k-free}}$-algebras B, the formation of $\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y})$ and $\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y})/\,\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y})$ is compatible with base change from $U_{A^{\text{k-free}}}$ and $T_{A^{\text{k-free}}}/U_{A^{\text{k-free}}}$ respectively.

We choose a finite rank projective module V over $\mathbf{F}[x,x^{-1},y,y^{-1}]$ admitting a surjection $V \to T$. Thus, if we write $V_{A^{\mathrm{dist}}} := V \otimes_{\mathbf{F}[x^{\pm 1},y^{\pm 1}]} A^{\mathrm{dist}}$, then the induced morphism $V_{A^{\mathrm{dist}}} \to T_{A^{\mathrm{dist}}}$ is a (split) surjection of A^{dist}-modules.

Following the prescription of Subsection 3.2, we form the symmetric algebra $B^{\mathrm{twist}} := \mathbf{F}[x^{\pm 1}, y^{\pm 1}][V^\vee]$, and construct the family of extensions $\widetilde{\mathfrak{E}}$ over $\operatorname{Spec} B^{\mathrm{twist}}$. We may similarly form the symmetric algebras $B^{\mathrm{dist}} := A^{\mathrm{dist}}[T^\vee_{A^{\mathrm{dist}}}]$ and $B^{\text{k-free}} := A^{\text{k-free}}[T^\vee_{A^{\text{k-free}}}]$, and construct the families of extensions $\widetilde{\mathfrak{E}}^{\mathrm{dist}}$ and $\widetilde{\mathfrak{E}}^{\text{k-free}}$ over $\operatorname{Spec} B^{\mathrm{dist}}$ and $\operatorname{Spec} B^{\text{k-free}}$ respectively. Since $T_{A^{\text{k-free}}}/U_{A^{\text{k-free}}}$ is projective, the natural morphism $T^\vee_{A^{\text{k-free}}} \to U^\vee_{A^{\text{k-free}}}$ is surjective, and hence $C^{\text{k-free}} := A[U^\vee_{A^{\text{k-free}}}]$ is a quotient of $B^{\text{k-free}}$; geometrically, $\operatorname{Spec} C^{\text{k-free}}$ is a subbundle of the vector bundle $\operatorname{Spec} B^{\text{k-free}}$ over $\operatorname{Spec} A$.

We write $X := \operatorname{Spec} B^{\text{k-free}} \setminus \operatorname{Spec} C^{\text{k-free}}$; it is an open subscheme of the vector bundle $\operatorname{Spec} B^{\text{k-free}}$. The restriction of $\widetilde{\mathfrak{E}}'$ to X is the universal family of extensions over A which do not split after inverting u.

Remark 3.3.8. Since $\operatorname{Spec} A^{\mathrm{dist}}$ and $\operatorname{Spec} A^{\text{k-free}}$ are irreducible, each of the vector bundles $\operatorname{Spec} B^{\mathrm{dist}}$ and $\operatorname{Spec} B^{\text{k-free}}$ is also irreducible. In particular, $\operatorname{Spec} B^{\text{k-free}}$ is Zariski dense in $\operatorname{Spec} B^{\mathrm{dist}}$, and if X is non-empty, then it is Zariski dense in each of $\operatorname{Spec} B^{\text{k-free}}$ and $\operatorname{Spec} B^{\mathrm{dist}}$. Similarly, $\operatorname{Spec} B^{\mathrm{twist}} \times_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m} \operatorname{Spec} A^{\mathrm{dist}}$ is Zariski dense in $\operatorname{Spec} B^{\mathrm{twist}}$.

The surjection $V_{A^{\mathrm{dist}}} \to T_{A^{\mathrm{dist}}}$ induces a surjection of vector bundles $\pi : \operatorname{Spec} B^{\mathrm{twist}} \times_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m} \operatorname{Spec} A^{\mathrm{dist}} \to \operatorname{Spec} B^{\mathrm{dist}}$ over $\operatorname{Spec} A^{\mathrm{dist}}$, and there is a natural isomorphism

$$\pi^* \widetilde{\mathfrak{E}}^{\mathrm{dist}} \xrightarrow{\sim} \widetilde{\mathfrak{E}} \,\widehat{\otimes}_{\mathbf{F}[x^{\pm 1}, y^{\pm 1}]} A^{\mathrm{dist}}. \tag{3.3.9}$$

The rank two Breuil–Kisin module with descent data $\widetilde{\mathfrak{E}}$ is classified by a morphism $\xi : \operatorname{Spec} B^{\mathrm{twist}} \to \mathcal{C}^{\mathrm{dd},1}$; similarly, the rank two Breuil–Kisin module with descent data $\widetilde{\mathfrak{E}}^{\mathrm{dist}}$ is classified by a morphism $\xi^{\mathrm{dist}} : \operatorname{Spec} B^{\mathrm{dist}} \to \mathcal{C}^{\mathrm{dd},1}$. If we write $\xi_{A^{\mathrm{dist}}}$ for the restriction of ξ to the open subset $\operatorname{Spec} B^{\mathrm{twist}} \times_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m} \operatorname{Spec} A^{\mathrm{dist}}$ of $\operatorname{Spec} B^{\mathrm{twist}}$, then the isomorphism (3.3.9) shows that $\xi^{\mathrm{dist}} \circ \pi = \xi_{A^{\mathrm{dist}}}$. We also write $\xi^{\text{k-free}}$ for the restriction of ξ^{dist} to $\operatorname{Spec} B^{\text{k-free}}$, and ξ_X for the restriction of $\xi^{\text{k-free}}$ to X.

Lemma 3.3.10. *The scheme-theoretic images* (*in the sense of* [EG21, Def. 3.1.4]) *of* $\xi : \operatorname{Spec} B^{\mathrm{twist}} \to \mathcal{C}^{\mathrm{dd},1}$, $\xi^{\mathrm{dist}} : \operatorname{Spec} B^{\mathrm{dist}} \to \mathcal{C}^{\mathrm{dd},1}$, *and* $\xi^{\text{k-free}} : \operatorname{Spec} B^{\text{k-free}} \to \mathcal{C}^{\mathrm{dd},1}$ *all coincide; in particular, the scheme-theoretic image of* ξ *is independent of the choice of surjection* $V \to T$, *and the scheme-theoretic image of* $\xi^{\text{k-free}}$ *is independent of the choice of* $A^{\text{k-free}}$. *If* X *is non-empty, then the scheme-theoretic image of* $\xi_X : X \to \mathcal{C}^{\mathrm{dd},1}$ *also coincides with these other scheme-theoretic images, and is independent of the choice of* $A^{\text{k-free}}$.

Proof. This follows from the various observations about Zariski density made in Remark 3.3.8. ☐

Definition 3.3.11. We let $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ denote the scheme-theoretic image of $\xi^{\mathrm{dist}} : \operatorname{Spec} B^{\mathrm{dist}} \to \mathcal{C}^{\mathrm{dd},1}$, and we let $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ denote the scheme-theoretic image of the composite $\xi^{\mathrm{dist}} : \operatorname{Spec} B^{\mathrm{dist}} \to \mathcal{C}^{\mathrm{dd},1} \to \mathcal{Z}^{\mathrm{dd},1}$. Equivalently, $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ is the scheme-theoretic image of the composite $\operatorname{Spec} B^{\mathrm{dist}} \to \mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1}$ (*cf.* [EG21, Prop. 3.2.31]), and the scheme-theoretic image of $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ under the morphism $\mathcal{C}^{\mathrm{dd},1} \to \mathcal{Z}^{\mathrm{dd},1}$. (Note that Lemma 3.3.10 provides various other alternative descriptions of $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ (and therefore also $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$) as a scheme-theoretic image.)

Remark 3.3.12. Note that $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ and $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ are both reduced (because they are each defined as a scheme-theoretic image of $\operatorname{Spec} B^{\mathrm{dist}}$, which is reduced by definition).

As well as scheme-theoretic images, as in the preceding Lemma and Definition, we will need to consider images of underlying topological spaces. If $\mathcal{X}$ is an algebraic stack we let $|\mathcal{X}|$ be its underlying topological space, as defined in [Sta13, Tag 04Y8].

Lemma 3.3.13. *The image of the morphism on underlying topological spaces* $|\operatorname{Spec} B^{\text{twist}}| \to |\mathcal{C}^{\text{dd},1}|$ *induced by* ξ *is a constructible subset of* $|\mathcal{C}^{\text{dd},1}|$*, and is independent of the choice of* V*.*

Proof. The fact that the image of $|\operatorname{Spec} B^{\text{twist}}|$ is a constructible subset of $|\mathcal{C}^{\text{dd},1}|$ follows from the fact that ξ is a morphism of finite presentation between Noetherian stacks; see [Ryd11, App. D]. Suppose now that V' is another choice of finite rank projective $\mathbf{F}[x, x^{-1}, y, y^{-1}]$-module surjecting onto T. Since it is possible to choose a finite rank projective module surjecting onto the pullback of V, V' with respect to their maps to T, we see that it suffices to prove the independence claim of the lemma in the case when V' admits a surjection onto V (compatible with the maps of each of V and V' onto T). If we write $B' := \mathbf{F}[x^{\pm 1}, y^{\pm 1}][(V')^\vee]$, then the natural morphism $\operatorname{Spec} B' \to \operatorname{Spec} B^{\text{twist}}$ is a surjection, and the morphism $\xi' : \operatorname{Spec} B' \to \mathcal{C}^{\text{dd},1}$ is the composite of this surjection with the morphism ξ. Thus indeed the images of $|\operatorname{Spec} B'|$ and of $|\operatorname{Spec} B^{\text{twist}}|$ coincide as subsets of $|\mathcal{C}^{\text{dd},1}|$. ∎

Definition 3.3.14. We write $|\mathcal{C}(\mathfrak{M}, \mathfrak{N})|$ to denote the constructible subset of $|\mathcal{C}^{\text{dd},1}|$ described in Lemma 3.3.13.

Remark 3.3.15. We caution the reader that we don't define a substack $\mathcal{C}(\mathfrak{M}, \mathfrak{N})$ of $\mathcal{C}^{\text{dd},1}$. Rather, we have defined a closed substack $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ of $\mathcal{C}^{\text{dd},1}$, and a constructible subset $|\mathcal{C}(\mathfrak{M}, \mathfrak{N})|$ of $|\mathcal{C}^{\text{dd},1}|$. It follows from the constructions that $|\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})|$ is the closure in $|\mathcal{C}^{\text{dd},1}|$ of $|\mathcal{C}(\mathfrak{M}, \mathfrak{N})|$.

As in Subsection 3.2, there is a natural action of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ on T, and hence on each of $\operatorname{Spec} B^{\text{dist}}$, $\operatorname{Spec} B^{\text{k-free}}$ and X, given by the action of $\mathbf{G}_m$ as automorphisms on each of $\mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, x}$ and $\mathfrak{N}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, y}$ (which induces a corresponding action on T, hence on $T_{A^{\text{dist}}}$ and $T_{A^{\text{k-free}}}$, and hence on $\operatorname{Spec} B^{\text{dist}}$ and $\operatorname{Spec} B^{\text{k-free}}$). Thus we may form the corresponding quotient stacks $[\operatorname{Spec} B^{\text{dist}} / \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ and $[X / \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$, each of which admits a natural morphism to $\mathcal{C}^{\text{dd},1}$.

Remark 3.3.16. Note that we are making use of two independent copies of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$; one parameterises the different unramified twists of $\mathfrak{M}$ and $\mathfrak{N}$, and the other the automorphisms of (the pullbacks of) $\mathfrak{M}$ and $\mathfrak{N}$.

Definition 3.3.17. We say that the pair $(\mathfrak{M}, \mathfrak{N})$ is *strict* if $\operatorname{Spec} A^{\text{dist}} = \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$.

Before stating and proving the main result of this subsection, we prove some lemmas (the first two of which amount to recollections of standard — and simple — facts).

Lemma 3.3.18. *If* $\mathcal{X} \to \mathcal{Y}$ *is a morphism of stacks over* S*, with* $\mathcal{X}$ *algebraic and of finite type over* S*, and* $\mathcal{Y}$ *having diagonal which is representable by algebraic spaces and of finite type, then* $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ *is an algebraic stack of finite type over* S*.*

Proof. The fact that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic stack follows from [Sta13, Tag 04TF]. Since composites of morphisms of finite type are of finite type, in order to show that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is of finite type over S, it suffices to show that the natural morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is of finite type. Since this morphism is the base-change of the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$, this follows by assumption. ∎

Lemma 3.3.19. *The following conditions are equivalent:*
(1) $\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}) = 0$ *for all maximal ideals* $\mathfrak{m}$ *of* $A^{\text{k-free}}$.
(2) $U_{A^{\text{k-free}}} = 0$.
(3) $\operatorname{Spec} C^{\text{k-free}}$ *is the trivial vector bundle over* $\operatorname{Spec} A^{\text{k-free}}$.

Proof. Conditions (2) and (3) are equivalent by definition. Since the formation of $\text{ker-Ext}^1_{\mathcal{K}(A^{\text{k-free}})}(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y})$ is compatible with base change, and since $A^{\text{k-free}}$ is Jacobson, (1) is equivalent to the assumption that

$$\text{ker-Ext}^1_{\mathcal{K}(A^{\text{k-free}})}(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y}) = 0,$$

i.e. that $U_{A^{\text{k-free}}} = 0$, as required. $\square$

Lemma 3.3.20. *If the equivalent conditions of Lemma 3.3.19 hold, then the natural morphism*

$$\operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}} \\ \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$$

is an isomorphism.

Proof. Since $\mathcal{C}^{\text{dd},1} \to \mathcal{R}^{\text{dd},1}$ is separated (being proper) and representable, the diagonal morphism $\mathcal{C}^{\text{dd},1} \to \mathcal{C}^{\text{dd},1} \times_{\mathcal{R}^{\text{dd},1}} \mathcal{C}^{\text{dd},1}$ is a closed immersion, and hence the morphism in the statement of the lemma is a closed immersion. Thus, in order to show that it is an isomorphism, it suffices to show that it induces a surjection on R-valued points, for any $\mathbf{F}$-algebra R. Since the source and target are of finite type over $\mathbf{F}$, by Lemma 3.3.18, we may in fact restrict attention to finite type R-algebras.

A morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$ corresponds to an isomorphism class of tuples $(\alpha, \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

— α is a morphism $\alpha : \operatorname{Spec} R \to \operatorname{Spec} A^{\text{k-free}}$,

— $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in R,

— $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : \alpha^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to \alpha^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} \alpha^*\mathfrak{M} \to 0$ are both short exact.

Similarly, a morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$ corresponds to an isomorphism class of tuples $(\alpha, \mathfrak{E}, \mathfrak{E}', \beta, \iota, \iota', \pi, \pi')$, where

— α is a morphism $\alpha : \operatorname{Spec} R \to \operatorname{Spec} A^{\text{k-free}}$,

— $\mathfrak{E}$ and $\mathfrak{E}'$ are Breuil–Kisin modules with descent data and coefficients in R, and β is an isomorphism $\beta : \mathfrak{E}[1/u] \to \mathfrak{E}'[1/u]$ of etale φ-modules with descent data and coefficients in R,

— $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : \alpha^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to \alpha^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} \alpha^*\mathfrak{M} \to 0$ are both short exact.

Thus to prove the claimed surjectivity, we have to show that, given a tuple $(\alpha, \mathfrak{E}, \mathfrak{E}', \beta, \iota, \iota', \pi, \pi')$ associated to a morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$, the isomorphism β restricts to an isomorphism $\mathfrak{E} \to \mathfrak{E}'$.

By Lemma 3.3.19, the natural map $\operatorname{Ext}^1(\alpha^*\mathfrak{M}, \alpha^*\mathfrak{N}) \to \operatorname{Ext}^1_{\mathcal{K}(R)}(\alpha^*\mathfrak{M}[1/u], \alpha^*\mathfrak{N}[1/u])$ is injective; so the Breuil–Kisin modules $\mathfrak{E}$ and $\mathfrak{E}'$ are isomorphic. Arguing as in the proof of Corollary 3.2.7, we see that β is equivalent to the data of an R-point of $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$, corresponding to the automorphisms of $\alpha^*\mathfrak{M}[1/u]$ and $\alpha^*\mathfrak{N}[1/u]$ that it induces. These restrict to automorphisms of $\alpha^*\mathfrak{M}$ and $\alpha^*\mathfrak{N}$, so that (again by the proof of Corollary 3.2.7) β indeed restricts to an isomorphism $\mathfrak{E} \to \mathfrak{E}'$, as required. ☐

We now present the main result of this subsection.

Proposition 3.3.21. (1) *The morphism ξ^{dist} induces a morphism*

$$[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1}, \tag{3.3.22}$$

which is representable by algebraic spaces, of finite type, and unramified, whose fibres over finite type points are of degree ≤ 2. In the strict case, this induced morphism is in fact a monomorphism, while in general, the restriction ξ_X of ξ^{dist} induces a finite type monomorphism

$$[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \hookrightarrow \mathcal{C}^{\mathrm{dd},1}. \tag{3.3.23}$$

(2) *If* $\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$ *for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then the composite morphism*

$$[\operatorname{Spec} B^{\text{k-free}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1} \tag{3.3.24}$$

is a representable by algebraic spaces, of finite type, and unramified, with fibres of degree ≤ 2. In the strict case, this induced morphism is in fact a monomorphism, while in general, the composite morphism

$$[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \hookrightarrow \mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1} \tag{3.3.25}$$

is a finite type monomorphism.

Remark 3.3.26. The failure of (3.3.22) to be a monomorphism in general is due, effectively, to the possibility that an extension $\mathfrak{E}$ of some $\mathfrak{M}_{R,x}$ by $\mathfrak{N}_{R,y}$ and an extension $\mathfrak{E}'$ of some $\mathfrak{M}_{R,x'}$ by $\mathfrak{N}_{R,y'}$ might be isomorphic as Breuil–Kisin modules while nevertheless $(x, y) \neq (x', y')$. As we will see in the proof, whenever this happens the map $\mathfrak{N}_{\Lambda,y} \to \mathfrak{E} \to \mathfrak{E}' \to \mathfrak{M}_{\Lambda,x'}$ is nonzero, and then $\mathfrak{E}' \otimes_R \kappa(\mathfrak{m})[1/u]$ is split for some maximal ideal $\mathfrak{m}$ of R. This explains why, to obtain a monomorphism, we can restrict either to the strict case or to the substack of extensions that are non-split after inverting u.

Remark 3.3.27. We have stated this proposition in the strongest form that we are able to prove, but in fact its full strength is not required in the subsequent applications. In particular, we don't need the precise bounds on the degrees of the fibres.

Proof of Proposition 3.3.21. By Corollary 3.2.7 (which we can apply because Assumption 3.2.4 is satisfied, by Lemma 3.3.7) the natural morphism $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}$ is a finite type monomorphism, and hence so is its restriction to the open substack $[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ of its source.

Let us momentarily write $\mathcal{X}$ to denote either $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ or $[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$. To show that the finite type morphism $\mathcal{X} \to \mathcal{C}^{\mathrm{dd},1}$ is representable by algebraic spaces, resp. unramified, resp. a monomorphism, it suffices to show that the corresponding diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{C}^{\mathrm{dd},1}} \mathcal{X}$ is a monomorphism, resp. étale, resp. an isomorphism.

Now since $\mathcal{X} \to \operatorname{Spec} A^{\text{dist}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}$ is a monomorphism, the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{\operatorname{Spec} A^{\text{dist}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}} \mathcal{X}$ *is* an isomorphism, and so it is equivalent to show that the morphism of products

$$\mathcal{X} \times_{\operatorname{Spec} A^{\text{dist}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{C}^{\text{dd},1}} \mathcal{X}$$

is a monomorphism, resp. étale, resp. an isomorphism. This is in turn equivalent to showing the corresponding properties for the morphisms

$$\operatorname{Spec} B^{\text{dist}} \times_{\operatorname{Spec} A^{\text{dist}} \times \mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{dist}} \to \operatorname{Spec} B^{\text{dist}} \times_{\mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{dist}} \tag{3.3.28}$$

or

$$X \times_{\operatorname{Spec} A^{\text{dist}} \times \mathcal{C}^{\text{dd},1}} X \to X \times_{\mathcal{C}^{\text{dd},1}} X. \tag{3.3.29}$$

Now each of these morphisms is a base-change of the diagonal $\operatorname{Spec} A^{\text{dist}} \to \operatorname{Spec} A^{\text{dist}} \times_{\mathbf{F}} \operatorname{Spec} A^{\text{dist}}$, which is a closed immersion (affine schemes being separated), and so is itself a closed immersion. In particular, it is a monomorphism, and so we have proved the representability by algebraic spaces of each of (3.3.22) and (3.3.23). Since the source and target of each of these monomorphisms is of finite type over $\mathbf{F}$, by Lemma 3.3.18, in order to show that either of these monomorphisms is an isomorphism, it suffices to show that it induces a surjection on R-valued points, for arbitrary finite type $\mathbf{F}$-algebras R. Similarly, to check that the closed immersion (3.3.28) is étale, it suffices to verify that it is formally smooth, and for this it suffices to verify that it satisfies the infinitesimal lifting property with respect to square zero thickenings of finite type $\mathbf{F}$-algebras.

A morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\text{dist}} \times_{\mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{dist}}$ corresponds to an isomorphism class of tuples $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

— α, α' are morphisms $\alpha, \alpha' : \operatorname{Spec} R \to \operatorname{Spec} A^{\text{dist}}$,

— $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in R,

— $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : (\alpha')^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to (\alpha')^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to (\alpha')^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} (\alpha')^*\mathfrak{M} \to 0$ are both short exact.

We begin by proving that (3.3.28) satisfies the infinitesimal lifting criterion (when R is a finite type $\mathbf{F}$-algebra). Thus we assume given a square-zero ideal $I \subset R$, such that the induced morphism

$$\operatorname{Spec} R/I \to \operatorname{Spec} B^{\text{dist}} \times_{\mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{dist}}$$

factors through $\operatorname{Spec} B^{\text{dist}} \times_{\operatorname{Spec} A^{\text{dist}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{dist}}$. In terms of the data $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, we are assuming that α and α' coincide when restricted to $\operatorname{Spec} R/I$, and we must show that α and α' themselves coincide.

To this end, we consider the composite

$$\alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\beta} \mathfrak{E}' \xrightarrow{\pi'} (\alpha')^*\mathfrak{M}. \tag{3.3.30}$$

If we can show the vanishing of this morphism, then by reversing the roles of $\mathfrak{E}$ and $\mathfrak{E}'$, we will similarly deduce the vanishing of $\pi \circ \beta^{-1} \circ \iota'$, from which we can conclude that β induces an isomorphism between $\alpha^*\mathfrak{N}$ and $(\alpha')^*\mathfrak{N}$. Consequently, it also induces an isomorphism between $\alpha^*\mathfrak{M}$ and $(\alpha')^*\mathfrak{M}$, so it follows from Lemma 3.3.5 that $\alpha = \alpha'$, as required.

We show the vanishing of (3.3.30). Suppose to the contrary that it doesn't vanish, so that we have a non-zero morphism $\alpha^*\mathfrak{N} \to (\alpha')^*\mathfrak{M}$. It follows from Proposition 3.1.17 that, for some maximal ideal $\mathfrak{m}$ of R, there exists a non-zero morphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m}) \to (\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

By assumption α and α' coincide modulo I. Since $I^2 = 0$, there is an inclusion $I \subset \mathfrak{m}$, and so in particular we find that

$$(\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}) \xrightarrow{\sim} \alpha^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

Thus there exists a non-zero morphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m}) \to \alpha^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

Then, by Lemma 3.3.4, after inverting u we obtain an isomorphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u] \xrightarrow{\sim} \alpha^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m})[1/u],$$

contradicting the assumption that α maps $\operatorname{Spec} R$ into $\operatorname{Spec} A^{\mathrm{dist}}$. This completes the proof that (3.3.28) is formally smooth, and hence that (3.3.22) is unramified.

We next show that, in the strict case, the closed immersion (3.3.28) is an isomorphism, and thus that (3.3.22) is actually a monomorphism. As noted above, it suffices to show that (3.3.28) induces a surjection on R-valued points for finite type $\mathbf{F}$-algebras R, which in terms of the data $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, amounts to showing that necessarily $\alpha = \alpha'$. Arguing just as we did above, it suffices show the vanishing of (3.3.30).

Again, we suppose for the sake of contradiction that (3.3.30) does not vanish. It then follows from Proposition 3.1.17 that for some maximal ideal $\mathfrak{m}$ of R there exists a non-zero morphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m}) \to (\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

Then, by Lemma 3.3.4, after inverting u we obtain an isomorphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u] \xrightarrow{\sim} (\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m})[1/u]. \tag{3.3.31}$$

In the strict case, such an isomorphism cannot exist by assumption, and thus (3.3.30) must vanish.

We now turn to proving that (3.3.29) is an isomorphism. Just as in the preceding arguments, it suffices to show that (3.3.30) vanishes, and if not then we obtain an isomorphism (3.3.31). Since we are considering points of $X \times X$, we are given that the induced extension $\mathfrak{E}' \otimes_R \kappa(\mathfrak{m})[1/u]$ is non-split, so that the base change of the morphism (3.3.30) from $R[[u]]$ to $\kappa(\mathfrak{m})((u))$ must vanish. Consequently the composite $\beta \circ \iota$ induces a non-zero morphism $\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u] \to (\alpha')^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u]$, which, by Lemma 3.3.4, must in fact be an isomorphism. Comparing this isomorphism with the isomorphism (3.3.31), we find that $(\alpha')^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u]$ and $(\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m})[1/u]$ are isomorphic, contradicting the fact that α' maps $\operatorname{Spec} R$ to $\operatorname{Spec} A^{\mathrm{dist}}$. Thus in fact the composite (3.3.30) must vanish, and we have completed the proof that (3.3.23) is a monomorphism.

To complete the proof of part (1) of the proposition, we have to show that the fibres of (3.3.22) are of degree at most 2. We have already observed that $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}$ is a monomorphism, so it is enough to check that given a finite extension $\mathbf{F}'/\mathbf{F}$ and an isomorphism class of tuples $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

— α, α' are distinct morphisms $\alpha, \alpha' : \operatorname{Spec} \mathbf{F}' \to \operatorname{Spec} A^{\mathrm{dist}}$,

— $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in $\mathbf{F}'$,

— $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : (\alpha')^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to (\alpha')^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to (\alpha')^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} (\alpha')^*\mathfrak{M} \to 0$ are both short exact.

then α' is determined by the data of α and $\mathfrak{E}$. To see this, note that since we are assuming that $\alpha' \neq \alpha$, the arguments above show that (3.3.30) does not vanish, so that (since $\mathbf{F}'$ is a field), we have an isomorphism $\alpha^*\mathfrak{N}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{M}[1/u]$. Since we are over A^{dist}, it follows that $\mathfrak{E}[1/u] \cong \mathfrak{E}'[1/u]$ is split, and that we also have an isomorphism $\alpha^*\mathfrak{M}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{N}[1/u]$. Thus if α'' is another possible choice for α', we have $(\alpha'')^*\mathfrak{M}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{M}[1/u]$ and $(\alpha'')^*\mathfrak{N}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{N}[1/u]$, whence $\alpha'' = \alpha'$ by Lemma 3.3.5, as required.

We turn to proving (2), and thus assume that

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$.

Lemma 3.3.20 shows that

$$\operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}} \operatorname{Spec} B^{\text{k-free}} \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\mathrm{dd},1}} \operatorname{Spec} B^{\text{k-free}}$$

is an isomorphism, from which we deduce that

$$[\operatorname{Spec} B^{\text{k-free}} / \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\mathrm{dd},1}$$

is a monomorphism. Using this as input, the claims of (2) may be proved in an essentially identical fashion to those of (1). □

Corollary 3.3.32. *The dimension of $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ is equal to the rank of $T_{A^{\mathrm{dist}}}$ as a projective A^{dist}-module. If*

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then the dimension of $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ is also equal to this rank, while if

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) \neq 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then the dimension of $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ is strictly less than this rank.

Proof. The dimension of $[\operatorname{Spec} B^{\mathrm{dist}} / \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ is equal to the rank of $T_{A^{\mathrm{dist}}}$ (it is the quotient by a two-dimensional group of a vector bundle over a two-dimensional base of rank equal to the rank of $T_{A^{\mathrm{dist}}}$). By Lemma 3.3.10, $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ is the scheme-theoretic image of the morphism $[\operatorname{Spec} B^{\mathrm{dist}} / \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1}$ provided by Proposition 3.3.21(1), which (by that proposition) is representable by algebraic spaces and unramified. Since such a morphism is locally quasi-finite (in fact, in this particular case, we have shown that the fibres of this morphism have degree at most 2), [Sta13, Tag 0DS6] ensures that $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ has the claimed dimension.

If $\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$ for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then an identical argument using Proposition 3.3.21(2) implies the claim regarding the dimension of $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$.

Finally, suppose that

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) \neq 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$. Then the composite $[\operatorname{Spec} B^{\text{k-free}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1}$ has the property that for every point t in the source, the fibre over the image of t has a positive dimensional fibre. [Sta13, Tag 0DS6] then implies the remaining claim of the present lemma. ☐

4. Extensions of rank one Breuil–Kisin modules

4.1. **Rank one modules over finite fields, and their extensions.** We now wish to apply the results of the previous section to study the geometry of our various moduli stacks. In order to do this, it will be convenient for us to have an explicit description of the rank one Breuil–Kisin modules of height at most one with descent data over a finite field of characteristic p, and of their possible extensions. Many of the results in this section are proved (for $p > 2$) in [DS15, §1] in the context of Breuil modules, and in those cases it is possible simply to translate the relevant statements to the Breuil–Kisin module context.

Assume from now on that $e(K'/K)$ is divisible by $p^f - 1$, so that we are in the setting of [DS15, Remark 1.7]. (Note that the parallel in [DS15] of our field extension K'/K, with ramification and inertial indices e', f' and e, f respectively, is the extension K/L with indices e, f and e', f' respectively.)

Let $\mathbf{F}$ be a finite subfield of $\overline{\mathbf{F}}_p$ containing the image of some (so all) embedding(s) $k' \hookrightarrow \overline{\mathbf{F}}_p$. Recall that for each $g \in \operatorname{Gal}(K'/K)$ we write $g(\pi')/\pi' = h(g)$ with $h(g) \in \mu_{e(K'/K)}(K') \subset W(k')$. We abuse notation and denote the image of $h(g)$ in k' again by $h(g)$, so that we obtain a map $h\colon \operatorname{Gal}(K'/K) \to (k')^\times$. Note that h restricts to a character on the inertia subgroup $I(K'/K)$, and is itself a character when $e(K'/K) = p^f - 1$.

Lemma 4.1.1. *Every rank one Breuil–Kisin module of height at most one with descent data and $\mathbf{F}$-coefficients is isomorphic to one of the modules $\mathfrak{M}(r,a,c)$ defined by:*

- $\mathfrak{M}(r,a,c)_i = \mathbf{F}[[u]] \cdot m_i$,
- $\Phi_{\mathfrak{M}(r,a,c),i}(1 \otimes m_{i-1}) = a_i u^{r_i} m_i$,
- $\hat{g}(\sum_i m_i) = \sum_i h(g)^{c_i} m_i$ *for all* $g \in \operatorname{Gal}(K'/K)$,

where $a_i \in \mathbf{F}^\times$, $r_i \in \{0, \dots, e'\}$ and $c_i \in \mathbf{Z}/e(K'/K)$ are sequences satisfying $pc_{i-1} \equiv c_i + r_i \pmod{e(K'/K)}$, the sums in the third bullet point run from 0 to $f'-1$, and the r_i, c_i, a_i are periodic with period dividing f.

Furthermore, two such modules $\mathfrak{M}(r,a,c)$ and $\mathfrak{M}(s,b,d)$ are isomorphic if and only if $r_i = s_i$ and $c_i = d_i$ for all i, and $\prod_{i=0}^{f-1} a_i = \prod_{i=0}^{f-1} b_i$.

Proof. The proof is elementary; see e.g. [Sav08, Thm. 2.1, Thm. 3.5] for proofs of analogous results. ☐

We will sometimes refer to the element $m = \sum_i m_i \in \mathfrak{M}(r,a,c)$ as the standard generator of $\mathfrak{M}(r,a,c)$.

Remark 4.1.2. When $p > 2$ many of the results in this section (such as the above) can be obtained by translating [DS15, Lem. 1.3, Cor. 1.8] from the Breuil module context to the Breuil–Kisin module context. We briefly recall the dictionary between these two categories (*cf.* [Kis09, §1.1.10]). If A is a finite local $\mathbf{Z}_p$-algebra, write $S_A = S \otimes_{\mathbf{Z}_p} A$, where S is Breuil's ring. We regard S_A as a $\mathfrak{S}_A$-algebra via $u \mapsto u$, and we let $\varphi : \mathfrak{S}_A \to S_A$ be the composite of this map with φ on $\mathfrak{S}_A$. Then given a Breuil–Kisin module of height at most 1 with descent data $\mathfrak{M}$, we set $\mathcal{M} := S_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M}$. We have a map $1 \otimes \varphi_{\mathfrak{M}} : S_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M} \to S_A \otimes_{\mathfrak{S}_A} \mathfrak{M}$, and we set

$$\mathrm{Fil}^1 \mathcal{M} := \{x \in \mathcal{M} \ : \ (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^1 S_A \otimes_{\mathfrak{S}_A} \mathfrak{M} \subset S_A \otimes_{\mathfrak{S}_A} \mathfrak{M}\}$$

and define $\varphi_1 : \mathrm{Fil}^1 \mathcal{M} \to \mathcal{M}$ as the composite

$$\mathrm{Fil}^1 \mathcal{M} \xrightarrow{1\otimes\varphi_{\mathfrak{M}}} \mathrm{Fil}^1 S_A \otimes_{\mathfrak{S}_A} \mathfrak{M} \xrightarrow{\varphi_1 \otimes 1} S_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M} = \mathcal{M}.$$

Finally, we define $\hat{g}$ on $\mathcal{M}$ via $\hat{g}(s \otimes m) = g(s) \otimes \hat{g}(m)$. One checks without difficulty that this makes $\mathcal{M}$ a strongly divisible module with descent data (*cf.* the proofs of [Kis09, Proposition 1.1.11, Lemma 1.2.4]).

In the correspondence described above, the Breuil–Kisin module $\mathfrak{M}((r_i),(a_i),(c_i))$ corresponds to the Breuil module $\mathcal{M}((e'-r_i),(a_i),(pc_{i-1}))$ of [DS15, Lem. 1.3].

Definition 4.1.3. If $\mathfrak{M} = \mathfrak{M}(r,a,c)$ is a rank one Breuil–Kisin module as described in the preceding lemma, we set $\alpha_i(\mathfrak{M}) := (p^{f'-1} r_{i-f'+1} + \cdots + r_i)/(p^{f'}-1)$ (equivalently, $(p^{f-1} r_{i-f+1} + \cdots + r_i)/(p^f - 1)$). We may abbreviate $\alpha_i(\mathfrak{M})$ simply as α_i when $\mathfrak{M}$ is clear from the context.

It follows easily from the congruence $r_i \equiv pc_{i-1} - c_i \pmod{e(K'/K)}$ together with the hypothesis that $p^f - 1 \mid e(K'/K)$ that $\alpha_i \in \mathbf{Z}$ for all i. Note that the α_i's are the unique solution to the system of equations $p\alpha_{i-1} - \alpha_i = r_i$ for all i. Note also that $(p^f-1)(c_i - \alpha_i) \equiv 0 \pmod{e(K'/K)}$, so that $h^{c_i - \alpha_i}$ is a character with image in $k^\times$.

Lemma 4.1.4. *We have* $T(\mathfrak{M}(r,a,c)) = \left(\sigma_i \circ h^{c_i-\alpha_i} \cdot \mathrm{ur}_{\prod_{i=0}^{f-1} a_i}\right)|_{G_{K_\infty}}$*, where* ur_λ *is the unramified character of* G_K *sending geometric Frobenius to* λ*.*

Proof. Set $\mathfrak{N} = \mathfrak{M}(0,(a_i),0)$, so that $\mathfrak{N}$ is effectively a Breuil–Kisin module without descent data. Then for $\mathfrak{N}$ this result follows from the second paragraph of the proof [GLS14, Lem. 6.3]. (Note that the functor $T_{\mathfrak{S}}$ of *loc. cit.* is dual to our functor T; *cf.* [Fon90, A 1.2.7]. Note also that the fact that the base field is unramified in *loc. cit.* does not change the calculation.) If $n = \sum n_i$ is the standard generator of $\mathfrak{N}$ as in Lemma 4.1.1, let $\gamma \in \mathbf{Z}_p^{\mathrm{un}} \otimes_{\mathbf{Z}_p} (k' \otimes_{\mathbf{F}_p} \mathbf{F})$ be an element so that $\gamma n \in (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} \mathfrak{N}[1/u])^{\varphi=1}$.

Now for $\mathfrak{M}$ as in the statement of the lemma it is straightforward to verify that

$$\gamma \sum_{i=0}^{f'-1} [\underline{\pi}']^{-\alpha_i} \otimes m_i \in (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} \mathfrak{M}[1/u])^{\varphi=1},$$

and the result follows. □

One immediately deduces the following.

Corollary 4.1.5. *Let* $\mathfrak{M} = \mathfrak{M}(r,a,c)$ *and* $\mathfrak{N} = \mathfrak{M}(s,b,d)$ *be rank one Breuil–Kisin modules with descent data as above. We have* $T(\mathfrak{M}) = T(\mathfrak{N})$ *if and only if* $c_i - \alpha_i(\mathfrak{M}) \equiv d_i - \alpha_i(\mathfrak{N}) \pmod{e(K'/K)}$ *for some* i *(hence for all* i*) and* $\prod_{i=0}^{f-1} a_i = \prod_{i=0}^{f-1} b_i$*.*

Lemma 4.1.6. *In the notation of the previous Corollary, there is a nonzero map* $\mathfrak{M} \to \mathfrak{N}$ *(equivalently,* $\dim_{\mathbf{F}} \operatorname{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) = 1$*) if and only if* $T(\mathfrak{M}) = T(\mathfrak{N})$ *and* $\alpha_i(\mathfrak{M}) \geq \alpha_i(\mathfrak{N})$ *for each* i.

Proof. The proof is essentially the same as that of [DS15, Lem. 1.6]. (Indeed, when $p > 2$ this lemma can once again be proved by translating directly from [DS15] to the Breuil–Kisin module context.) $\square$

Using the material of Section 3.1, one can compute $\operatorname{Ext}^1(\mathfrak{M}, \mathfrak{N})$ for any pair of rank one Breuil–Kisin modules $\mathfrak{M}, \mathfrak{N}$ of height at most one. We begin with the following explicit description of the complex $C^\bullet(\mathfrak{N})$ of Section 3.1.

Definition 4.1.7. We write $\mathcal{C}_u^0 = \mathcal{C}_u^0(\mathfrak{M}, \mathfrak{N}) \subset \mathbf{F}((u))^{\mathbf{Z}/f\mathbf{Z}}$ for the space of f-tuples (μ_i) such that each nonzero term of μ_i has degree congruent to $c_i - d_i \pmod{e(K'/K)}$, and set $\mathcal{C}^0 = \mathcal{C}_u^0 \cap \mathbf{F}[[u]]^{\mathbf{Z}/f\mathbf{Z}}$.

We further define $\mathcal{C}_u^1 = \mathcal{C}_u^1(\mathfrak{M}, \mathfrak{N}) \subset \mathbf{F}((u))^{\mathbf{Z}/f\mathbf{Z}}$ to be the space of f-tuples (h_i) such that each nonzero term of h_i has degree congruent to $r_i + c_i - d_i \pmod{e(K'/K)}$, and set $\mathcal{C}^1 = \mathcal{C}_u^1 \cap \mathbf{F}[[u]]^{\mathbf{Z}/f\mathbf{Z}}$. There is a map $\partial \colon \mathcal{C}_u^0 \to \mathcal{C}_u^1$ defined by

$$\partial(\mu_i) = (-a_i u^{r_i} \mu_i + b_i \varphi(\mu_{i-1}) u^{s_i})$$

Evidently this restricts to a map $\partial \colon \mathcal{C}^0 \to \mathcal{C}^1$.

Lemma 4.1.8. *There is an isomorphism of complexes*

$$[\mathcal{C}^0 \xrightarrow{\partial} \mathcal{C}^1] \xrightarrow{\sim} C^\bullet(\mathfrak{N})$$

in which $(\mu_i) \in \mathcal{C}^0$ *is sent to the map* $m_i \mapsto \mu_i n_i$ *in* $C^0(\mathfrak{N})$*, and* $(h_i) \in \mathcal{C}^1$ *is sent to the map* $(1 \otimes m_{i-1}) \mapsto h_i n_i$ *in* $C^1(\mathfrak{N})$.

Proof. Each element of $\operatorname{Hom}_{\mathfrak{S}_{\mathbf{F}}}(\mathfrak{M}, \mathfrak{N})$ has the form $m_i \mapsto \mu_i n_i$ for some f'-tuple $(\mu_i)_{i \in \mathbf{Z}/f'\mathbf{Z}}$ of elements of $\mathbf{F}[[u]]$. The condition that this map is $\operatorname{Gal}(K'/K)$-equivariant is easily seen to be equivalent to the conditions that (μ_i) is periodic with period dividing f, and that each nonzero term of μ_i has degree congruent to $c_i - d_i \pmod{e(K'/K)}$. (For the former consider the action of a lift to $g \in \operatorname{Gal}(K'/K)$ satisfying $h(g) = 1$ of a generator of $\operatorname{Gal}(k'/k)$, and for the latter consider the action of $I(K'/K)$; *cf.* the proof of [DS15, Lem. 1.5].) It follows that the map $\mathcal{C}^0 \to C^0(\mathfrak{N})$ in the statement of the Lemma is an isomorphism. An essentially identical argument shows that the given map $\mathcal{C}^1 \to C^1(\mathfrak{N})$ is an isomorphism.

To conclude, it suffices to observe that if $\alpha \in C^0(\mathfrak{N})$ is given by $m_i \mapsto \mu_i n_i$ with $(\mu_i)_i \in \mathcal{C}^0$ then $\delta(\alpha) \in C^1(\mathfrak{N})$ is the map given by

$$(1 \otimes m_{i-1}) \mapsto (-a_i u^{r_i} \mu_i + b_i \varphi(\mu_{i-1}) u^{s_i}) n_i,$$

which follows by a direct calculation. $\square$

It follows from Corollary 3.1.8 that $\operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) \cong \operatorname{coker} \partial$. If $h \in \mathcal{C}^1$, we write $\mathfrak{P}(h)$ for the element of $\operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ represented by h under this isomorphism.

Remark 4.1.9. Let $\mathfrak{M} = \mathfrak{M}(r, a, c)$ and $\mathfrak{N} = \mathfrak{M}(s, b, d)$ be rank one Breuil–Kisin modules with descent data as in Lemma *4.1.1*. It follows from the proof of Lemma 3.1.5, and in particular the description of the map (3.1.6) found there, that the extension $\mathfrak{P}(h)$ is given by the formulas

— $\mathfrak{P}_i = \mathbf{F}[[u]] \cdot m_i + \mathbf{F}[[u]] \cdot n_i$,

— $\Phi_{\mathfrak{P},i}(1 \otimes n_{i-1}) = b_i u^{s_i} n_i$, $\Phi_{\mathfrak{P},i}(1 \otimes m_{i-1}) = a_i u^{r_i} m_i + h_i n_i$.

— $\hat{g}(\sum_i m_i) = \sum_i h(g)^{c_i} m_i$, $\hat{g}(\sum_i n_i) = h(g)^{d_i} \sum_i n_i$ for all $g \in \mathrm{Gal}(K'/K)$.

From this description it is easy to see that the extension $\mathfrak{P}(h)$ has height at most 1 if and only if each h_i is divisible by $u^{r_i+s_i-e'}$.

Theorem 4.1.10. *The dimension of* $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ *is given by the formula*

$$\Delta + \sum_{i=0}^{f-1} \#\Big\{ j \in [0, r_i) : j \equiv r_i + c_i - d_i \pmod{e(K'/K)} \Big\}$$

where $\Delta = \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ *is* 1 *if there is a nonzero map* $\mathfrak{M} \to \mathfrak{N}$ *and* 0 *otherwise, while the subspace consisting of extensions of height at most* 1 *has dimension*

$$\Delta + \sum_{i=0}^{f-1} \#\Big\{ j \in [\max(0, r_i + s_i - e'), r_i) : j \equiv r_i + c_i - d_i \pmod{e(K'/K)} \Big\}.$$

Proof. When $p > 2$, this result (for extensions of height at most 1) can be obtained by translating [DS15, Thm. 1.11] from Breuil modules to Breuil–Kisin modules. We argue in the same spirit as [DS15] using the generalities of Section 3.1.

Choose N as in Lemma 3.1.10(2). For brevity we write $C^\bullet$ in lieu of $C^\bullet(\mathfrak{N})$. We now use the description of $C^\bullet$ provided by Lemma 4.1.8. As we have noted, C^0 consists of the maps $m_i \mapsto \mu_i n_i$ with $(\mu_i) \in \mathcal{C}^0$. Since $(\varphi^*_{\mathfrak{M}})^{-1}(v^N C^1)$ contains precisely the maps $m_i \mapsto \mu_i n_i$ in C^0 such that $v^N \mid u^{r_i}\mu_i$, we compute that $\dim_{\mathbf{F}} C^0/\big((\varphi^*_{\mathfrak{M}})^{-1}(v^N C^1)\big)$ is the quantity

$$Nf - \sum_{i=0}^{f-1} \#\Big\{ j \in [e(K'/K)N - r_i, e(K'/K)N) : j \equiv c_i - d_i \pmod{e(K'/K)} \Big\}.$$

We have $\dim_{\mathbf{F}} C^1/v^N C^1 = Nf$, so our formula for the dimension of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ now follows from Lemma 3.1.10. $\square$

Remark 4.1.11. One can show exactly as in [DS15] that each element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ can be written uniquely in the form $\mathfrak{P}(h)$ for $h \in \mathcal{C}^1$ with $\deg(h_i) < r_i$, except that when there exists a nonzero morphism $\mathfrak{M} \to \mathfrak{N}$, the polynomials h_i for $f \mid i$ may also have a term of degree $\alpha_0(\mathfrak{M}) - \alpha_0(\mathfrak{N}) + r_0$ in common. Since we will not need this fact we omit the proof.

4.2. **Extensions of shape J.** We now begin the work of showing, for each non-scalar tame type τ, that $\mathcal{C}^{\tau,\mathrm{BT},1}$ has 2^f irreducible components, indexed by the subsets J of $\{0, 1, \ldots, f-1\}$. We will also describe the irreducible components of $\mathcal{Z}^{\tau,1}$. The proof of this hinges on examining the extensions considered in Theorem 4.1.10, and then applying the results of Subsection 3.3. We will show that most of these families of extensions have positive codimension in $\mathcal{C}^{\tau,\mathrm{BT},1}$, and are thus negligible from the point of view of determining irreducible components. By a base change argument, we will also be able to show that we can neglect the irreducible Breuil–Kisin modules. The rest of Section 3 is devoted to establishing the necessary bounds on the dimension of the various families of extensions, and to studying the map from $\mathcal{C}^{\tau,\mathrm{BT},1}$ to $\mathcal{R}^{\mathrm{dd},1}$.

We now introduce notation that we will use for the remainder of the paper. We fix a tame inertial type $\tau = \eta \oplus \eta'$ with coefficients in $\overline{\mathbf{Q}}_p$. We allow the case of scalar types (that is, the case $\eta = \eta'$). Assume that the subfield $\mathbf{F}$ of $\overline{\mathbf{F}}_p$ is large enough so that the reductions modulo $\mathfrak{m}_{\mathbf{Z}_p}$ of η and η' (which by abuse of notation we continue to denote η, η') have image in $\mathbf{F}$. We also fix a uniformiser π of K.

Remark 4.2.1. We stress that when we write $\tau = \eta \oplus \eta'$, we are implicitly ordering η, η'. Much of the notation in this section depends on distinguishing η, η', as do some of the constructions later in paper (in particular, those using the map to the Dieudonné stack of Section 2.4).

As in Subsection 2.4, we make the following "standard choice" for the extension K'/K: if τ is a tame principal series type, we take $K' = K(\pi^{1/(p^f-1)})$, while if τ is a tame cuspidal type, we let L be an unramified quadratic extension of K, and set $K' = L(\pi^{1/(p^{2f}-1)})$. In either case K'/K is a Galois extension and η, η' both factor through $I(K'/K)$. In the principal series case, we have $e' = (p^f-1)e$, $f' = f$, and in the cuspidal case we have $e' = (p^{2f}-1)e$, $f' = 2f$. Either way, we have $e(K'/K) = p^{f'} - 1$.

In either case, it follows from Lemma 4.1.1 that a Breuil–Kisin module of rank one with descent data from K' to K is described by the data of the quantities r_i, a_i, c_i for $0 \le i \le f-1$, and similarly from Lemma 4.1.8 that extensions between two such Breuil–Kisin modules are described by the h_i for $0 \le i \le f-1$. This common description will enable us to treat the principal series and cuspidal cases largely in parallel.

The character $h|_{I_K}$ of Section 4.1 is identified via the Artin map $\mathcal{O}_L^\times \to I_L^{\mathrm{ab}} = I_K^{\mathrm{ab}}$ with the reduction map $\mathcal{O}_L^\times \to (k')^\times$. Thus for each $\sigma \in \mathrm{Hom}(k', \overline{\mathbf{F}}_p)$ the map $\sigma \circ h|_{I_L}$ is the fundamental character ω_σ defined in Section 1.4. Define $k_i, k_i' \in \mathbf{Z}/(p^{f'}-1)\mathbf{Z}$ for all i by the formulas $\eta = \sigma_i \circ h^{k_i}|_{I(K'/K)}$ and $\eta' = \sigma_i \circ h^{k_i'}|_{I(K'/K)}$. In particular we have $k_i = p^i k_0$, $k_i' = p^i k_0'$ for all i.

Definition 4.2.2. Let $\mathfrak{M} = \mathfrak{M}(r, a, c)$ and $\mathfrak{N} = \mathfrak{M}(s, b, d)$ be Breuil–Kisin modules of rank one with descent data. We say that the pair $(\mathfrak{M}, \mathfrak{N})$ has *type* τ provided that for all i:

— the multisets $\{c_i, d_i\}$ and $\{k_i, k_i'\}$ are equal, and

— $r_i + s_i = e'$.

Lemma 4.2.3. *The following are equivalent.*

(1) The pair $(\mathfrak{M}, \mathfrak{N})$ *has type* τ.

(2) Some element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ *of height at most one satisfies the strong determinant condition and is of type* τ.

(3) Every element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ *has height at most one, satisfies the strong determinant condition, and is of type* τ.

(Accordingly, we will sometimes refer to the condition that $r_i + s_i = e'$ *for all* i *as the determinant condition.)*

Proof. Suppose first that the pair $(\mathfrak{M}, \mathfrak{N})$ has type τ. The last sentence of Remark 4.1.9 shows that every element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ has height at most one. Let $\mathfrak{P}$ be such an element. The condition on the multisets $\{c_i, d_i\}$ guarantees that $\mathfrak{P}$ has unmixed type τ. By [CEGS20b, Prop. 4.2.12] we see that $\dim_{\mathbf{F}}(\mathrm{im}_{\mathfrak{P},i}/E(u)\mathfrak{P}_i)_{\tilde{\eta}}$ is independent of $\tilde{\eta}$. From the condition that $r_i + s_i = e'$ we know that the sum over all $\tilde{\eta}$ of these dimensions is equal to e'; since they are all equal, each is equal to e, and [CEGS20b, Lem. 4.2.11] tells us that $\mathfrak{P}$ satisfies the strong determinant condition. This proves that (1) implies (3).

Certainly (3) implies (2), so it remains to check that (2) implies (1). Suppose that $\mathfrak{P} \in \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ has height at most one, satisfies the strong determinant condition, and has type τ. The condition that $\{c_i, d_i\} = \{k_i, k'_i\}$ follows from $\mathfrak{P}$ having type τ, and the condition that $r_i + s_i = e'$ follows from the last part of [CEGS20b, Lem. 4.2.11]. □

Definition 4.2.4. If $(\mathfrak{M}, \mathfrak{N})$ is a pair of type τ (resp. $\mathfrak{P}$ is an extension of type τ), we define the *shape* of $(\mathfrak{M}, \mathfrak{N})$ (resp. of $\mathfrak{P}$) to be the subset $J := \{i \,|\, c_i = k_i\} \subseteq \mathbf{Z}/f'\mathbf{Z}$, unless τ is scalar, in which case we define the shape to be the subset $\varnothing$. (Equivalently, J is in all cases the complement in $\mathbf{Z}/f'\mathbf{Z}$ of the set $\{i \,|\, c_i = k'_i\}$.)

Observe that in the cuspidal case the equality $c_i = c_{i+f}$ means that $i \in J$ if and only if $i + f \notin J$, so that the set J is determined by its intersection with any f consecutive integers modulo $f' = 2f$.

In the cuspidal case we will say that a subset $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ is a shape if it satisfies $i \in J$ if and only if $i + f \notin J$; in the principal series case, we may refer to any subset $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ as a shape.

We define the *refined shape* of the pair $(\mathfrak{M}, \mathfrak{N})$ (resp. of $\mathfrak{P}$) to consist of its shape J, together with the f-tuple of invariants $r := (r_i)_{i=0}^{f-1}$. If (J, r) is a refined shape that arises from some pair (or extension) of type τ, then we refer to (J, r) as a refined shape for τ.

We say the pair $(i-1, i)$ is a *transition* for J if $i - 1 \in J$, $i \notin J$ or vice-versa. (In the first case we sometimes say that the pair $(i-1, i)$ is a transition out of J, and in the latter case a transition into J.) Implicit in many of our arguments below is the observation that in the cuspidal case $(i-1, i)$ is a transition if and only if $(i + f - 1, i + f)$ is a transition.

4.2.5. *An explicit description of refined shapes.* The refined shapes for τ admit an explicit description. If $\mathfrak{P}$ is of shape J, for some fixed $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ then, since c_i, d_i are fixed, we see that the r_i and s_i appearing in $\mathfrak{P}$ are determined modulo $e(K'/K) = p^{f'} - 1$. Furthermore, we see that $r_i + s_i \equiv 0 \pmod{p^{f'} - 1}$, so that these values are consistent with the determinant condition; conversely, if we make any choice of the r_i in the given residue class modulo $(p^{f'} - 1)$, then the s_i are determined by the determinant condition, and the imposed values are consistent with the descent data. There are of course only finitely many choices for the r_i, and so there are only finitely many possible refined shapes for τ.

To make this precise, recall that we have the congruence

$$r_i \equiv pc_{i-1} - c_i \pmod{p^{f'} - 1}.$$

We will write $[n]$ for the least non-negative residue class of n modulo $e(K'/K) = p^{f'} - 1$.

If both $i - 1$ and i lie in J, then we have $c_{i-1} = k_{i-1}$ and $c_i = k_i$. The first of these implies that $pc_{i-1} = k_i$, and therefore $r_i \equiv 0 \pmod{p^{f'} - 1}$. The same conclusion holds if neither $i-1$ and i lie in J. Therefore if $(i-1, i)$ is not a transition we may write

$$r_i = (p^{f'} - 1)y_i \quad \text{and} \quad s_i = (p^{f'} - 1)(e - y_i).$$

with $0 \le y_i \le e$.

Now suppose instead that $(i-1,i)$ is a transition. (In particular the type τ is not scalar.) This time $pc_{i-1} = d_i$ (instead of $pc_{i-1} = c_i$), so that $r_i \equiv d_i - c_i \pmod{p^{f'}-1}$. In this case we write

$$r_i = (p^{f'}-1)y_i - [c_i - d_i] \quad \text{and} \quad s_i = (p^{f'}-1)(e+1-y_i) - [d_i - c_i]$$

with $1 \le y_i \le e$.

Conversely, for fixed shape J one checks that each choice of integers y_i in the ranges described above gives rise to a refined shape for τ.

If $(i-1,i)$ is not a transition and $(h_i) \in \mathcal{C}^1_u(\mathfrak{M},\mathfrak{N})$ then non-zero terms of h_i have degree congruent to $r_i + c_i - d_i \equiv c_i - d_i \pmod{p^{f'}-1}$. If instead $(i-1,i)$ is a transition and $(h_i) \in \mathcal{C}^1_u(\mathfrak{M},\mathfrak{N})$ then non-zero terms of h_i have degree congruent to $r_i + c_i - d_i \equiv 0 \pmod{p^{f'}-1}$. In either case, comparing with the preceding paragraphs we see that $\#\{j \in [0, r_i) : j \equiv r_i + c_i - d_i \pmod{e(K'/K)}\}$ is exactly y_i.

By Theorem 4.1.10, we conclude that for a fixed choice of the r_i the dimension of the corresponding Ext^1 is $\Delta + \sum_{i=0}^{f-1} y_i$ (with Δ as in the statement of *loc. cit.*). We say that the refined shape $\left(J, (r_i)_{i=0}^{f-1}\right)$ is *maximal* if the r_i are chosen to be maximal subject to the above conditions, or equivalently if the y_i are all chosen to be e; for each shape J, there is a unique maximal refined shape (J, r).

4.2.6. *The sets $\mathcal{P}_\tau$.* To each tame type τ we now associate a set $\mathcal{P}_\tau$, which will be a subset of the set of shapes in $\mathbf{Z}/f'\mathbf{Z}$. (In Appendix A we will recall, following [Dia07], that the set $\mathcal{P}_\tau$ parameterises the Jordan–Hölder factors of the reduction mod p of $\sigma(\tau)$.)

We write $\eta(\eta')^{-1} = \prod_{j=0}^{f'-1} (\sigma_j \circ h)^{\gamma_j}$ for uniquely defined integers $0 \le \gamma_j \le p-1$ not all equal to $p-1$, so that

$$[k_i - k'_i] = \sum_{j=0}^{f'-1} p^j \gamma_{i-j} \tag{4.2.6}$$

with subscripts taken modulo f'.

If τ is scalar then we set $\mathcal{P}_\tau = \{\varnothing\}$. Otherwise we let $\mathcal{P}_\tau$ be the collection of shapes $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ satisfying the conditions:

— if $i-1 \in J$ and $i \notin J$ then $\gamma_i \ne p-1$, and

— if $i-1 \notin J$ and $i \in J$ then $\gamma_i \ne 0$.

When τ is a cuspidal type, so that $\eta' = \eta^q$, the integers γ_j satisfy $\gamma_{i+f} = p-1-\gamma_i$ for all i; thus the condition that if $(i-1,i)$ is a transition out of J then $\gamma_i \ne p-1$ translates exactly into the condition that if $(i+f-1, i+f)$ is a transition into J then $\gamma_{i+f} \ne 0$.

4.2.7. *Moduli stacks of extensions.* We now apply the constructions of stacks and topological spaces of Definitions 3.3.11 and 3.3.14 to the families of extensions considered in Section 4.2.

Definition 4.2.8. If (J,r) is a refined shape for τ, then we let $\mathfrak{M}(J,r) := \mathfrak{M}(r,1,c)$ and let $\mathfrak{N}(J,r) := \mathfrak{M}(s,1,d)$, where c, d, and s are determined from J, r, and τ according to the discussion of (4.2.5); for instance we take $c_i = k_i$ when $i \in J$ and $c_i = k'_i$ when $i \notin J$. For the unique maximal shape (J,r) refining J, we write simply $\mathfrak{M}(J)$ and $\mathfrak{N}(J)$.

Definition 4.2.9. If (J, r) is a refined shape for τ, then following Definition 3.3.11, we may construct the reduced closed substack $\overline{\mathcal{C}}\big(\mathfrak{M}(J, r), \mathfrak{N}(J, r)\big)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$, as well as the reduced closed substack $\overline{\mathcal{Z}}\big(\mathfrak{M}(J, r), \mathfrak{N}(J, r)\big)$ of $\mathcal{Z}^{\tau,1}$. We introduce the notation $\overline{\mathcal{C}}(J, r)$ and $\overline{\mathcal{Z}}(J, r)$ for these two stacks, and note that (by definition) $\overline{\mathcal{Z}}(J, r)$ is the scheme-theoretic image of $\overline{\mathcal{C}}(J, r)$ under the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$.

Remark 4.2.10. As noted in the final sentence of Definition 3.3.11, Lemma 3.3.10 shows that $\overline{\mathcal{C}}(J, r)$ contains all extensions of refined shape (J, r) over extensions of $\mathbf{F}$, and not only those corresponding to a maximal ideal of A^{dist}.

Theorem 4.2.11. *If (J, r) is any refined shape for τ, then $\dim \overline{\mathcal{C}}(J, r) \le [K : \mathbf{Q}_p]$, with equality if and only if (J, r) is maximal.*

Proof. This follows from Corollary 3.3.32, Theorem 4.1.10, and Proposition 3.1.15. (See also the discussion following Definition 4.2.4, and note that over $\operatorname{Spec} A^{\mathrm{dist}}$, we have $\Delta = 0$ by definition.) □

Definition 4.2.12. If $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ is a shape, and if r is chosen so that (J, r) is a maximal refined shape for τ, then we write $\overline{\mathcal{C}}(J)$ to denote the closed substack $\overline{\mathcal{C}}(J, r)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$, and $\overline{\mathcal{Z}}(J)$ to denote the closed substack $\overline{\mathcal{Z}}(J, r)$ of $\mathcal{Z}^{\tau,1}$. Again, we note that by definition $\overline{\mathcal{Z}}(J)$ is the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ in $\mathcal{Z}^{\tau,1}$.

We will see later that the $\overline{\mathcal{C}}(J)$ are precisely the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$; in particular, their finite type points can correspond to irreducible Galois representations. While we do not need it in the sequel, we note the following definition and result, describing the underlying topological spaces of the loci of reducible Breuil–Kisin modules of fixed refined shape.

Definition 4.2.13. For each refined type (J, r), we write $|\mathcal{C}(J, r)^{\tau}|$ for the constructible subset $|\mathcal{C}(\mathfrak{M}(J, r), \mathfrak{N}(J, r))|$ of $|\mathcal{C}^{\tau,\mathrm{BT},1}|$ of Definition 3.3.14 (where $\mathfrak{M}(J, r)$, $\mathfrak{N}(J, r)$ are the Breuil–Kisin modules of Definition 4.2.8). We write $|\mathcal{Z}(J, r)^{\tau}|$ for the image of $|\mathcal{C}(J, r)^{\tau}|$ in $|\mathcal{Z}^{\tau,1}|$ (which is again a constructible subset).

Lemma 4.2.14. *The $\overline{\mathbf{F}}_p$-points of $|\mathcal{C}(J, r)^{\tau}|$ are precisely the reducible Breuil–Kisin modules with $\overline{\mathbf{F}}_p$-coefficients of type τ and refined shape (J, r).*

Proof. This is immediate from the definition. □

5. Components of Breuil–Kisin and Galois moduli stacks

Now that we have constructed the morphisms $\overline{\mathcal{C}}(J) \to \overline{\mathcal{Z}}(J)$ for each J, we can begin our study of the components of the stacks $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$. The first step in Subsection 5.1 is to determine precisely for which J the scheme-theoretic image $\overline{\mathcal{Z}}(J)$ has dimension smaller than $[K : \mathbf{Q}_p]$, and hence is *not* a component of $\mathcal{Z}^{\tau,1}$. In Section 5.2 we study the irreducible locus in $\mathcal{C}^{\tau,\mathrm{BT},1}$ and prove that it lies in a closed substack of positive codimension. We are then ready to establish our main results in Subsections 5.3 and 5.4.

5.1. ker-Ext^1 and vertical components.

In this section we will establish some basic facts about $\text{ker-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$, and use these results to study the images of our irreducible components in $\mathcal{Z}^{\tau,1}$. Let $\mathfrak{M} = \mathfrak{M}(r, a, c)$ and $\mathfrak{N} = \mathfrak{M}(s, b, c)$ be Breuil–Kisin modules as in Section 4.1.

Recall from (3.1.31) that the dimension of $\text{ker-Ext}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ is bounded above by the dimension of $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$; more precisely, by Lemma 2.2.4 we find in this setting that

$$\begin{aligned}\dim_{\mathbf{F}} \text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) &= \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N}) \\ -(\dim_{\mathbf{F}} \mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N})) &- \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})).\end{aligned} \tag{5.1.1}$$

A map $f:\mathfrak{M}\to\mathfrak{N}[1/u]/\mathfrak{N}$ has the form $f(m_i)=\mu_i n_i$ for some f'-tuple of elements $\mu_i\in\mathbf{F}((u))/\mathbf{F}[[u]]$. By the same argument as in the first paragraph of the proof of Lemma 4.1.8, such a map belongs to $C^0(\mathfrak{N}[1/u]/\mathfrak{N})$ (i.e., it is $\mathrm{Gal}(K'/K)$-equivariant) if and only if the μ_i are periodic with period dividing f, and each nonzero term of μ_i has degree congruent to $c_i-d_i \pmod{e(K'/K)}$. One computes that $\delta(f)(1\otimes m_{i-1})=(u^{s_i}\varphi(\mu_{i-1})-u^{r_i}\mu_i)n_i$ and so $f\in C^0(\mathfrak{N}[1/u]/\mathfrak{N})$ lies in $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$ precisely when

$$a_i u^{r_i}\mu_i = b_i\varphi(\mu_{i-1})u^{s_i} \tag{5.1.2}$$

for all i.

Remark 5.1.3. Let $f\in\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$ be given as above. Choose any lifting $\tilde{\mu}_i$ of μ_i to $\mathbf{F}((u))$. Then (with notation as in Definition 4.1.7) the tuple $(\tilde{\mu}_i)$ is an element of $\mathcal{C}^0_u$, and we define $h_i=\partial(\tilde{\mu}_i)$. Then h_i lies in $\mathbf{F}[[u]]$ for all i, so that $(h_i)\in\mathcal{C}^1$, and a comparison with Lemma 4.1.8 shows that f maps to the extension class in $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ represented by $\mathfrak{P}(h)$.

Recall that Lemma 3.1.32 implies that nonzero terms appearing in μ_i have degree at least $-\lfloor e'/(p-1)\rfloor$. From this we obtain the following trivial bound on ker-Ext.

Lemma 5.1.4. *We have* $\dim_{\mathbf{F}}\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})\le\lceil e/(p-1)\rceil f$.

Proof. The degrees of nonzero terms of μ_i all lie in a single congruence class modulo $e(K'/K)$, and are bounded below by $-e'/(p-1)$. Therefore there are at most $\lceil e/(p-1)\rceil$ nonzero terms, and since the μ_i are periodic with period dividing f the lemma follows. ☐

Remark 5.1.5. It follows directly from Corollary 5.1.4 that if $p>3$ and $e\ne 1$ then we have $\dim_{\mathbf{F}}\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})\le[K:\mathbf{Q}_p]/2$, for then $\lceil e/(p-1)\rceil\le e/2$. Moreover these inequalities are strict if $e>2$.

We will require a more precise computation of $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ in the setting of Section 4.2 where the pair $(\mathfrak{M},\mathfrak{N})$ has maximal refined shape (J,r). We now return to that setting and its notation.

Let τ be a tame type. We will find the following notation to be helpful. We let $\gamma_i^*=\gamma_i$ if $i-1\notin J$, and $\gamma_i^*=p-1-\gamma_i$ if $i-1\in J$. (Here the integers γ_i are as in Section 4.2.6. In the case of scalar types this means that we have $\gamma_i^*=0$ for all i.) Since $p[k_{i-1}-k'_{i-1}]-[k_i-k'_i]=(p^{f'}-1)\gamma_i$, an elementary but useful calculation shows that

$$p[d_{i-1}-c_{i-1}]-[c_i-d_i]=\gamma_i^*(p^{f'}-1), \tag{5.1.6}$$

when $(i-1,i)$ is a transition, and that in this case $\gamma_i^*=0$ if and only if $[d_{i-1}-c_{i-1}]<p^{f'-1}$. Similarly, if τ is not a scalar type and $(i-1,i)$ is not a transition then

$$p[d_{i-1}-c_{i-1}]+[c_i-d_i]=(\gamma_i^*+1)(p^{f'}-1). \tag{5.1.7}$$

The main computational result of this section is the following.

Proposition 5.1.8. *Let (J,r) be any maximal refined shape for τ, and suppose that the pair $(\mathfrak{M},\mathfrak{N})$ has refined shape (J,r). Then* $\dim_{\mathbf{F}} \text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ *is equal to*

$$\#\{0 \le i < f \,:\, \textit{the pair } (i-1,i) \textit{ is a transition and } \gamma_i^* = 0\},$$

except that when $e=1$, $\prod_i a_i = \prod_i b_i$, and the quantity displayed above is f, then the dimension of $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ *is equal to $f-1$.*

Proof. The argument has two parts. First we show that $\dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$ is precisely the displayed quantity in the statement of the Proposition; then we check that $\dim_{\mathbf{F}} \mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N})) - \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ is equal to 1 in the exceptional case of the statement, and 0 otherwise. The result then follows from (5.1.1).

Let $f : m_i \mapsto \mu_i n_i$ be an element of $C^0(\mathfrak{N}[1/u]/\mathfrak{N})$. Since $u^{e'}$ kills μ_i, and all nonzero terms of μ_i have degree congruent to $c_i - d_i \pmod{p^{f'}-1}$, certainly all nonzero terms of μ_i have degree at least $-e' + [c_i - d_i]$. On the other hand since the shape (J,r) is maximal we have $r_i = e' - [c_i - d_i]$ when $(i-1,i)$ is a transition and $r_i = e'$ otherwise. In either case u^{r_i} kills μ_i, so that (5.1.2) becomes simply the condition that u^{s_i} kills $\varphi(\mu_{i-1})$.

If $(i-1,i)$ is not a transition then $s_i = 0$, and we conclude that $\mu_{i-1} = 0$. Suppose instead that $(i-1,i)$ is a transition, so that $s_i = [c_i - d_i]$. Then all nonzero terms of μ_{i-1} have degree at least $-s_i/p > -p^{f'-1} > -e(K'/K)$. Since those terms must have degree congruent to $c_{i-1} - d_{i-1} \pmod{p^{f'}-1}$, it follows that μ_{i-1} has at most one nonzero term (of degree $-[d_{i-1} - c_{i-1}]$), and this only if $[d_{i-1} - c_{i-1}] < p^{f'-1}$, or equivalently $\gamma_i^* = 0$ (as noted above). Conversely if $\gamma_i^* = 0$ then

$$u^{s_i}\varphi(u^{-[d_{i-1}-c_{i-1}]}) = u^{[c_i-d_i]-p[d_{i-1}-c_{i-1}]} = u^{-\gamma_i^*(p^{f'}-1)}$$

vanishes in $\mathbf{F}((u))/\mathbf{F}[[u]]$. We conclude that μ_{i-1} may have a single nonzero term if and only if $(i-1,i)$ is a transition and $\gamma_i^* = 0$, and this completes the first part of the argument.

Turn now to the second part. Looking at Corollary 4.1.5 and Lemma 4.1.6, to compare $\mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N}))$ and $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ we need to compute the quantities $\alpha_i(\mathfrak{M}) - \alpha_i(\mathfrak{N})$. By definition this quantity is equal to

$$\frac{1}{p^{f'}-1}\sum_{j=1}^{f'} p^{f'-j}\left(r_{i+j} - s_{i+j}\right). \tag{5.1.9}$$

Suppose first that τ is non-scalar. When $(i+j-1,i+j)$ is a transition, we have $r_{i+j} - s_{i+j} = (e-1)(p^{f'}-1) + [d_{i+j} - c_{i+j}] - [c_{i+j} - d_{i+j}]$, and otherwise we have $r_{i+j} - s_{i+j} = e(p^{f'}-1) = (e-1)(p^{f'}-1) + [d_{i+j} - c_{i+j}] + [c_{i+j} - d_{i+j}]$. Substituting these expressions into (5.1.9), adding and subtracting $\frac{1}{p^{f'}-1}p^{f'}[d_i - c_i]$, and regrouping gives

$$-[d_i - c_i] + (e-1)\cdot\frac{p^{f'}-1}{p-1} + \frac{1}{p^{f'}-1}\sum_{j=1}^{f'} p^{f'-j}\left(p[d_{i+j-1} - c_{i+j-1}] \mp [c_{i+j} - d_{i+j}]\right),$$

where the sign is $-$ if $(i+j-1, i+j)$ is a transition and $+$ if not. Applying the formulas (5.1.6) and (5.1.7) we conclude that

$$\alpha_i(\mathfrak{M}) - \alpha_i(\mathfrak{N}) = -[d_i - c_i] + (e-1)\cdot \frac{p^{f'}-1}{p-1} + \sum_{j=1}^{f'} p^{f'-j}\gamma^*_{i+j} + \sum_{j\in S_i} p^{f'-j} \tag{5.1.10}$$

where the set S_i consists of $1 \le j \le f$ such that $(i+j-1, i+j)$ is not a transition. Finally, a moment's inspection shows that the same formula still holds if τ is scalar (recalling that $J = \varnothing$ in that case).

Suppose that we are in the exceptional case of the proposition, so that $e = 1$, $\gamma^*_i = 0$ for all i, and every pair $(i-1, i)$ is a transition. The formula (5.1.10) gives $\alpha_i(\mathfrak{M}) - \alpha_i(\mathfrak{N}) = -[d_i - c_i]$. Since also $\prod_i a_i = \prod_i b_i$ the conditions of Corollary 4.1.5 are satisfied, so that $T(\mathfrak{M}) = T(\mathfrak{N})$; but on the other hand $\alpha_i(\mathfrak{M}) < \alpha_i(\mathfrak{N})$, so that by Lemma 4.1.6 there are no nonzero maps $\mathfrak{M} \to \mathfrak{N}$, and $\dim_{\mathbf{F}} \mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}), T(\mathfrak{N})) - \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) = 1$.

If instead we are not in the exceptional case of the proposition, then either $\prod_i a_i \ne \prod_i b_i$, or else (5.1.10) gives $\alpha_i(\mathfrak{M}) - \alpha_i(\mathfrak{N}) > -[d_i - c_i]$ for all i. Suppose that $T(\mathfrak{M}) \cong T(\mathfrak{N})$. It follows from Corollary 4.1.5 that $\alpha_i(\mathfrak{M}) - \alpha_i(\mathfrak{N}) \equiv -[d_i - c_i] \pmod{e(K'/K)}$. Combined with the previous inequality we deduce that $\alpha_i(\mathfrak{M}) - \alpha_i(\mathfrak{N}) > 0$, and Lemma 4.1.6 guarantees the existence of a nonzero map $\mathfrak{M} \to \mathfrak{N}$. We deduce that in any event $\dim_{\mathbf{F}} \mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}), T(\mathfrak{N})) = \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$, completing the proof. ☐

Corollary 5.1.11. *Let (J, r) be any maximal refined shape for τ, and suppose that the pair $(\mathfrak{M}, \mathfrak{N})$ has refined shape (J, r). If $J \in \mathcal{P}_\tau$ then $\dim_{\mathbf{F}} \text{ker-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) = 0$. Indeed this is an if and only if except possibly when $K = \mathbf{Q}_p$, the type τ is cuspidal, and $T(\mathfrak{M}(J, r)) \cong T(\mathfrak{N}(J, r))$.*

Proof. The first statement is immediate from Proposition 5.1.8, comparing the definition of γ^*_i with the defining condition on elements of $\mathcal{P}_\tau$; in fact this gives an if and only if unless we are in the exceptional case in Proposition 5.1.8 and $f - 1 = 0$. In that case $e = f = 1$, so $K = \mathbf{Q}_p$. In the principal series case for $K = \mathbf{Q}_p$ there can be no transitions, so the type is cuspidal. Then $\gamma^*_i = 0$ for $i = 0, 1$ and an elementary analysis of (5.1.6) shows that there exists $x \in \mathbf{Z}/(p-1)\mathbf{Z}$ such that $c_i = 1 + x(p+1)$, $d_i = p + x(p+1)$ for $i = 0, 1$. Then $r_i = p - 1$ and $s_i = p(p-1)$, and Lemma 4.1.4 gives $T(\mathfrak{M}(J, r)) \cong T(\mathfrak{N}(J, r))$. ☐

Recall that $\overline{\mathcal{Z}}(J)$ is by definition the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ in $\mathcal{Z}^{\tau,1}$. In the remainder of this section, we show that the $\overline{\mathcal{Z}}(J)$ with $J \in \mathcal{P}_\tau$ are pairwise distinct irreducible components of $\mathcal{Z}^{\tau,1}$. In Section 5.3 below we will show that they in fact exhaust the irreducible components of $\mathcal{Z}^{\tau,1}$.

Theorem 5.1.12. *$\overline{\mathcal{Z}}(J)$ has dimension at most $[K : \mathbf{Q}_p]$, with equality occurring if and only if $J \in \mathcal{P}_\tau$. Consequently, the $\overline{\mathcal{Z}}(J)$ with $J \in \mathcal{P}_\tau$ are irreducible components of $\mathcal{Z}^{\tau,1}$.*

Proof. The first part is immediate from Corollary 3.3.32, Proposition 3.1.15, Corollary 5.1.11 and Theorem 4.2.11 (noting that the exceptional case of Corollary 5.1.11 occurs away from max-Spec A^{dist}). Since $\mathcal{Z}^{\tau,1}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$ by Theorem 2.3.10, and the $\overline{\mathcal{Z}}(J)$ are closed and irreducible by construction, the second part follows from the first together with [Sta13, Tag 0DS2]. ☐

We also note the following result.

Proposition 5.1.13. *If $J \in \mathcal{P}_\tau$, then there is a dense open substack $\mathcal{U}$ of $\overline{\mathcal{C}}(J)$ such that the canonical morphism $\overline{\mathcal{C}}(J) \to \overline{\mathcal{Z}}(J)$ restricts to an open immersion on $\mathcal{U}$.*

Proof. This follows from Proposition 3.3.21 and Corollary 5.1.11. $\square$

For later use, we note the following computation. Recall that we write $\mathfrak{N}(J) = \mathfrak{N}(J, r)$ for the maximal shape (J, r) refining J, and that $\tau = \eta \oplus \eta'$.

Proposition 5.1.14. *For each shape J we have*

$$T(\mathfrak{N}(J)) \cong \eta \cdot \left(\prod_{i=0}^{f'-1} (\sigma_i \circ h)^{t_i} \right)^{-1} |_{G_{K_\infty}}$$

where

$$t_i = \begin{cases} \gamma_i + \delta_{J^c}(i) & \text{if } i-1 \in J \\ 0 & \text{if } i-1 \notin J. \end{cases}$$

Here δ_{J^c} is the characteristic function of the complement of J in $\mathbf{Z}/f'\mathbf{Z}$, and we are abusing notation by writing η for the function $\sigma_i \circ h^{k_i}$, which agrees with η on I_K.

In particular the map $J \mapsto T(\mathfrak{N}(J))$ is injective on $\mathcal{P}_\tau$.

Remark 5.1.15. In the cuspidal case it is not *a priori* clear that the formula in Proposition 5.1.14 gives a character of G_{K_∞} (rather than a character only when restricted to G_{L_∞}), but this is an elementary (if somewhat painful) calculation using the definition of the γ_i's and the relation $\gamma_i + \gamma_{i+f} = p - 1$.

Proof. We begin by explaining how the final statement follows from the rest of the Proposition. First observe that if $J \in \mathcal{P}_\tau$ then $0 \le t_i \le p-1$ for all i. Indeed the only possibility for a contradiction would be if $\gamma_i = p-1$ and $i \notin J$, but then the definition of $\mathcal{P}_\tau$ requires that we cannot have $i - 1 \in J$. Next, note that we never have $t_i = p-1$ for all i. Indeed, this would require $J = \mathbf{Z}/f'\mathbf{Z}$ and $\gamma_i = p-1$ for all i, but by definition the γ_i are not all equal to $p-1$.

The observations in the previous paragraph imply that (for $J \in \mathcal{P}_\tau$) the character $T(\mathfrak{N}(J))$ uniquely determines the integers t_i, and so it remains to show that the integers t_i determine the set J. If $t_i = 0$ for all i, then either $J = \varnothing$ or $J = \mathbf{Z}/f'\mathbf{Z}$ (for otherwise there is a transition out of J, and $\delta_{J^c}(i) \neq 0$ for some $i-1 \in J$). But if $J = \mathbf{Z}/f'\mathbf{Z}$ then $\gamma_i = 0$ for all i and τ is scalar; but for scalar types we have $\mathbf{Z}/f'\mathbf{Z} \notin \mathcal{P}_\tau$, a contradiction. Thus $t_i = 0$ for all i implies $J = \varnothing$.

For the rest of this part of the argument, we may therefore suppose $t_i \neq 0$ for some i, which forces $i - 1 \in J$. The entire set J will then be determined by recursion if we can show that knowledge of t_i along with whether or not $i \in J$, determines whether or not $i-1 \in J$. Given the defining formula for t_i, the only possible ambiguity is if $t_i = 0$ and $\gamma_i + \delta_{J^c}(i) = 0$, so that $\gamma_i = 0$ and $i \in J$. But the definition of $\mathcal{P}_\tau$ requires $i - 1 \in J$ in this case. This completes the proof.

We now turn to proving the formula for $T(\mathfrak{N}(J))$. We will use Lemma 4.1.4 applied at $i = 0$, for which we have to compute $\alpha_0 - d_0$ writing $\alpha_0 = \alpha_0(\mathfrak{N})$. Recall that we have already computed $\alpha_0(\mathfrak{M}(J)) - \alpha_0(\mathfrak{N}(J))$ in the proof of Proposition 5.1.8. Since $\alpha_0(\mathfrak{M}(J)) + \alpha_0(\mathfrak{N}(J)) = e(p^{f'} - 1)/(p-1)$, taking the difference between these formulas gives

$$2\alpha_0 = [d_0 - c_0] - \sum_{j=1}^{f'} p^{f'-j}\gamma_j^* + \sum_{j \in S_0^c} p^{f'-j}$$

where S_0^c consists of those $1 \le j \le f$ such that $(j-1, j)$ is a transition. Subtract $2[d_0]$ from both sides, and add the expression $-[k_0 - k_0'] + \sum_{j=1}^{f'} p^{f'-j}\gamma_j$ (which vanishes by definition) to the right-hand side. Note that $[d_0 - c_0] - [k_0 - k_0'] - 2[d_0]$ is equal to $-2[k_0]$ if $0 \notin J$, and to $e(K'/K) - 2[k_0 - k_0'] - 2[k_0']$ if $0 \in J$. Since $\gamma_j - \gamma_j^* = 2\gamma_j - (p-1)$ if $j - 1 \in J$ and is 0 otherwise, the preceding expression rearranges to give (after dividing by 2)

$$\alpha_0 - [d_0] = -\kappa_0 + \sum_{j-1 \in J} p^{f'-j}\gamma_j + \sum_{j-1 \in J, j \notin J} p^{f'-j} = -\kappa_0 + \sum_{j=1}^{f'} p^{f'-j} t_j$$

where $\kappa_0 = [k_0]$ if $0 \notin J$ and $\kappa_0 = [k_0 - k_0'] + [k_0']$ if $0 \in J$. Since in either case $\kappa_0 \equiv k_0 \pmod{e(K'/K)}$ the result now follows from Lemma 4.1.4. ☐

Definition 5.1.16. Let $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F}')$ be representation. Then we say that a Breuil–Kisin module $\mathfrak{M}$ with $\mathbf{F}'$-coefficients is a *Breuil–Kisin model of* $\overline{r}$ *of type* τ if $\mathfrak{M}$ is an $\mathbf{F}'$-point of $\mathcal{C}^{\tau,\mathrm{BT},1}$, and $T_{\mathbf{F}'}(\mathfrak{M}) \cong \overline{r}|_{G_{K_\infty}}$.

Recall that for each continuous representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, there is an associated (nonempty) set of Serre weights $W(\overline{r})$ whose precise definition is recalled in Appendix A.

Theorem 5.1.17. *The* $\overline{\mathcal{Z}}(J)$*, with* $J \in \mathcal{P}_\tau$*, are pairwise distinct closed substacks of* $\mathcal{Z}^{\tau,1}$*. For each* $J \in \mathcal{P}_\tau$*, there is a dense set of finite type points of* $\overline{\mathcal{Z}}(J)$ *with the property that the corresponding Galois representations have* $\overline{\sigma}(\tau)_J$ *as a Serre weight, and which furthermore admit a unique Breuil–Kisin model of type* τ*.*

Proof. Recall from Definition 3.3.11 that $\overline{\mathcal{Z}}(J)$ is defined to be the scheme-theoretic image of a morphism $\operatorname{Spec} B^{\mathrm{dist}} \to \mathcal{Z}^{\mathrm{dd},1}$. As in the proof of Lemma 3.3.13, since the source and target of this morphism are of finite presentation over $\mathbf{F}$, its image is a dense constructible subset of its scheme-theoretic image, and so contains a dense open subset, which we may interpret as a dense open substack $\mathcal{U}$ of $\overline{\mathcal{Z}}(J)$. From the definition of B^{dist}, the finite type points of $\mathcal{U}$ correspond to reducible Galois representations admitting a model of type τ and refined shape (J, r), for which (J, r) is maximal.

That the $\overline{\mathcal{Z}}(J)$ are pairwise distinct is immediate from the above and Proposition 5.1.14. Combining this observation with Theorem 4.2.11, we see that by deleting the intersections of $\overline{\mathcal{Z}}(J)$ with the $\overline{\mathcal{Z}}(J', r')$ for all refined shapes $(J', r') \neq (J, r)$, we obtain a dense open substack $\mathcal{U}'$ whose finite type points have the property that every Breuil–Kisin model of type τ of the corresponding Galois representation has shape (J, r). The unicity of such a Breuil–Kisin model then follows from Corollary 5.1.11.

It remains to show that every such Galois representation $\overline{r}$ has $\overline{\sigma}(\tau)_J$ as a Serre weight. Suppose first that τ is a principal series type. We claim that (writing

$\overline{\sigma}(\tau)_J = \overline{\sigma}_{\vec{t},\vec{s}} \otimes (\eta' \circ \det)$ as in Appendix A) we have

$$T(\mathfrak{N}(J))|_{I_K} = \eta'|_{I_K} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+t_i}.$$

To see this, note that by Proposition 5.1.14 it is enough to show that $\eta|_{I_K} = \eta'|_{I_K} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+2t_i}$, which follows by comparing the central characters of $\overline{\sigma}(\tau)_J$ and $\overline{\sigma}(\tau)$ (or from a direct computation with the quantities s_i, t_i).

Since $\det \overline{r}|_{I_K} = \eta\eta'\overline{\varepsilon}^{-1}$, we have

$$\overline{r}|_{I_K} \cong \eta'|_{I_K} \otimes \begin{pmatrix} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+t_i} & * \\ 0 & \overline{\varepsilon}^{-1} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{t_i} \end{pmatrix}.$$

The result then follows from Lemma A.6, using Lemma A.5(2) and the fact that the fibre of the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{R}^{\mathrm{dd},1}$ above $\overline{r}$ is nonempty to see that $\overline{r}$ is not très ramifiée.

The argument in the cuspidal case proceeds analogously, noting that if the character θ (as in Appendix A) corresponds to $\widetilde{\theta}$ under local class field theory then $\widetilde{\theta}|_{I_K} = \eta' \prod_{i=0}^{f'-1} \omega_{\sigma'_i}^{t_i}$, and that from central characters we have $\eta\eta' = (\widetilde{\theta}|_{I_K})^2 \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i}$. □

Remark 5.1.18. With more work, we could use the results of [GLS15] and our results on dimensions of families of extensions to strengthen Theorem 5.1.17, showing that there is a dense set of finite type points of $\overline{\mathcal{Z}}(J)$ with the property that the corresponding Galois representations have $\overline{\sigma}(\tau)_J$ as their *unique* non-Steinberg Serre weight. In fact, we will prove this as part of our work on the geometric Breuil–Mézard conjecture in [CEGS20a] (which uses Theorem 5.1.17 as an input).

5.2. **Irreducible Galois representations.** We now show that the points of $\mathcal{C}^{\tau,\mathrm{BT},1}$ which are irreducible (that is, cannot be written as an extension of rank one Breuil–Kisin modules) lie in a closed substack of positive codimension. We begin with the following useful observation.

Lemma 5.2.1. *The rank two Breuil–Kisin modules with descent data and $\overline{\mathbf{F}}_p$-coefficients which are irreducible (that is, which cannot be written as an extension of rank* 1 *Breuil–Kisin modules with descent data) are precisely those whose corresponding étale φ-modules are irreducible, or equivalently whose corresponding G_K-representations are irreducible.*

Proof. Let $\mathfrak{M}$ be a Breuil–Kisin module with descent data corresponding to a finite type point of $\mathcal{C}_{\mathrm{dd}}^{\tau,\mathrm{BT},1}$, let $M = \mathfrak{M}[1/u]$, and let $\overline{\rho}$ be the G_K-representation corresponding to M. As noted in the proof of Lemma 2.2.6, $\overline{\rho}$ is reducible if and only if $\overline{\rho}|_{G_{K_\infty}}$ is reducible, and by Lemma 2.2.4, this is equivalent to M being reducible. That this is in turn equivalent to $\mathfrak{M}$ being reducible may be proved in the same way as [GLS14, Lem. 5.5]. □

Recall that L/K denotes the unramified quadratic extension; then the irreducible representations $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ are all induced from characters of G_L. Bearing in mind Lemma 5.2.1, this means that we can study irreducible Breuil–Kisin modules via a consideration of base-change of Breuil–Kisin modules from K to L, and our previous study of reducible Breuil–Kisin modules. Since this will require us to consider Breuil–Kisin modules (and moduli stacks thereof) over both K and L, we will have to introduce additional notation in order to indicate over which of the two fields we might be working. We do this simply by adding a subscript 'K' or 'L' to our current notation. We will also omit other decorations which are being held fixed throughout the present discussion. Thus we write $\mathcal{C}_K^{\tau}$ to denote the moduli stack that was previously denoted $\mathcal{C}^{\tau,\mathrm{BT},1}$, and $\mathcal{C}_L^{\tau_{|L}}$ to denote the corresponding moduli stack for Breuil–Kisin modules over L, with the type taken to be the restriction $\tau_{|L}$ of τ from K to L. (Note that whether τ is principal series or cuspidal, the restriction $\tau_{|L}$ is principal series.)

As usual we fix a uniformiser π of K, which we also take to be our fixed uniformiser of L. Also, throughout this section we take $K' = L(\pi^{1/(p^{2f}-1)})$, so that K'/L is the standard choice of extension for τ and π regarded as a type and uniformiser for L.

If $\mathfrak{P}$ is a Breuil–Kisin module with descent data from K' to L, then we let $\mathfrak{P}^{(f)}$ be the Breuil–Kisin module $W(k')[[u]] \otimes_{\mathrm{Gal}(k'/k),W(k')[[u]]} \mathfrak{P}$, where the pullback is given by the non-trivial automorphism of k'/k, and the descent data on $\mathfrak{P}^{(f)}$ is given by $\hat{g}(s \otimes m) = \hat{g}(s) \otimes \hat{g}^{p^f}(m)$ for $s \in W(k')[[u]]$ and $m \in \mathfrak{P}$. In particular, we have $\mathfrak{M}(r,a,c)^{(f)} = \mathfrak{M}(r',a',c')$ where $r'_i = r_{i+f}$, $a'_i = a_{i+f}$, and $c'_i = c_{i+f}$.

We let σ denote the non-trivial automorphism of L over K, and write $G := \mathrm{Gal}(L/K) = \langle \sigma \rangle$, a cyclic group of order two. There is an action α of G on $\mathcal{C}_L$ defined via $\alpha_\sigma : \mathfrak{P} \mapsto \mathfrak{P}^{(f)}$. More precisely, this induces an action of $G := \langle \sigma \rangle$ on $\mathcal{C}_L^{\tau_{|L}}$ in the strict [2] sense that

$$\alpha_\sigma \circ \alpha_\sigma = \mathrm{id}_{\mathcal{C}_L^{\tau_{|L}}} .$$

We now define the fixed point stack for this action.

