Progress of Theoretical Physics, Vol. 122, No. 5, November 2009

Quantum States Allowing Minimum Uncertainty Product of ϕ and L_z

Tiago PEREIRA[1,*)] and Domingos H. U. MARCHETTI[2,**)]

[1] *Center for Mathematics, Federal University of ABC, Santo André, Brazil*
[2] *Institute for Physics, University of São Paulo, São Paulo, Brazil*

(Received March 24, 2009; Revised July 24, 2009)

We provide necessary and sufficient conditions for states to have an arbitrarily small uncertainty product of the azimuthal angle ϕ and its canonical moment L_z. We illustrate our results with analytical examples.

Subject Index: 064

§1. Introduction

The Newtonian determinism states that the present state of the universe determines its future precisely. At the beginning of the past century the advent of quantum mechanics exposed the determinism to great delusion. It turned out that in the quantum world the uncertainty prevails. Heisenberg was the first to recognize the antagonism between classical and quantum mechanics.[1)] He noticed that for the position and its conjugate momentum the more concentrated the distribution of the position, the more uniform is the distribution of the momentum and vice-versa. The Heisenberg relation states that it is impossible to predict, with arbitrary certainty, the outcomes of measurements of two canonically conjugate observables.

The uncertainty relation was subsequently generalized by Robertson.[2)] The variance of an observable A for a given state ψ is

$$\sigma_A^2 = \langle A\psi, A\psi \rangle - |\langle \psi, A\psi \rangle|^2 \ ,$$

and the Heisenberg-Robertson (HR) uncertainty relation, in its most well known form, reads:

$$\sigma_A \sigma_B \geq \frac{\hbar}{2} \left| \langle \psi, i[A,B]\psi \rangle \right| \ , \tag{1·1}$$

where $[A, B]$ is the commutator of observables A and B.

The uncertainty principle has been one of the most intricate points in quantum mechanics.[3),4)] Besides its philosophical meaning it plays a major role in experimental physics of atomic scale as, for example, in the Bose-Einstein condensation,[5)] and electrons jump at random from one energy state which they could never reach except by fluctuations in their energy. Another manifestation of the uncertainty principle in the energy spectrum can be seen in the spectral linewidth that characterizes the width of a spectral line.[6),7)]

*) E-mail: tiago.pereira@ufabc.edu.br
**) E-mail: marchett@if.usp.br

An old problem concerning the uncertainty principle (and whether the uncertainty relation (1·1) expresses it adequately) appears if the quantum system is described in terms of angle variables.[8),9)] When the Cartesian coordinates (x, y, z) are changed to spherical ones (r, θ, ϕ), Eq. (1·1) no longer provides a lower bound for the product of uncertainty of the azimuthal angle operator ϕ and its canonical conjugate momentum L_z.[*),8),10)] The intuitive idea is that fluctuations on ϕ bigger than 2π do not have much physical meaning. Hence, for a wave sufficiently localized in the Fourier space the amount σ_{L_z} becomes very small, while σ_ϕ remains bounded. As a result one may have a lower bound smaller than the one of Eq. (1·1). Recently this problem has attracted a great deal of attention.[11)–15)]

The HR uncertainty relation for the angle and position has been criticized on several grounds and other mathematical formulations of the uncertainty principle have been proposed (see 11), 13) and 16) for a contextualization). Examples of such attempts include the entropic relations for the observables;[17)–20)] the introduction of a unitary operator for phase ϕ;[21),22)] evaluation of the commutator for functions that just belong to the domains of the angle and angular momentum operators;[23),24)] the exchange of the angle with an absolutely continuous periodic function;[25)] and expressing the lower bound as state dependent.[16),26)]

Despite of these alternatives, expressing the uncertainty principle for angular operators by lower-bounding the product of the standard deviations is widely used. In particular, experimental confirmation of the uncertainty principle for the angular momentum and position has been carried out for intelligent states (states that saturates the uncertainty relation for ϕ and L_z observables).[14)] Also recently, the relation between these intelligent states and the constrained minimum uncertainty product for the angular operator has shown to be important.[15)]

Motivated by the state-dependence of standard measures of uncertainty and the fact that some state features may be prepared or detected experimentally we investigate the class of states that allows for an arbitrarily small uncertainty product.

In this paper, we provide necessary and sufficient conditions on these families that allow for an arbitrarily small uncertainty product. We demonstrate that arbitrarily small uncertainty product is attained if, and only if, a single nonvanishing Fourier coefficient $C_k(\alpha)$ decays, as a function of α, slower than the others $C_n(\alpha)$ with $n \neq k$. Furthermore, we provide explicit examples of our result. Our main contribution to this subject is to translate this intuitive reasoning on the uncertainty product into a rigorous statement.

This paper is organized as follows: In §2 we discuss some problems associated with the HR relation. Our hypotheses on the states are given in §3. Our main result concerning the states which allow for an arbitrarily small uncertainty product is given in §4. In §5 we deduce the equations for σ_ϕ and σ_{L_z}. We provide examples of our result in §6 for the exponential decay and in §7 for the polynomial decay of the Fourier coefficients of the states. Section 8 contains a proof of our main result.

[*)] The eigenfunctions in spherical coordinates ψ are chosen to be regular at the origin, so that $\langle g(\phi), \psi\rangle = \int g(\phi)|\psi(r, \theta\phi)|^2 dV = \int_0^{2\pi} g(\phi)\rho(\phi)d\phi$ is well defined for all square integrable states $\psi = \sum c_{n,l,m}\psi_{n,l,m}(r, \theta, \phi)$ and any continuous function g. From now on we shall restrict ourselves to the marginal distribution $\rho(\phi)$.

Finally, in §9 we give our conclusions.

§2. Pitfalls and apparent paradox

Let us start by introducing the operators ϕ and its canonical conjugate L_z. The phase is introduced as the angular displacement of the vector position:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right).$$

The angle operator is usually defined as a multiplication operator either by the variable ϕ or by

$$Y(\phi) = (\phi - \pi) \bmod 2\pi + \pi .$$

See for example Ref. 26). When ϕ is defined on the lift, that is, without the mod 2π, it is continuous but no longer periodic. Since ϕ and $\phi + 2\pi$ correspond to the same physical situation, the mod 2π operation in the range $[-\pi, \pi]$ is preferred. Here, we adopt ϕ as a multiplication operator by ