

## Exponential energy growth in adiabatically changing Hamiltonian systems

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(Received 7 October 2014; published 9 January 2015)

We show that the mixed phase space dynamics of a typical smooth Hamiltonian system universally leads to a sustained exponential growth of energy at a slow periodic variation of parameters. We build a model for this process in terms of geometric Brownian motion with a positive drift, and relate it to the steady entropy increase after each period of the parameters variation.

DOI: [10.1103/PhysRevE.91.010901](https://doi.org/10.1103/PhysRevE.91.010901)

PACS number(s): 05.45.-a, 05.70.Ln, 05.40.Jc

In his celebrated work [1] Fermi proposed a mechanism for particles in cosmic rays to achieve anomalously high energies. His idea can be put in a more general framework: A fast particle can accelerate due to collisions with massive, slowly moving objects [2]. Most of the research of this phenomenon has focused on the energy growth of particles in billiards with a moving boundary, where the bulk of numerical studies shows the growth of the particle's kinetic energy which is at most polynomial in time [3,4]. This regime can be easily destroyed by a small dissipation [5]. However, in recent examples a robust exponential energy growth was achieved by breaking the ergodicity of the billiard motion [4,6].

In this Rapid Communication we investigate a general question: How can a slow (adiabatic) periodic variation of parameters of an arbitrary Hamiltonian system lead to a sustained energy growth? A theory proposed in Ref. [4] shows that for a billiard with slowly moving boundaries the obstacle for a fast energy growth is the ergodicity of the particle dynamics in the static billiard, which creates the so-called Anosov-Kasuga adiabatic invariant [7]. This adiabatic invariant is not billiard specific, so it imposes restrictions to the energy growth in any Hamiltonian system with slowly changing parameters if the frozen dynamics is ergodic on every energy level. However, apart from special classes of systems, such as geodesic flows and billiards, a typical Hamiltonian system is not ergodic.

We demonstrate that the nonergodicity of a chaotic Hamiltonian system must universally lead to the exponential growth of energy at a slow periodic oscillation of parameters. The key mechanism is the following: Chaos in a nonergodic Hamiltonian system has a mixed, “imperfect” nature such that regions of chaotic dynamics in the phase space coexist with stability islands. Adiabatic changes of parameters lead to transitions between these regions. Different initial conditions give rise to different itineraries of these transitions, and different itineraries give different values of the energy gain and loss per period of the parameters oscillation. We show that, on average over all possible itineraries, the entropy of the system linearly increases after each period, yielding the exponential energy growth. We stress that the steady entropy growth at the adiabatic change of parameters is impossible in the ergodic regime, so the effect is solely due to the imperfect nature of the Hamiltonian chaos.

Consider a family of Hamiltonians  $H(p, q; \tau)$  and assume that the parameter  $\tau$  changes periodically with time. Assume the Hamiltonians in the family are homogeneous, i.e., invariant

with respect to energy scaling, so the dynamics in each energy level is the same. A typical example is the motion in a homogeneous polynomial potential [see, e.g., Eq. (4)]. Another example is the Boltzmann gas of hard spheres. We do not lose generality by focusing on the homogeneous case because we study the process of an unbounded energy growth, and even for a general potential the only terms relevant at high energies are those of the highest order.

We will further suppose that the system defined by the Hamiltonian  $H(p, q; \tau)$  has, at each frozen value of  $\tau$  from a certain region, more than one ergodic component (on each energy level). We assume that the change of  $\tau$  leads to a change in the structure of the decomposition into ergodic components, which causes a typical orbit to undergo random transitions between them.

As  $\tau$  changes, the energy  $E = H(p, q, \tau)$  is no longer preserved by the system:  $\dot{E} = (\partial H / \partial \tau) \dot{\tau}$ . By the homogeneity, it follows that  $\partial H / \partial \tau$  is of the same order as  $H$  at large  $E$ , so the speed of change of  $\ln E$  is comparable with  $\dot{\tau}$ . We assume that parameters of the system change adiabatically, i.e., the change in  $\tau$  and  $\ln E$  is much slower than the dynamics in the  $(p, q)$  phase space.

Our first claim is that at high energies we can model the energy changes by the multiplicative random walk

$$E_{n+1} = E_n \zeta_n, \quad (1)$$

where  $E_n = H(p, q, \tau(nT))$  is the energy after  $n$  periods  $T$ ; the energy gains  $\zeta_n$  form a sequence of independent, identically distributed random variables. The multiplicative character of law (1) is due to  $\partial H / \partial \tau \sim H$ , while the randomness of the factors  $\zeta_n$  is due to the transitions between different ergodic components during the period.

Model (1) describes a random walk for  $\ln E_n$ . It follows that the distribution of  $\ln E_n$  tends to a Gaussian with the mean  $n\rho$ , where  $\rho = \mathbb{E} \ln \zeta_n$ . In particular, for a typical realization of the random walk,  $\lim_{n \rightarrow \infty} \frac{1}{n} \ln E_n = \rho$ , so that the energy  $E_n$  changes exponentially at the rate  $\rho$ . Note that  $\mathbb{E} E_{n+1} = (\mathbb{E} \zeta_n) \mathbb{E} E_n$ , so the expected value of the energy changes at a faster rate  $\rho^+ = \ln \mathbb{E} \zeta_n$ , which means that a small minority of realizations far outperform the rest. Note that similar multiplicative processes provide a basic model for describing nonthermal behavior in various applications [8].

The second claim is that for adiabatically perturbed Hamiltonian systems, which have several distinct ergodic components in the energy level, the multiplicative random

walk model (1) has a positive bias:

$$\rho = \mathbb{E} \ln \zeta_n > 0. \quad (2)$$

Hence, *the energy grows exponentially both for typical initial conditions and on average*. Note that the nonergodicity plays an important role here. In the ergodic case the bias  $\rho$  vanishes and model (1) becomes invalid (the energy grows at most polynomially in this case).

Our third claim is that in a typical situation the distribution  $\nu$  of  $\ln E$  is close to a Gaussian already after the first period of parameters oscillation, so, for all  $n$ ,

$$\nu(\ln E_n) \approx \mathcal{N}(n\rho, \sqrt{n}\sigma), \quad (3)$$

where  $\sigma^2 = \mathbb{E}(\ln \zeta_n - \rho)^2 = 2(\rho^+ - \rho)$ . In other words, the energy growth is modeled by a particular class of multiplicative random processes, the so-called geometric Brownian motion (GBM).

We start with a detailed numerical verification of the above claims for an example of a particle in a quartic potential (4). Then, we develop an averaging theory for nonergodic Hamiltonians, which, in particular, implies law (2). The numerical experiments are performed with

$$H(\mathbf{p}, \mathbf{q}, \tau(t)) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{a(t)}{4}(q_1^4 + q_2^4) + \frac{b(t)}{2}q_1^2q_2^2, \quad (4)$$

where  $\mathbf{q} = (q_1, q_2)$ ,  $\mathbf{p} = (p_1, p_2)$ , and  $\tau(t) = (a(t), b(t))$ . For frozen values of the parameters this system has been thoroughly studied [9]. For example, for  $a = 0.01$ ,  $b = 1$  the system is apparently chaotic. If  $b = 0$ , the two degrees of freedom are uncoupled and the system is integrable. Thus, typical values of  $a$  and  $b$  lead to mixed phase space dynamics [see Fig. 2(a)].

Numerical integration is performed using an explicit fourth-order symplectic method [10] with integration step  $10^{-4}$ . Initial conditions (ensemble of  $2 \times 10^4$  points) are uniformly distributed at the energy level  $E_0 = 3 \times 10^5$ .