Definition 5.2.2. We let the fixed point stack $(\mathcal{C}_L^{\tau_{|L}})^G$ denote the stack whose A-valued points consist of an A-valued point $\mathfrak{M}$ of $\mathcal{C}_L^{\tau_{|L}}$, together with an isomorphism $\imath : \mathfrak{M} \xrightarrow{\sim} \mathfrak{M}^{(f)}$ which satisfies the cocycle condition that the composite

$$\mathfrak{M} \xrightarrow{\imath} \mathfrak{M}^{(f)} \xrightarrow{\imath^{(f)}} (\mathfrak{M}^{(f)})^{(f)} = \mathfrak{M}$$

is equal to the identity morphism $\mathrm{id}_{\mathfrak{M}}$.

We now give another description of $(\mathcal{C}_L^{\tau_{|L}})^G$, in terms of various fibre products, which is technically useful. This alternate description involves two steps. In the first step, we define fixed points of the automorphism α_σ, without imposing the additional condition that the fixed point data be compatible with the relation $\sigma^2 = 1$ in G. Namely, we define

$$(\mathcal{C}_L^{\tau_{|L}})^{\alpha_\sigma} := \mathcal{C}_L^{\tau_{|L}} \underset{\mathcal{C}_L^{\tau_{|L}} \times \mathcal{C}_L^{\tau_{|L}}}{\times} \mathcal{C}_L^{\tau_{|L}}$$

where the first morphism $\mathcal{C}_L^{\tau_{|L}} \to \mathcal{C}_L^{\tau_{|L}} \times \mathcal{C}_L^{\tau_{|L}}$ is the diagonal, and the second is $\mathrm{id} \times \alpha_\sigma$. Working through the definitions, one finds that an A-valued point of $(\mathcal{C}_L^{\tau_{|L}})^{\alpha_\sigma}$ consists of a pair $(\mathfrak{M}, \mathfrak{M}')$ of objects of $\mathcal{C}_L^{\tau_{|L}}$ over A, equipped with isomorphisms $\alpha : \mathfrak{M} \xrightarrow{\sim} \mathfrak{M}'$ and $\beta : \mathfrak{M} \xrightarrow{\sim} (\mathfrak{M}')^{(f)}$. The morphism

$$(\mathfrak{M}, \mathfrak{M}', \alpha, \beta) \mapsto (\mathfrak{M}, \imath),$$

2. From a 2-categorical perspective, it is natural to relax the notion of group action on a stack so as to allow natural transformations, rather than literal equalities, when relating multiplication in the group to the compositions of the corresponding equivalences of categories arising in the definition of an action. An action in which actual equalities hold is then called *strict*. Since our action is strict, we are spared from having to consider the various 2-categorical aspects of the situation that would otherwise arise.

where $\imath := (\alpha^{-1})^{(f)} \circ \beta : \mathfrak{M} \to \mathfrak{M}^{(f)}$, induces an isomorphism between $(\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma}$ and the stack classifying points $\mathfrak{M}$ of $\mathcal{C}_L^{\tau|_L}$ equipped with an isomorphism $\imath : \mathfrak{M} \to \mathfrak{M}^{(f)}$. (However, no cocycle condition has been imposed on $\imath$.)

Let $I_{\mathcal{C}_L^{\tau|_L}}$ denote the inertia stack of $\mathcal{C}_L^{\tau|_L}$. We define a morphism

$$(\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma} \to I_{\mathcal{C}_L^{\tau|_L}}$$

via

$$(\mathfrak{M}, \imath) \mapsto (\mathfrak{M}, \imath^{(f)} \circ \imath),$$

where, as in Definition 5.2.2, we regard the composite $\imath^{(f)} \circ \imath$ as an automorphism of $\mathfrak{M}$ via the identity $(\mathfrak{M}^{(f)})^{(f)} = \mathfrak{M}$. Of course, we also have the identity section $e : \mathcal{C}_L^{\tau|_L} \to I_{\mathcal{C}_L^{\tau|_L}}$. We now define

$$(\mathcal{C}_L^{\tau|_L})^G := (\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma} \underset{I_{\mathcal{C}_L^{\tau|_L}}}{\times} \mathcal{C}_L^{\tau|_L}.$$

If we use the description of $(\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma}$ as classifying pairs $(\mathfrak{M}, \imath)$, then (just unwinding definitions) this fibre product classifies tuples $(\mathfrak{M}, \imath, \mathfrak{M}', \alpha)$, where α is an isomorphism $\mathfrak{M} \xrightarrow{\sim} \mathfrak{M}'$ which furthermore identifies $\imath^{(f)} \circ \imath$ with $\mathrm{id}_{\mathfrak{M}'}$. Forgetting $\mathfrak{M}'$ and α then induces an isomorphism between $(\mathcal{C}_L^{\tau|_L})^G$, as defined via the above fibre product, and the stack defined in Definition 5.2.2.

To compare this fixed point stack to $\mathcal{C}_K^\tau$, we make the following observations. Given a Breuil–Kisin module with descent data from K' to K, we obtain a Breuil–Kisin module with descent data from K' to L via the obvious forgetful map. Conversely, given a Breuil–Kisin module $\mathfrak{P}$ with descent data from K' to L, the additional data required to enrich this to a Breuil–Kisin module with descent data from K' to K can be described as follows as follows: let $\theta \in \mathrm{Gal}(K'/K)$ denote the unique element which fixes $\pi^{1/(p^{2f}-1)}$ and acts nontrivially on L. Then to enrich the descent data on $\mathfrak{P}$ to descent data from K' to K, it is necessary and sufficient to give an additive map $\hat{\theta} : \mathfrak{P} \to \mathfrak{P}$ satisfying $\hat{\theta}(sm) = \theta(s)\hat{\theta}(m)$ for all $s \in \mathfrak{S}_{\mathbf{F}}$ and $m \in \mathfrak{P}$, and such that $\hat{\theta}\hat{g}\hat{\theta} = \hat{g}^{p^f}$ for all $g \in \mathrm{Gal}(K'/L)$.

In turn, the data of the additive map $\hat{\theta} : \mathfrak{P} \to \mathfrak{P}$ is equivalent to giving the data of the map $\theta(\hat{\theta}) : \mathfrak{P} \to \mathfrak{P}^{(f)}$ obtained by composing $\hat{\theta}$ with the Frobenius on L/K. The defining properties of $\hat{\theta}$ are equivalent to asking that this map is an isomorphism of Breuil–Kisin modules with descent data satisfying the cocycle condition of Definition 5.2.2; accordingly, we have a natural morphism $\mathcal{C}_K^\tau \to (\mathcal{C}_L^{\tau|_L})^G$, and a restriction morphism

$$\mathcal{C}_K^\tau \to \mathcal{C}_L^{\tau|_L}. \tag{5.2.3}$$

The following simple lemma summarises the basic facts about base-change in the situation we are considering.

Lemma 5.2.4. *There is an isomorphism $\mathcal{C}_K^\tau \xrightarrow{\sim} (\mathcal{C}_L^{\tau|_L})^G$.*

Proof. This follows immediately from the preceding discussion. □

Remark 5.2.5. In the proof of Theorem 5.2.9 we will make use of the following analogue of Lemma 5.2.4 for étale φ-modules. Write $\mathcal{R}_K$, $\mathcal{R}_L$ for the moduli stacks of Definition 2.3.7, i.e. for the moduli stacks of rank 2 étale φ-modules with descent data respectively to K or to L. Then we have an action of G on $\mathcal{R}_L$

defined via $M \mapsto M^{(f)} := W(k') \otimes_{\mathrm{Gal}(k'/k),W(k')} M$, and we define the fixed point stack $(\mathcal{R}_L)^G$ exactly as in Definition 5.2.2: namely an A-valued point of $(\mathcal{R}_L)^G$ consists of an A-valued point M of $\mathcal{R}_L$, together with an isomorphism $\iota : M \xrightarrow{\sim} M^{(f)}$ satisfying the cocycle condition. The preceding discussion goes through in this setting, and shows that there is an isomorphism $\mathcal{R}_K \xrightarrow{\sim} (\mathcal{R}_L)^G$.

We also note that the morphisms $\mathcal{C}_K^{\tau} \to \mathcal{C}_L^{\tau_{|L}}$ and $\mathcal{C}_K^{\tau} \to \mathcal{R}_K$ induce a monomorphism

$$\mathcal{C}_K^{\tau} \hookrightarrow \mathcal{C}_L^{\tau_{|L}} \times_{\mathcal{R}_L} \mathcal{R}_K \tag{5.2.6}$$

One way to see this is to rewrite this morphism (using the previous discussion) as a morphism

$$(\mathcal{C}_L^{\tau_{|L}})^G \to \mathcal{C}_L^{\tau_{|L}} \times_{\mathcal{R}_L} (\mathcal{R}_L)^G,$$

and note that the descent data via G on an object classified by the source of this morphism is determined by the induced descent data on its image in $(\mathcal{R}_L)^G$.

We now use the Lemma 5.2.4 to study the locus of finite type points of $\mathcal{C}_K^{\tau}$ which correspond to irreducible Breuil–Kisin modules. Any irreducible Breuil–Kisin module over K becomes reducible when restricted to L, and so may be described as an extension

$$0 \to \mathfrak{N} \to \mathfrak{P} \to \mathfrak{M} \to 0,$$

where $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules of rank one with descent data from K' to L, and $\mathfrak{P}$ is additionally equipped with an isomorphism $\mathfrak{P} \cong \mathfrak{P}^{(f)}$, satisfying the cocycle condition of Definition 5.2.2.

Note that the characters $T(\mathfrak{M})$, $T(\mathfrak{N})$ of G_{L_∞} are distinct and cannot be extended to characters of G_K. Indeed, this condition is plainly necessary for an extension $\mathfrak{P}$ to arise as the base change of an irreducible Breuil–Kisin module (see the proof of Lemma 2.2.6). Conversely, if $T(\mathfrak{M})$, $T(\mathfrak{N})$ of G_{L_∞} are distinct and cannot be extended to characters of G_K, then for any $\mathfrak{P} \in \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ whose descent data can be enriched to give descent data from K' to K, this enrichment is necessarily irreducible. In particular, the existence of such a $\mathfrak{P}$ implies that the descent data on $\mathfrak{M}$ and $\mathfrak{N}$ cannot be enriched to give descent data from K' to K.

We additionally have the following observation.

Lemma 5.2.7. *If $\mathfrak{M}, \mathfrak{N}$ are such that there is an extension*

$$0 \to \mathfrak{N} \to \mathfrak{P} \to \mathfrak{M} \to 0$$

whose descent data can be enriched to give an irreducible Breuil–Kisin module over K, then there exists a nonzero map $\mathfrak{N} \to \mathfrak{M}^{(f)}$.

Proof. The composition $\mathfrak{N} \to \mathfrak{P} \xrightarrow{\hat{\theta}} \mathfrak{P} \to \mathfrak{M}$, in which first and last arrows are the natural inclusions and projections, must be nonzero (or else $\hat{\theta}$ would give descent data on $\mathfrak{N}$ from K' to K). It is not itself a map of Breuil–Kisin modules, because $\hat{\theta}$ is semilinear, but is a map of Breuil–Kisin modules when viewed as a map $\mathfrak{N} \to \mathfrak{M}^{(f)}$. □

We now consider (for our fixed $\mathfrak{M}$, $\mathfrak{N}$, and working over L rather than over K) the scheme $\operatorname{Spec} B^{\mathrm{dist}}$ as in Subsection 3.3. Following Lemma 5.2.7, we assume that there exists a nonzero map $\mathfrak{N} \to \mathfrak{M}^{(f)}$. The observations made above show that we are in the strict case, and thus that $\operatorname{Spec} A^{\mathrm{dist}} = \mathbf{G}_m \times \mathbf{G}_m$ and that furthermore we may (and do) set $V = T$. We consider the fibre product with the restriction morphism (5.2.3)

$$Y(\mathfrak{M},\mathfrak{N}) := \operatorname{Spec} B^{\mathrm{dist}} \times_{\mathcal{C}_L^{\tau|_L}} \mathcal{C}_K^{\tau}.$$

Let $\mathbf{G}_m \hookrightarrow \mathbf{G}_m \times \mathbf{G}_m$ be the diagonal closed immersion, and let $(\operatorname{Spec} B^{\mathrm{dist}})_{|\mathbf{G}_m}$ denote the pull-back of $\operatorname{Spec} B^{\mathrm{dist}}$ along this closed immersion. By Lemma 5.2.7, the projection $Y(\mathfrak{M},\mathfrak{N}) \to \operatorname{Spec} B^{\mathrm{dist}}$ factors through $(\operatorname{Spec} B^{\mathrm{dist}})_{|\mathbf{G}_m}$, and combining this with Lemma 5.2.4 we see that $Y(\mathfrak{M},\mathfrak{N})$ may also be described as the fibre product

$$(\operatorname{Spec} B^{\mathrm{dist}})_{|\mathbf{G}_m} \times_{\mathcal{C}_L^{\tau|_L}} (\mathcal{C}_L^{\tau|_L})^G.$$

Recalling the warning of Remark 3.3.16, Proposition 3.3.21 now shows that there is a monomorphism

$$[(\operatorname{Spec} B^{\mathrm{dist}})_{|\mathbf{G}_m}/\mathbf{G}_m \times \mathbf{G}_m] \hookrightarrow \mathcal{C}_L^{\tau|_L},$$

and thus, by Lemma 3.2.8, that there is an isomorphism

$$(\operatorname{Spec} B^{\mathrm{dist}})_{|\mathbf{G}_m} \times_{\mathcal{C}_L^{\tau|_L}} (\operatorname{Spec} B^{\mathrm{dist}})_{|\mathbf{G}_m} \xrightarrow{\sim} (\operatorname{Spec} B^{\mathrm{dist}})_{|\mathbf{G}_m} \times \mathbf{G}_m \times \mathbf{G}_m.$$

(An inspection of the proof of Proposition 3.3.21 shows that in fact this result is more-or-less proved directly, as the key step in proving the proposition.) An elementary manipulation with fibre products then shows that there is an isomorphism

$$Y(\mathfrak{M},\mathfrak{N}) \times_{(\mathcal{C}_L^{\tau|_L})^G} Y(\mathfrak{M},\mathfrak{N}) \xrightarrow{\sim} Y(\mathfrak{M},\mathfrak{N}) \times \mathbf{G}_m \times \mathbf{G}_m,$$

and thus, by another application of Lemma 3.2.8, we find that there is a monomorphism

$$[Y(\mathfrak{M},\mathfrak{N})/\mathbf{G}_m \times \mathbf{G}_m] \hookrightarrow (\mathcal{C}_L^{\tau|_L})^G. \tag{5.2.8}$$

We define $\mathcal{C}_{\mathrm{irred}}$ to be the union over all such pairs $(\mathfrak{M},\mathfrak{N})$ of the scheme-theoretic images of the various projections $Y(\mathfrak{M},\mathfrak{N}) \to (\mathcal{C}_L^{\tau|_L})^G$. Note that this image depends on $(\mathfrak{M},\mathfrak{N})$ up to simultaneous unramified twists of $\mathfrak{M}$ and $\mathfrak{N}$, and there are only finitely many such pairs $(\mathfrak{M},\mathfrak{N})$ up to such unramified twist. By definition, $\mathcal{C}_{\mathrm{irred}}$ is a closed substack of $\mathcal{C}_K^{\tau}$ which contains every finite type point of $\mathcal{C}_K^{\tau}$ corresponding to an irreducible Breuil–Kisin module.

The following is the main result of this section.

Theorem 5.2.9. *The closed substack $\mathcal{C}_{\mathrm{irred}}$ of $\mathcal{C}_K^{\tau} = \mathcal{C}^{\tau,\mathrm{BT},1}$, which contains every finite type point of $\mathcal{C}_K^{\tau}$ corresponding to an irreducible Breuil–Kisin module, has dimension strictly less than $[K : \mathbf{Q}_p]$.*

Proof. As noted above, there are only finitely many pairs $(\mathfrak{M},\mathfrak{N})$ up to unramified twist, so it is enough to show that for each of them, the scheme-theoretic image of the monomorphism (5.2.8) has dimension less than $[K : \mathbf{Q}_p]$.

By [Sta13, Tag 0DS6], to prove the present theorem, it then suffices to show that $\dim Y(\mathfrak{M},\mathfrak{N}) \le [K : \mathbf{Q}_p] + 1$ (since $\dim \mathbf{G}_m \times \mathbf{G}_m = 2$). To establish this, it suffices to show, for each point $x \in \mathbf{G}_m(\mathbf{F}')$, where $\mathbf{F}'$ is a finite extension of $\mathbf{F}$, that the dimension of the fibre $Y(\mathfrak{M},\mathfrak{N})_x$ is bounded by $[K : \mathbf{Q}_p]$. After relabelling, as we may, the field $\mathbf{F}'$ as $\mathbf{F}$ and the Breuil–Kisin modules $\mathfrak{M}_x$ and $\mathfrak{N}_x$ as $\mathfrak{M}$ and $\mathfrak{N}$, we may suppose that in fact $\mathbf{F}' = \mathbf{F}$ and $x = 1$.

Manipulating the fibre product appearing in the definition of $Y(\mathfrak{M},\mathfrak{N})$, we find that

$$Y(\mathfrak{M},\mathfrak{N})_1 = \operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \times_{\mathcal{C}_L^{\tau|_N}} \mathcal{C}_K^{\tau}, \tag{5.2.10}$$

where the fibre product is taken with respect to the morphism $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \to \mathcal{C}^\tau_L$ that associates the corresponding rank two extension to an extension of rank one Breuil–Kisin modules, and the restriction morphism (5.2.3).

In order to bound the dimension of $Y(\mathfrak{M},\mathfrak{N})_1$, it will be easier to first embed it into another, larger, fibre product, which we now introduce. Namely, the monomorphism (5.2.6) induces a monomorphism

$$Y(\mathfrak{M},\mathfrak{N})_1 \hookrightarrow Y'(\mathfrak{M},\mathfrak{N})_1 := \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \times_{\mathcal{R}_L} \mathcal{R}_K.$$

Any finite type point of this fibre product lies over a fixed isomorphism class of finite type points of $\mathcal{R}_K$ (corresponding to some fixed irreducible Galois representation); we let P be a choice of such a point. The restriction of P then lies in a fixed isomorphism class of finite type points of $\mathcal{R}_L$ (namely, the isomorphism class of the direct sum $\mathfrak{M}[1/u]\oplus\mathfrak{N}[1/u] \cong \mathfrak{M}[1/u]\oplus\mathfrak{M}^{(f)}[1/u]$). Thus the projection $Y'(\mathfrak{M},\mathfrak{N})_1 \to \mathcal{R}_K$ factors through the residual gerbe of P, while the morphism $Y'(\mathfrak{M},\mathfrak{N})_1 \to \mathcal{R}_L$ factors through the residual gerbe of $\mathfrak{M}[1/u]\oplus\mathfrak{N}[1/u] \cong \mathfrak{M}[1/u]\oplus\mathfrak{M}^{(f)}[1/u]$. Since P corresponds via Lemma 2.2.4 to an irreducible Galois representation, we find that $\mathrm{Aut}(P) = \mathbf{G}_m$. Since $\mathfrak{M}[1/u]\oplus\mathfrak{N}[1/u]$ corresponds via Lemma 2.2.4 to the direct sum of two non-isomorphic Galois characters, we find that $\mathrm{Aut}(\mathfrak{M}[1/u]\oplus\mathfrak{N}[1/u]) = \mathbf{G}_m \times \mathbf{G}_m$.