*Strong chaos and polynomial energy growth.* If we vary parameters in the region where the frozen Hamiltonian remains strongly chaotic, we observe only a slow energy growth. For instance, Fig. 1 shows that for  $a = 0.01$  and  $b(t) = 1.5 + \cos(2\pi t/T)$  with  $T = 400$  the ensemble averaged energy

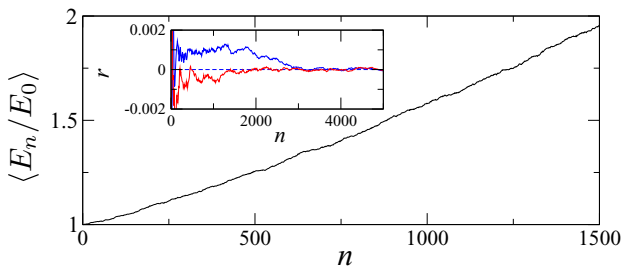


FIG. 1. (Color online) Quadratic energy growth in ergodic regime. We change parameters so that the frozen system remains chaotic, with no visible stability islands. The ensemble averaged energy versus time is shown. In the inset the rates  $r(n) = \frac{1}{n} \ln [E(n)/E(0)]$  are shown for two trajectories for a larger number of periods. Both rates tend to zero, corroborating the lack of exponential acceleration.

growth  $\langle E_n/E_0 \rangle$  vs the number of periods  $n$  behaves as a quadratic polynomial.

*Ergodicity breaking leads to exponential acceleration.* The most of our numerical experiments correspond to the case where the parameters go through chaotic and integrable regions in the parameter space, along the cycle displayed in Fig. 2(a). The cycle is described by  $a(t) = A \cos(2\pi t/T)$  if  $A \cos(2\pi t/T) > a_0$  and  $a(t) = a_0$  otherwise, along with  $b(t) = A \sin(2\pi t/T)$  if  $A \sin(2\pi t/T) > 0$  and  $b = 0$  otherwise. In Figs. 2(b) and 3, we show results for  $a_0 = 0.1$ ,  $A = 1$ , and  $T = 400$ .

For each initial condition, we record the energy gain after  $n$  periods:  $E_n/E_0 = \prod_{k=1}^n E_k/E_{k-1}$ , and compute the ensemble rate  $r_{\text{en}}(n) = \frac{1}{n} \sum_{k=1}^n \ln \frac{E_k}{E_{k-1}}$ . The characteristic signature of GBM is that two distinct growth rates are observed when the averaging  $\langle \cdot \rangle$  is performed over a finite ensemble [4,8]. Namely, since the standard deviation of  $E_n$  in Eq. (1) grows much faster than  $\mathbb{E}(E_n)$ , it follows that for finite ensembles there is a crossover from the ensemble rate  $\rho^+ = \ln \mathbb{E} \zeta$  to a lower rate  $\rho = \mathbb{E} \ln \zeta$  as  $n$  grows. In Fig. 2(b) we clearly observe this crossover to a lower (still positive) rate of the exponential energy growth. Thus, the ensemble rate  $r_{\text{en}}(n)$  observed at the initial stage of the acceleration process can be identified with the parameter  $\rho^+$  of the GBM. The data shown in Fig. 2(b) give  $r_{\text{en}}(n)$  that quickly stabilizes to  $\rho^+ \approx 0.0368$  and holds over the first 70 cycles.

In order to make a qualitative check of our GBM model, we investigate the behavior of the distribution of  $\ln E_n$ . As seen in Fig. 3, this distribution is indeed close to Gaussian, in accordance with Eq. (3). The values of  $\rho$  and the standard deviation  $\sigma$  are estimated from the numerics as  $\rho \approx \frac{1}{n} \langle \ln(E_n/E_0) \rangle$  and  $\sigma \approx \frac{1}{\sqrt{n}} \langle \ln(E_n/E_0) - n\rho \rangle$ . In our experiment the values of  $\rho$  and  $\sigma$  stabilize already at the first cycle, giving  $\rho = 0.0212$  and  $\sigma^2 = 0.032$ . We performed the same numerical experiments with  $a_0$  and  $A$  varying in  $a_0 \in [10^{-6}, 10^2]$  and  $A \in [1, 10]$ . In all experiments we observed the log-Gaussian character of the distribution of energies established after the first cycle, with parameters  $\rho > 0$  and  $\sigma$  independent of  $n$ . We also observed the constant ensemble rate  $r_{\text{en}} \approx \rho^+$  at the initial stage of the acceleration process. As a test for the Gaussianity we checked the relation  $\rho = \rho^+ - \sigma^2/2$  [which is a consequence of Eq. (3)]. Figure 2(c) shows the results of this test for  $A = 1$ . As we see, this relation holds for the entire range of values of  $\rho$ ; the same holds true for other values of  $A$ . We conclude that the observed energy growth is governed by the GBM with a positive drift.