Thus we obtain monomorphisms

$$\begin{aligned} (5.2.11)\quad & Y(\mathfrak{M},\mathfrak{N})_1 \hookrightarrow Y'(\mathfrak{M},\mathfrak{N})_1 \\ & \hookrightarrow \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \times_{[\mathrm{Spec}\, F'//\mathbf{G}_m\times\mathbf{G}_m]} [\mathrm{Spec}\, F'//\mathbf{G}_m] \cong \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \times \mathbf{G}_m. \end{aligned}$$

In Proposition 5.2.12 we obtain a description of the image of $Y(\mathfrak{M},\mathfrak{N})_1$ under this monomorphism which allows us to bound its dimension by $[K:\mathbf{Q}_p]$, as required. $\square$

We now prove the bound on the dimension of $Y(\mathfrak{M},\mathfrak{N})_1$ that we used in the proof of Theorem 5.2.9. Before establishing this bound, we make some further remarks. To begin with, we remind the reader that we are working with Breuil–Kisin modules, étale φ-modules, etc., over L rather than K, so that e.g. the structure parameters of $\mathfrak{M},\mathfrak{N}$ are periodic modulo $f' = 2f$ (not modulo f), and the pair $(\mathfrak{M},\mathfrak{N})$ has type $\tau|_L$. We will readily apply various pieces of notation that were introduced above in the context of the field K, adapted in the obvious manner to the context of the field L. (This applies in particular to the notation $\mathfrak{C}^1_u$, $\mathfrak{C}^0_u$, etc. introduced in Definition 4.1.7.)

We write m, n for the standard generators of $\mathfrak{M}$ and $\mathfrak{N}$. The existence of the nonzero map $\mathfrak{N} \to \mathfrak{M}^{(f)}$ implies that $\alpha_i(\mathfrak{N}) \geq \alpha_{i+f}(\mathfrak{M})$ for all i, and also that $\prod_i a_i = \prod_i b_i$. Thanks to the latter we will lose no generality by assuming that $a_i = b_i = 1$ for all i. Let $\tilde{m}$ be the standard generator for $\mathfrak{M}^{(f)}$. The map $\mathfrak{N} \to \mathfrak{M}^{(f)}$ will (up to a scalar) have the form $n_i \mapsto u^{x_i}\tilde{m}_i$ for integers x_i satisfying $px_{i-1} - x_i = s_i - r_{i+f}$ for all i; thus $x_i = \alpha_i(\mathfrak{N}) - \alpha_{i+f}(\mathfrak{M})$ for all i. Since the characters $T(\mathfrak{M})$ and $T(\mathfrak{N})$ are conjugate we must have $x_i \equiv d_i - c_{i+f} \pmod{p^{f'}-1}$ for all i (*cf.* Lemma 4.1.4). Moreover, the strong determinant condition $s_i + r_i = e'$ for all i implies that $x_i = x_{i+f}$.

We stress that we make no claims about the optimality of the following result; we merely prove “just what we need” for our applications. Indeed the estimates of [Hel09, Car17] suggest that improvement should be possible.

Proposition 5.2.12. *We have* $\dim Y(\mathfrak{M},\mathfrak{N})_1 \leq [K:\mathbf{Q}_p]$.

Remark 5.2.13. Since the image of $Y(\mathfrak{M},\mathfrak{N})_1$ in $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ lies in $\ker\text{-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ with fibres that can be seen to have dimension at most one, many cases of Proposition 5.2.12 will already follow from Remark 5.1.5 (applied with L in place of K).

Proof of Proposition 5.2.12. Let $\mathfrak{P}=\mathfrak{P}(h)$ be an element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ whose descent data can be enriched to give descent data from K' to K, and let $\widetilde{\mathfrak{P}}$ be such an enrichment. By Lemma 5.2.7 (and the discussion preceding that lemma) the étale φ-module $\mathfrak{P}[\frac{1}{u}]$ is isomorphic to $\mathfrak{M}[\frac{1}{u}]\oplus\mathfrak{M}^{(f)}[\frac{1}{u}]$. All extensions of the G_{L_∞}-representation $T(\mathfrak{M}[\frac{1}{u}]\oplus\mathfrak{M}^{(f)}[\frac{1}{u}])$ to a representation of G_{K_∞} are isomorphic (and given by the induction of $T(\mathfrak{M}[\frac{1}{u}])$ to G_{K_∞}), so the same is true of the étale φ-modules with descent data from K' to K that enrich the descent data on $\mathfrak{M}[\frac{1}{u}]\oplus\mathfrak{M}^{(f)}[\frac{1}{u}]$. One such enrichment, which we denote P, has $\hat{\theta}$ that interchanges m and $\tilde{m}$. Thus $\widetilde{\mathfrak{P}}[\frac{1}{u}]$ is isomorphic to P.

As in the proof of Lemma 5.2.7, the hypothesis that $T(\mathfrak{M})\not\cong T(\mathfrak{N})$ implies that any non-zero map (equivalently, isomorphism) of étale φ-modules with descent data $\lambda:\widetilde{\mathfrak{P}}[\frac{1}{u}]\to P$ takes the submodule $\mathfrak{N}[\frac{1}{u}]$ to $\mathfrak{M}^{(f)}[\frac{1}{u}]$. We may scale the map λ so that it restricts to the map $n_i\to u^{x_i}\tilde{m}_i$ on $\mathfrak{N}$. Then there is an element $\xi\in\mathbf{F}^\times$ so that λ induces multiplication by ξ on the common quotients $\mathfrak{M}[\frac{1}{u}]$. That is, the map λ may be assumed to have the form

$$\begin{pmatrix} n_i \\ m_i \end{pmatrix}\mapsto\begin{pmatrix} u^{x_i} & 0 \\ \nu_i & \xi \end{pmatrix}\begin{pmatrix} \tilde{m}_i \\ m_i \end{pmatrix} \tag{5.2.14}$$

for some $(\nu_i)\in\mathbf{F}((u))^{f'}$. The condition that the map λ commutes with the descent data from K' to L is seen to be equivalent to the condition that nonzero terms in ν_i have degree congruent to $c_i-d_i+x_i \pmod{p^{f'}-1}$; or equivalently, if we define $\mu_i:=\nu_i u^{-x_i}$ for all i, that the tuple $\mu=(\mu_i)$ is an element of the set $\mathcal{C}^0_u=\mathcal{C}^0_u(\mathfrak{M},\mathfrak{N})$ of Definition 4.1.7.

The condition that λ commutes with φ can be checked to give

$$\varphi\begin{pmatrix} n_{i-1} \\ m_{i-1} \end{pmatrix}=\begin{pmatrix} u^{s_i} & 0 \\ \varphi(\nu_{i-1})u^{r_{i+f}-x_i}-\nu_i u^{r_i-x_i} & u^{r_i} \end{pmatrix}\begin{pmatrix} n_i \\ m_i \end{pmatrix}.$$

The extension $\mathfrak{P}$ is of the form $\mathfrak{P}(h)$, for some $h\in\mathcal{C}^1$ as in Definition 4.1.7. The lower-left entry of the first matrix on the right-hand side of the above equation must then be h_i. Since $r_{i+f}-x_i=s_i-px_{i-1}$, the resulting condition can be rewritten as

$$h_i=\varphi(\mu_{i-1})u^{s_i}-\mu_i u^{r_i},$$

or equivalently that $h=\partial(\mu)$. Comparing with Remark 5.1.3, we recover the fact that the extension class of $\mathfrak{P}$ is an element of $\ker\text{-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$, and the tuple μ determines an element of the space $\mathcal{H}$ defined as follows.

Definition 5.2.15. The map $\partial\colon\mathcal{C}^0_u\to\mathcal{C}^1_u$ induces a map $\mathcal{C}^0_u/\mathcal{C}^0\to\mathcal{C}^1_u/\partial(\mathcal{C}^0)$, which we also denote ∂. We let $\mathcal{H}\subset\mathcal{C}^0_u/\mathcal{C}^0$ denote the subspace consisting of elements μ such that $\partial(\mu)\in\mathcal{C}^1/\partial(\mathcal{C}^0)$.

By the discussion following Lemma 4.1.8, an element $\mu\in\mathcal{H}$ determines an extension $\mathfrak{P}(\partial(\mu))$. Indeed, Remark 5.1.3 and the proof of (3.1.31) taken together show that there is a natural isomorphism, in the style of Lemma 4.1.8, between the morphism $\partial:\mathcal{H}\to\mathcal{C}^1/\partial(\mathcal{C}^0)$ and the connection map $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})\to\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$, with $\operatorname{im}\partial$ corresponding to $\ker\text{-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$.

Conversely, let h be an element of $\partial(\mathcal{C}_u^0) \cap \mathcal{C}^1$, and set $\nu_i = u^{x_i}\mu_i$. The condition that there is a Breuil–Kisin module $\widetilde{\mathfrak{P}}$ with descent data from K' to K and $\xi \in \mathbf{F}^\times$ such that $\lambda : \widetilde{\mathfrak{P}}[\frac{1}{u}] \to P$ defined as above is an isomorphism is precisely the condition that the map $\hat{\theta}$ on P pulls back via λ to a map that preserves $\mathfrak{P}$. One computes that this pullback is

$$\hat{\theta}\begin{pmatrix} n_i \\ m_i \end{pmatrix} = \xi^{-1}\begin{pmatrix} -\nu_{i+f} & u^{x_i} \\ (\xi^2 - \nu_i\nu_{i+f})u^{-x_i} & \nu_i \end{pmatrix}\begin{pmatrix} n_{i+f} \\ m_{i+f} \end{pmatrix}$$

recalling that $x_i = x_{i+f}$.

We deduce that $\hat{\theta}$ preserves $\mathfrak{P}$ precisely when the ν_i are integral and $\nu_i\nu_{i+f} \equiv \xi^2 \pmod{u^{x_i}}$ for all i. For i with $x_i = 0$ the latter condition is automatic given the former, which is equivalent to the condition that μ_i and μ_{i+f} are both integral. If instead $x_i > 0$, then we have the nontrivial condition $\nu_{i+f} \equiv \xi^2\nu_i^{-1} \pmod{u^{x_i}}$; in other words that μ_i, μ_{i+f} have u-adic valuation exactly $-x_i$, and their principal parts determine one another via the equation $\mu_{i+f} \equiv \xi^2(u^{2x_i}\mu_i)^{-1} \pmod 1$.

Let $\mathbf{G}_{m,\xi}$ be the multiplicative group with parameter ξ. We now (using the notation of Definition 5.2.15) define $\mathcal{H}' \subset \mathcal{C}_u^0/\mathcal{C}^0 \times \mathbf{G}_{m,\xi}$ to be the subvariety consisting of the pairs (μ, ξ) with exactly the preceding properties; that is, we regard $\mathcal{C}_u^0/\mathcal{C}^0$ as an Ind-affine space in the obvious way, and define $\mathcal{H}'$ to be the pairs (μ, ξ) satisfying

— if $x_i = 0$ then $\mathrm{val}_i\,\mu = \mathrm{val}_{i+f}\,\mu = \infty$, and

— if $x_i > 0$ then $\mathrm{val}_i\,\mu = \mathrm{val}_{i+f}\,\mu = -x_i$ and $\mu_{i+f} \equiv \xi^2(u^{2x_i}\mu_i)^{-1} \pmod{u^0}$

where we write $\mathrm{val}_i\,\mu$ for the u-adic valuation of μ_i, putting $\mathrm{val}_i\,\mu = \infty$ when μ_i is integral.

Putting all this together with (5.2.10), we find that the map

$$\mathcal{H}' \cap (\mathcal{H} \times \mathbf{G}_{m,\xi}) \to Y(\mathfrak{M}, \mathfrak{N})_1$$

sending (μ, ξ) to the pair $(\mathfrak{P}, \widetilde{\mathfrak{P}})$ is a well-defined surjection, where $\mathfrak{P} = \mathfrak{P}(\partial(\mu))$, $\widetilde{\mathfrak{P}}$ is the enrichment of $\mathfrak{P}$ to a Breuil–Kisin module with descent data from K' to K in which $\hat{\theta}$ is pulled back to $\mathfrak{P}$ from P via the map λ as in (5.2.14). (Note that $Y(\mathfrak{M}, \mathfrak{N})_1$ is reduced and of finite type, for example by (5.2.11), so the surjectivity can be checked on $\overline{\mathbf{F}}_p$-points.) In particular $\dim Y(\mathfrak{M}, \mathfrak{N})_1 \le \dim \mathcal{H}'$.

Note that $\mathcal{H}'$ will be empty if for some i we have $x_i > 0$ but $x_i + c_i - d_i \not\equiv 0 \pmod{p^{f'} - 1}$ (so that ν_i cannot be a u-adic unit). Otherwise, the dimension of $\mathcal{H}'$ is easily computed to be $D = 1 + \sum_{i=0}^{f-1} \lceil x_i/(p^{f'} - 1)\rceil$ (indeed if d is the number of nonzero x_i's, then $\mathcal{H}' \cong \mathbf{G}_m^{d+1} \times \mathbf{G}_a^{D-d-1}$), and since $x_i \le e'/(p-1)$ we find that $\mathcal{H}'$ has dimension at most $1 + \lceil e/(p-1)\rceil f$. This establishes the bound $\dim Y(\mathfrak{M}, \mathfrak{N})_1 \le 1 + \lceil e/(p-1)\rceil f$.

Since $p > 2$ this bound already establishes the theorem when $e > 1$. If instead $e = 1$ the above bound gives $\dim Y(\mathfrak{M}, \mathfrak{N}) \le [K : \mathbf{Q}_p] + 1$. Suppose for the sake of contradiction that equality holds. This is only possible if $\mathcal{H}' \cong \mathbf{G}_m^{f+1}$, $\mathcal{H}' \subset \mathcal{H} \times \mathbf{G}_{m,\xi}$, and $x_i = [d_i - c_i] > 0$ for all i. Define $\mu^{(i)} \in \mathcal{C}_u^0$ to be the element such that $\mu_i = u^{-[d_i - c_i]}$, and $\mu_j = 0$ for $j \ne i$. Let $\mathbf{F}''/\mathbf{F}$ be any finite extension such that $\#\mathbf{F}'' > 3$. For each nonzero $z \in \mathbf{F}''$ define $\mu_z = \sum_{j \ne i, i+f} \mu^{(i)} + z\mu^{(i)} + z^{-1}\mu^{(i+f)}$, so that $(\mu_z, 1)$ is an element of $\mathcal{H}'(\mathbf{F}'')$. Since $\mathcal{H}' \subset \mathcal{H} \times \mathbf{G}_{m,\xi}$ and $\mathcal{H}$ is linear, the differences between the μ_z for varying z lie in $\mathcal{H}(\mathbf{F}'')$, and (e.g. by considering

$\mu_1 - \mu_{-1}$ and $\mu_1 - \mu_z$ for any $z \in \mathbf{F}''$ with $z \neq z^{-1}$) we deduce that each $\mu^{(i)}$ lies in $\mathcal{H}$. In particular each $\partial(\mu^{(i)})$ lies in $\mathcal{C}^1$.

If $(i-1,i)$ were not a transition then (since $e=1$) we would have either $r_i = 0$ or $s_i = 0$. The former would contradict $\partial(\mu^{(i)}) \in \mathcal{C}^1$ (since the ith component of $\partial(\mu^{(i)})$ would be $u^{-[d_i - c_i]}$, of negative degree), and similarly the latter would contradict $\partial(\mu^{(i-1)}) \in \mathcal{C}^1$. Thus $(i-1,i)$ is a transition for all i. In fact the same observations show more precisely that $r_i \geq x_i = [d_i - c_i]$ and $s_i \geq px_{i-1} = p[d_{i-1} - c_{i-1}]$. Summing these inequalities and subtracting e' we obtain $0 \geq p[d_{i-1} - c_{i-1}] - [c_i - d_i]$, and comparing with (5.1.6) shows that we must also have $\gamma_i^* = 0$ for all i. Since $e = 1$ and $(i-1,i)$ is a transition for all i the refined shape of the pair $(\mathfrak{M}, \mathfrak{N})$ is automatically maximal; but then we are in the exceptional case of Proposition 5.1.8, which (recalling the proof of that Proposition) implies that $T(\mathfrak{M}) \cong T(\mathfrak{N})$. This is the desired contradiction. ☐

5.3. **Irreducible components.** We can now use our results on families of extensions of characters to classify the irreducible components of the stacks $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$. In the article [CEGS20a] we will combine these results with results coming from Taylor–Wiles patching (in particular the results of [GK14, EG14]) to describe the closed points of each irreducible component of $\mathcal{Z}^{\tau,1}$ in terms of the weight part of Serre's conjecture.

Corollary 5.3.1. *Each irreducible component of $\mathcal{C}^{\tau,\mathrm{BT},1}$ is of the form $\overline{\mathcal{C}}(J)$ for some J; conversely, each $\overline{\mathcal{C}}(J)$ is an irreducible component of $\mathcal{C}^{\tau,\mathrm{BT},1}$.*

Remark 5.3.2. Note that at this point we have not established that different sets J give distinct irreducible components $\overline{\mathcal{C}}(J)$; we will prove this in Section 5.4 below by a consideration of Dieudonné modules.

Proof of Corollary 5.3.1. By Theorem 2.3.6(2), $\mathcal{C}^{\tau,\mathrm{BT},1}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$. By construction, the $\overline{\mathcal{C}}(J)$ are irreducible substacks of $\mathcal{C}^{\tau,\mathrm{BT},1}$, and by Theorem 4.2.11 they also have dimension $[K : \mathbf{Q}_p]$, so they are in fact irreducible components by [Sta13, Tag 0DS2].