*General setting.* A Hamiltonian  $H(q, p)$  is *homogeneous* if for every  $E > 0$  there exists a coordinate transformation  $\Phi$  that keeps the system the same, sends the energy level  $H = 1$  to  $H = E$ , and has a constant Jacobian  $J(E) = E^\alpha$ ,  $\alpha > 0$ . Let the positive energy levels be compact, so  $J(E) = \frac{V(E)}{V(1)}$  where  $V(E)$  is the volume of the  $(q, p)$  space between the levels  $H = E$  and  $H = 0$ . We will label the points in the phase space  $(p, q)$  by the coordinates  $(x, E)$  where  $E = H(p, q)$  is the energy and  $x = \Phi^{-1}(p, q)$  is the projection to the level  $H = 1$ .

Consider a family of adiabatically changing homogeneous Hamiltonians  $H(p, q; \tau)$ . If the frozen system is ergodic in every energy level with respect to the Liouville measure

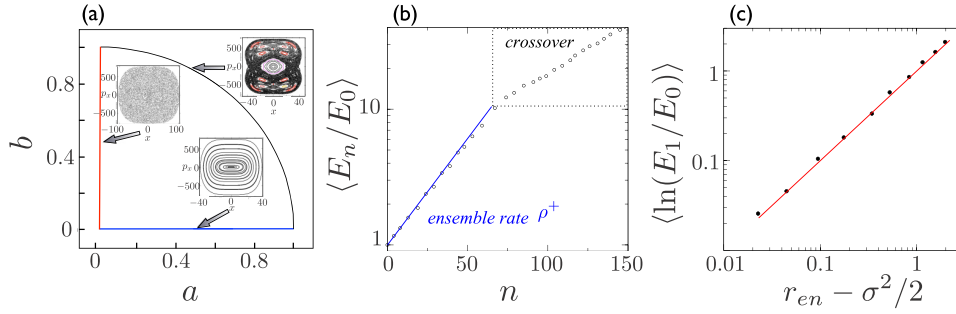


FIG. 2. (Color online) Exponential energy growth. (a) Parameter space for the quartic Hamiltonian system (4). Parameters are chosen such that the frozen Hamiltonian exhibits chaotic dynamics (along  $a = 0.01$ ), quasiperiodic motion (along  $b = 0$ ), and mixed behavior (along the connecting arc). The insets show the dynamics of the frozen Hamiltonian. Performing the cycle leads to the exponential growth of energy. In (b) we show the ensemble energy growth with the energy given in logarithmic scale. For the first 70 cycles the higher ensemble rate  $r_{\text{en}} = 0.0368$  holds. Further, a crossover to a lower rate starts. In (c) a prediction of the GBM model for the relation between  $\rho^+$ ,  $\rho$ , and  $\sigma$  is verified (the rates are varied by changing the value of the parameter  $a_0$  at fixed  $A = 1$ ). The red solid line is the identity.

$\mu = \delta(E - H(p, q, \tau)) dp dq$ , then a theorem by Anosov is applied [7] that guarantees that averaging over this measure gives a good approximation of the slow evolution of the energy for a large set of initial conditions.

By analogy, if the frozen system is *not ergodic*, we assume that the slow evolution of the energy is given by

$$\dot{E} = \int \frac{\partial H}{\partial \tau} \delta(E - H) \mu_\tau(dx) \dot{\tau}, \quad (5)$$

where  $\mu_\tau$  is, at each  $\tau$ , a certain ergodic measure on the space of fast variables  $x$ . We call the  $\tau$ -dependent family  $\mu_\tau$  in Eq. (5) an *averaging protocol*. It can be different for different initial conditions, though we assume that it is independent of the initial energy  $E_0$  (by the homogeneity of the frozen Hamiltonians, this assumption is natural at large  $E_0$ ). Thus, we split the space of initial conditions  $x$  into cells  $M_1, \dots, M_k$  that give rise to distinct averaging protocols over the period  $T$  of the oscillation of parameters. For each cell, the majority of points exhibits the same energy evolution over the period.

Let  $E_0$  and  $E_1$  be two sufficiently large values of energy. Take a cell  $M_k$ . If the points with initial conditions  $E = E_0$ ,  $x \in M_k$  move to the energy level  $E = \bar{E}_0 = e^\lambda E_0$  after the period  $T$ , then the points with initial conditions  $E = E_1$ ,  $x \in M_k$  move to the level  $E = \bar{E}_1 = e^\lambda E_1$ , by the homogeneity of Eq. (5). The original, nonaveraged system preserves volume

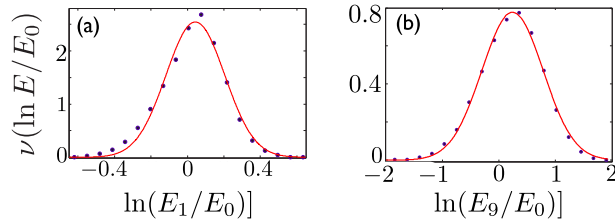


FIG. 3. (Color online) Distribution of energies is log-normal. The distribution of energies for an ensemble of  $2 \times 10^4$  particles starting with initial conditions randomly distributed in the energy shell  $E_0$ . Already after one period the distribution of the logarithm of  $E_n$  is close to the Gauss law.

in the  $(p, q)$  space. Therefore, the volume between the levels  $E = E_0$  and  $E = E_1$  occupied by the points with  $x \in M_k$  equals the volume between the levels  $E = \bar{E}_0$  and  $E = \bar{E}_1$ , occupied by the points with  $x \in \bar{M}_k$ , where  $\bar{M}_k$  is the image of the cell  $M_k$  after the period. This gives

$$\alpha \lambda_k = \ln[v(M_k)/v(\bar{M}_k)]. \quad (6)$$

We assume that after each period the orbit does not necessarily return to the original cell, i.e., there is a probability  $Q_{kj}$  to start at the cell  $M_j$  and, after time  $T$ , land at the cell  $M_k$ . We also assume that the distribution within the cell  $M_k$  is made uniform by the fast dynamics. Thus, we have a Markov process  $\mathcal{T}$  that redistributes phase points between the cells after each period. Under general conditions, this process leads to exponential convergence of the density in the  $x$  space at the beginning of each period to  $\phi(x) = P_k/v(M_k)$  for  $x \in M_k$ , where  $P_k = \sum_j Q_{kj} P_j$  is the probability of being in  $M_k$ , and  $v$  is the volume in the  $x$  space (at the level  $H = 1$ ).

Define the entropy  $S$  at the beginning of the period as an averaged value of  $\ln[V(E)/V(1)] = \ln J(E) = \alpha \ln E$ :

$$S = \alpha \int \sum_k P_k \ln E \theta_k(E) dE,$$

where  $\theta_k(E)$  is the energy density at  $x \in M_k$ . Let us show that  $S$  is a *nondecreasing* function of the number of cycles. The change of  $S$  over one period is  $\Delta S = \sum_k \alpha \lambda_k P_k$ . By Eq. (6), we obtain

$$\Delta S = \sum_k P_k \ln \left[ \frac{v_k}{\bar{v}_k} \right] = \Sigma - \bar{\Sigma}, \quad (7)$$

where  $\Sigma = \sum_k P_k \ln \left[ \frac{v_k}{P_k} \right] = - \int \phi(x) \ln \phi(x) dx$ ,  $\bar{\Sigma} = \sum_j P_j \ln \left[ \frac{\bar{v}_j}{P_j} \right] = - \int \bar{\phi}(x) \ln \bar{\phi}(x) dx$ , i.e.,  $\Sigma$  is the Gibbs entropy of the density  $\phi(x) = \frac{P_k}{v_k}$  at  $x \in M_k$  (the stationary density for the Markov process  $\mathcal{T}$ ), and  $\bar{\Sigma}$  is the Gibbs entropy of  $\bar{\phi}(x) = \frac{P_j}{\bar{v}_j}$  at  $x \in \bar{M}_j$ . The density  $\bar{\phi}$  is the image of  $\phi$  by the slow evolution over the oscillation period, just before it relaxes back to  $\phi$  due to the fast dynamics. The relaxation must increase the Gibbs entropy, so  $\Delta S \geq 0$ .