By Theorem 5.2.9 and Theorem 4.2.11, we see that there is a closed substack $\mathcal{C}_{\mathrm{small}}$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$ of dimension strictly less than $[K : \mathbf{Q}_p]$, with the property that every finite type point of $\mathcal{C}^{\tau,\mathrm{BT},1}$ is a point of at least one of the $\overline{\mathcal{C}}(J)$ or of $\mathcal{C}_{\mathrm{small}}$ (or both). Indeed, every extension of refined shape (J,r) lies on $\overline{\mathcal{C}}(J,r)$, by Remark 4.2.10, so we can take $\mathcal{C}_{\mathrm{small}}$ to be the union of the stack $\mathcal{C}_{\mathrm{irred}}$ of Theorem 5.2.9 and the stacks $\overline{\mathcal{C}}(J,r)$ for non-maximal shapes (J,r). Since $\mathcal{C}^{\tau,\mathrm{BT},1}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$, it follows that the $\overline{\mathcal{C}}(J)$ exhaust the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$, as required. ☐

We now deduce a classification of the irreducible components of $\mathcal{Z}^{\tau,1}$. In the paper [CEGS20a] we will give a considerable refinement of this, giving a precise description of the finite type points of the irreducible components in terms of the weight part of Serre's conjecture.

Corollary 5.3.3. *The irreducible components of $\mathcal{Z}^{\tau,1}$ are precisely the $\overline{\mathcal{Z}}(J)$ for $J \in \mathcal{P}_\tau$, and if $J \neq J'$ then $\overline{\mathcal{Z}}(J) \neq \overline{\mathcal{Z}}(J')$.*

Proof. By Theorem 5.1.12, if $J \in \mathcal{P}_\tau$ then $\overline{\mathcal{Z}}(J)$ is an irreducible component of $\mathcal{Z}^{\tau,1}$. Furthermore, these $\overline{\mathcal{Z}}(J)$ are pairwise distinct by Theorem 5.1.17.

Since the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$ is scheme-theoretically dominant, it follows from Corollary 5.3.1 that each irreducible component of $\mathcal{Z}^{\tau,1}$ is dominated by some $\overline{\mathcal{C}}(J)$. Applying Theorem 5.1.12 again, we see that if $J \notin \mathcal{P}_\tau$ then $\overline{\mathcal{C}}(J)$ does not dominate an irreducible component, as required. $\square$

5.4. **Dieudonné modules and the morphism to the gauge stack.** We now study the images of the irreducible components $\overline{\mathcal{C}}(J)$ in the gauge stack $\mathcal{G}_\eta$; this amounts to computing the Dieudonné modules and Galois representations associated to the extensions of Breuil–Kisin modules that we considered in Section 3. Suppose throughout this subsection that τ is a non-scalar type, and that (J, r) is a maximal refined shape. Recall that in the cuspidal case this entails that $i \in J$ if and only if $i + f \notin J$.

Lemma 5.4.1. *Let $\mathfrak{P} \in \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ be an extension of type τ and refined shape (J, r). Then for $i \in \mathbf{Z}/f'\mathbf{Z}$ we have $F = 0$ on $D(\mathfrak{P})_{\eta,i-1}$ if $i \in J$, while $V = 0$ on $D(\mathfrak{P})_{\eta,i}$ if $i \notin J$.*

Proof. Recall that $D(\mathfrak{P}) = \mathfrak{P}/u\mathfrak{P}$. Let w_i be the image of m_i in $D(\mathfrak{P})$ if $i \in J$, and let w_i be the image of n_i in $D(\mathfrak{P})$ if $i \notin J$. It follows easily from the definitions that $D(\mathfrak{P})_{\eta,i}$ is generated over $\mathbf{F}$ by w_i.

Recall that the actions of F, V on $D(\mathfrak{P})$ are as specified in Definition 2.1.7. In particular F is induced by φ, while V is $c^{-1}\mathfrak{V}$ mod u where $\mathfrak{V}$ is the unique map on $\mathfrak{P}$ satisfying $\mathfrak{V} \circ \varphi = E(u)$, and $c = E(0)$. For the Breuil–Kisin module $\mathfrak{P}$, we have

$$\varphi(n_{i-1}) = b_i u^{s_i} n_i, \qquad \varphi(m_{i-1}) = a_i u^{r_i} m_i + h_i n_i,$$

and so one checks (using that $E(u) = u^{e'}$ in $\mathbf{F}$) that

$$\mathfrak{V}(m_i) = a_i^{-1} u^{s_i} m_{i-1} - a_i^{-1} b_i^{-1} h_i n_{i-1}, \qquad \mathfrak{V}(n_i) = b_i^{-1} u^{r_i} n_{i-1}.$$

From Definition 4.2.4 and the discussion immediately following it, we recall that if $(i-1, i)$ is not a transition then $r_i = e'$, $s_i = 0$, and h_i is divisible by u (the latter because nonzero terms of h_i have degrees congruent to $r_i + c_i - d_i \pmod{p^{f'} - 1}$, and $c_i \not\equiv d_i$ since τ is non-scalar). On the other hand if $(i-1, i)$ is a transition, then $r_i, s_i > 0$, and nonzero terms of h_i have degrees divisible by $p^{f'} - 1$; in that case we write h_i^0 for the constant coefficient of h_i, and we remark that h_i^0 does not vanish identically on $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$.

Suppose, for instance, that $i - 1 \in J$ and $i \in J$. Then w_{i-1} and w_i are the images in $D(\mathfrak{P})$ of m_{i-1} and m_i. From the above formulas we see that $u^{r_i} = u^{e'}$ and h_i are both divisible by u, while on the other hand $u^{s_i} = 1$. We deduce that $F(w_{i-1}) = 0$ and $V(w_i) = c^{-1} a_i^{-1} w_{i-1}$. Computing along similar lines, it is easy to check the following four cases.

(1) $i - 1 \in J, i \in J$. Then $F(w_{i-1}) = 0$ and $V(w_i) = c^{-1} a_i^{-1} w_{i-1}$.

(2) $i - 1 \notin J, i \notin J$. Then $F(w_{i-1}) = b_i w_i$, $V(w_i) = 0$.

(3) $i - 1 \in J$, $i \notin J$. Then $F(w_{i-1}) = h_i^0 w_i$, $V(w_i) = 0$.

(4) $i - 1 \notin J$, $i \in J$. Then $F(w_{i-1}) = 0$, $V(w_i) = -c^{-1} a_i^{-1} b_i^{-1} h_i^0 w_{i-1}$.

In particular, if $i \in J$ then $F(w_i) = 0$, while if $i \notin J$ then $V(w_{i+1}) = 0$. $\square$

Since $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}$ by Theorem 2.3.6, it follows from Lemma 2.4.9 that the natural morphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{G}_\eta$ is determined by an f-tuple of effective Cartier divisors $\{\mathcal{D}_j\}_{0 \le j < f}$ lying in the special fibre $\mathcal{C}^{\tau,\mathrm{BT},1}$. Concretely, $\mathcal{D}_j$ is the zero locus of X_j, which is the zero locus of $F : D_{\eta,j} \to D_{\eta,j+1}$. The zero locus of Y_j (which is the zero locus of $V : D_{\eta,j+1} \to D_{\eta,j}$) is another Cartier divisor $\mathcal{D}'_j$. Since $\mathcal{C}^{\tau,\mathrm{BT},1}$ is reduced, we conclude that each of $\mathcal{D}_j$ and $\mathcal{D}'_j$ is simply a union of irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$, each component appearing precisely once in precisely one of either $\mathcal{D}_j$ or $\mathcal{D}'_j$.

Proposition 5.4.2. *$\mathcal{D}_j$ is equal to the union of the irreducible components $\overline{\mathcal{C}}(J)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$ for those J that contain $j+1$.*

Proof. Lemma 5.4.1 shows that if $j+1 \in J$, then $X_j = 0$, while if $j+1 \notin J$, then $Y_j = 0$. In the latter case, by an inspection of case (3) of the proof of Lemma 5.4.1, we have $X_j = 0$ if and only if $j \in J$ and $h^0_{j+1} = 0$. Since h^0_{j+1} does not vanish identically on an irreducible component, we see that the irreducible components on which X_j vanishes identically are precisely those for which $j+1 \in J$, as claimed. ☐

Theorem 5.4.3. *The algebraic stack $\mathcal{C}^{\tau,\mathrm{BT},1}$ has precisely 2^f irreducible components, namely the irreducible substacks $\overline{\mathcal{C}}(J)$.*

Proof. By Corollary 5.3.1, we need only show that if $J \neq J'$ then $\overline{\mathcal{C}}(J) \neq \mathcal{C}(J')$; but this is immediate from Proposition 5.4.2. ☐

Appendix A. Serre weights and tame types

We begin by recalling some results from [Dia07] on the Jordan–Hölder factors of the reductions modulo p of lattices in principal series and cuspidal representations of $\mathrm{GL}_2(k)$, following [EGS15, §3] (but with slightly different normalisations than those of *loc. cit.*).

Let τ be a tame inertial type. Recall from Section 1.4 that we associate a representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ to τ as follows: if $\tau \simeq \eta \oplus \eta'$ is a tame principal series type, then we set $\sigma(\tau) := \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \eta' \otimes \eta$, while if $\tau = \eta \oplus \eta^q$ is a tame cuspidal type, then $\sigma(\tau)$ is the inflation to $\mathrm{GL}_2(\mathcal{O}_K)$ of the cuspidal representation of $\mathrm{GL}_2(k)$ denoted by $\Theta(\eta)$ in [Dia07]. (Here we have identified η, η' with their composites with Art_K.)

Write $\overline{\sigma}(\tau)$ for the semisimplification of the reduction modulo p of (a $\mathrm{GL}_2(\mathcal{O}_K)$-stable $\mathcal{O}$-lattice in) $\sigma(\tau)$. The action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\overline{\sigma}(\tau)$ factors through $\mathrm{GL}_2(k)$, so the Jordan–Hölder factors $\mathrm{JH}(\overline{\sigma}(\tau))$ of $\overline{\sigma}(\tau)$ are Serre weights. By the results of [Dia07], these Jordan–Hölder factors of $\overline{\sigma}(\tau)$ are pairwise non-isomorphic, and are parametrised by a certain set $\mathcal{P}_\tau$ that we now recall.

Suppose first that $\tau = \eta \oplus \eta'$ is a tame principal series type. Set $f' = f$ in this case. We define $0 \le \gamma_i \le p-1$ (for $i \in \mathbf{Z}/f\mathbf{Z}$) to be the unique integers not all equal to $p-1$ such that $\eta(\eta')^{-1} = \prod_{i=0}^{f-1} \omega_{\sigma_i}^{\gamma_i}$. If instead $\tau = \eta \oplus \eta'$ is a cuspidal type, set $f' = 2f$. We define $0 \le \gamma_i \le p-1$ (for $i \in \mathbf{Z}/f'\mathbf{Z}$) to be the unique integers such that $\eta(\eta')^{-1} = \prod_{i=0}^{f'-1} \omega_{\sigma'_i}^{\gamma_i}$. Here the σ'_i are the embeddings $l \to \mathbf{F}$, where l is the quadratic extension of k, σ'_0 is a fixed choice of embedding extending σ_0, and $(\sigma'_{i+1})^p = \sigma'_i$ for all i.

If τ is scalar then we set $\mathcal{P}_\tau = \{\varnothing\}$. Otherwise we have $\eta \neq \eta'$, and we let $\mathcal{P}_\tau$ be the collection of subsets $J \subset \mathbf{Z}/f'\mathbf{Z}$ satisfying the conditions:

— if $i-1 \in J$ and $i \notin J$ then $\gamma_i \neq p-1$, and

— if $i-1 \notin J$ and $i \in J$ then $\gamma_i \neq 0$

and, in the cuspidal case, satisfying the further condition that $i \in J$ if and only if $i+f \notin J$.

The Jordan–Hölder factors of $\overline{\sigma}(\tau)$ are by definition Serre weights, and are parametrised by $\mathcal{P}_\tau$ as follows (see [EGS15, §3.2, 3.3]). For any $J \subseteq \mathbf{Z}/f'\mathbf{Z}$, we let δ_J denote the characteristic function of J, and if $J \in \mathcal{P}_\tau$ we define $s_{J,i}$ by

$$s_{J,i} = \begin{cases} p-1-\gamma_i-\delta_{J^c}(i) & \text{if } i-1 \in J \\ \gamma_i - \delta_J(i) & \text{if } i-1 \notin J, \end{cases}$$

and we set $t_{J,i} = \gamma_i + \delta_{J^c}(i)$ if $i-1 \in J$ and 0 otherwise.

In the principal series case we let $\overline{\sigma}(\tau)_J := \overline{\sigma}_{\vec{t},\vec{s}} \otimes \eta' \circ \det$; the $\overline{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\overline{\sigma}(\tau)$.

In the cuspidal case, one checks that $s_{J,i} = s_{J,i+f}$ for all i, and also that the character $\eta' \cdot \prod_{i=0}^{f'-1} (\sigma'_i)^{t_{J,i}} : l^\times \to \mathbf{F}^\times$ factors as $\theta \circ N_{l/k}$ where $N_{l/k}$ is the norm map. We let $\overline{\sigma}(\tau)_J := \overline{\sigma}_{0,\vec{s}} \otimes \theta \circ \det$; the $\overline{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\overline{\sigma}(\tau)$.

Remark A.1. The parameterisations above are easily deduced from those given in [EGS15, §3.2, 3.3] for the Jordan–Hölder factors of the representations $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \eta' \otimes \eta$ and $\Theta(\eta)$. (Note that there is a minor mistake in [EGS15, §3.1]: since the conventions of [EGS15] regarding the inertial Langlands correspondence agree with those of [GK14], the explicit identification of $\sigma(\tau)$ with a principal series or cuspidal type in [EGS15, §3.1] is missing a dual. The explicit parameterisation we are using here is of course independent of this issue.

This mistake has the unfortunate effect that various explicit formulae in [EGS15, §7] need to be modified in a more or less obvious fashion; note that since $\sigma(\tau)$ is self dual up to twist, all formulae can be fixed by making twists and/or exchanging η and η'. In particular, the definition of the strongly divisible module before [EGS15, Rem. 7.3.2] is incorrect as written, and can be fixed by either reversing the roles of η, η' or changing the definition of the quantity $c^{(j)}$ defined there.)

Remark A.2. In the cuspidal case, write η in the form $(\sigma'_0)^{(q+1)b+1+c}$ where $0 \le b \le q-2$, $0 \le c \le q-1$. Set $t'_{J,i} = t_{J,i+f}$ for integers $1 \le i \le f$. Then one can check that $\overline{\sigma}(\tau)_J = \overline{\sigma}_{\vec{t}',\vec{s}} \otimes (\sigma_0^{(q+1)b+\delta_J(0)} \circ \det)$.

We now recall some facts about the set of Serre weights $W(\overline{r})$ associated to a representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$.

Definition A.3. *We say that a crystalline representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has* type $\overline{\sigma}_{\vec{t},\vec{s}}$ *provided that for each embedding $\sigma_j : k \hookrightarrow \mathbf{F}$ there is an embedding $\widetilde{\sigma}_j : K \hookrightarrow \overline{\mathbf{Q}}_p$ lifting σ_j such that the $\widetilde{\sigma}_j$-labeled Hodge–Tate weights of r are $\{-s_j - t_j, 1 - t_j\}$, and the remaining $(e-1)f$ pairs of Hodge–Tate weights of r are all $\{0, 1\}$. (In particular the representations of type $\overline{\sigma}_{\vec{0},\vec{0}}$ (the trivial weight) are the same as those of Hodge type 0.)*

Definition A.4. *Given a representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ we define $W(\overline{r})$ to be the set of Serre weights $\overline{\sigma}$ such that $\overline{r}$ has a crystalline lift of type $\overline{\sigma}$.*

There are several definitions of the set $W(\overline{r})$ in the literature, which by the papers [BLGG13, GK14, GLS15] are known to be equivalent (up to normalisation). While the preceding definition is perhaps the most compact, it is the description of

$W(\overline{r})$ via the Breuil–Mézard conjecture that appears to be the most amenable to generalisation; see [GHS18] for much more discussion.

Recall that $\overline{r}$ is *très ramifiée* if it is a twist of an extension of the trivial character by the mod p cyclotomic character, and if furthermore the splitting field of its projective image is *not* of the form $K(\alpha_1^{1/p}, \dots, \alpha_s^{1/p})$ for some $\alpha_1, \dots, \alpha_s \in \mathcal{O}_K^\times$.

Lemma A.5. *(1) If τ is a tame type, then $\overline{r}$ has a potentially Barsotti–Tate lift of type τ if and only if $W(\overline{r}) \cap \mathrm{JH}(\overline{\sigma}(\tau)) \neq 0$.*

(2) The following conditions are equivalent:

(a) $\overline{r}$ admits a potentially Barsotti–Tate lift of some tame type.

(b) $W(\overline{r})$ contains a non-Steinberg Serre weight.

(c) $\overline{r}$ is not très ramifiée.

Proof. This is [CEGS20b, Lem. A.4]. □

Lemma A.6. *Suppose that $\overline{\sigma}_{\vec{t},\vec{s}}$ is a non-Steinberg Serre weight. Suppose that $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is a reducible representation satisfying*

$$\overline{r}|_{I_K} \cong \begin{pmatrix} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+t_i} & * \\ 0 & \overline{\varepsilon}^{-1} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{t_i} \end{pmatrix},$$

and that $\overline{r}$ is not très ramifiée. Then $\overline{\sigma}_{\vec{t},\vec{s}} \in W(\overline{r})$.

Proof. Write $\overline{r}$ as an extension of characters $\overline{\chi}$ by $\overline{\chi}'$. It is straightforward from the classification of crystalline characters as in [GHS18, Lem. 5.1.6] that there exist crystalline lifts χ, χ' of $\overline{\chi}, \overline{\chi}'$ so that χ, χ' have Hodge–Tate weights $1 - t_j$ and $-s_j - t_j$ respectively at one embedding lifting each σ_j and Hodge–Tate weights 1 and 0 respectively at the others. In the case that $\overline{r}$ is not the twist of an extension of $\overline{\varepsilon}^{-1}$ by 1 the result follows because the corresponding $H^1_f(G_K, \chi' \otimes \chi^{-1})$ agrees with the full $H^1(G_K, \chi' \otimes \chi^{-1})$ (as a consequence of the usual dimension formulas for H^1_f, [Nek93, Prop. 1.24]).

If $\overline{r}$ is twist of an extension of $\overline{\varepsilon}^{-1}$ by 1, the assumption that $\overline{\sigma}_{\vec{t},\vec{s}}$ is non-Steinberg implies $s_j = 0$ for all j. The hypothesis that $\overline{r}$ is not très ramifiée guarantees that $\overline{r} \otimes \prod_{i=0}^{f-1} \omega_{\sigma_i}^{-t_i}$ is finite flat, so has a Barsotti–Tate lift, and we deduce that $\overline{\sigma}_{\vec{t},\vec{0}} \in W(\overline{r})$. □

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