Indeed, let  $\psi_{kj} = v(\bar{M}_j \cap M_k)$ . Then  $\bar{v}_j = \sum_k \psi_{kj}$ ,  $v_k = \sum_j \psi_{kj}$ , and the transition probability from  $M_j$  to  $M_k$  is  $Q_{kj} = \psi_{kj}/\bar{v}_j$ , so  $v_k = \sum_k Q_{kj}\bar{v}_j$ . Let  $\gamma_{kj} = Q_{kj}P_j/P_k$ . As  $\sum_j \gamma_{kj} = 1$ , we have

$$\begin{aligned} \sum_j \gamma_{kj} \ln(\bar{v}_j/P_j) &= \ln \left[ \prod_j \left( \frac{\bar{v}_j}{P_j} \right)^{\gamma_{kj}} \right] \\ &\leq \ln \left( \sum_j \frac{\bar{v}_j}{P_j} \gamma_{kj} \right) \\ &= \ln \left( \frac{1}{P_k} \sum_j Q_{kj} \bar{v}_j \right) = \ln(v_k/P_k), \end{aligned}$$

so  $\sum_j Q_{kj}P_j \ln(\bar{v}_j/P_j) \leq P_k \ln(v_k/P_k)$ , hence  $\bar{\Sigma} \leq \Sigma$  (as  $\sum_k Q_{kj} = 1$ ). Thus, the entropy is a nondecreasing function of time, which proves our claim (2).

If the frozen system is ergodic for each  $\tau$ , i.e.  $\mu_\tau(dx) = dx$  in Eq. (5), then the Anosov-Kasuga averaging theorem gives  $\Delta S = 0$ , because  $\frac{d}{dt} V(E, \tau) = \frac{\partial V}{\partial E} \dot{E} + \frac{\partial V}{\partial \tau} \dot{\tau} = 0$  in this case (see Ref. [7]). Thus, that energy stays bounded and changes periodically with  $\tau$  [to keep  $V(E, \tau)$  constant].

If there is no ergodicity for some  $\tau$ , there is no restriction on the growth of entropy, and one should expect  $\Delta S > 0$ . As there is no dependence on energy in the right-hand side of Eq. (7), we have the same increment in entropy over each period of  $\tau$ , so  $S$  must grow linearly in time, which yields an exponential growth of energy at a rate  $\Delta S/(\alpha T)$  for a typical initial condition.

The distribution of the energy gain upon one cycle can be different in different settings (see Refs. [4,6] for billiard examples). However, in these special examples the number of different averaging protocols is small. In a typical Hamiltonian system with the mixed phase space many elliptic islands can coexist, so the adiabatic change of parameters can make an orbit visit many different ergodic components with essentially random itinerary. Averaging over each ergodic component [the measure  $\mu_\tau$  in Eq. (5) at a frozen  $\tau$ ] results in the energy multiplied by a random factor. Since the number of transitions between different components during one cycle of parameters oscillation is large, the logarithmic energy gain per one period gets close to Gaussian law (3).

Our results suggest a model for an adiabatically changing Hamiltonian system as a gas of noninteracting particles (corresponding to different initial conditions), hence, there is no equilibrium distribution in energies. In the typical nonergodic case we may think of particles as being in different states (corresponding to different ergodic measures  $\mu_\tau$ ). Thus, our gas is a mixture of different fractions; the adiabatic change of parameters causes the particles to states, so the relative densities of the fractions vary. This naturally leads to the entropy increase with each period of parameter oscillations, which implies the exponential energy growth.

*Acknowledgments.* We acknowledge the hospitality of the Weizmann Institute where part of this work was written. We are grateful to V. Gelfreich and V. Rom-Kedar for useful discussions. T.P. was supported by UK Leverhulme Trust Grant No. RPG-279 and European Union FP7 Marie Curie Project 303180. D.T. was supported by Grant No. 14-41-00044 of RSF (Russia).

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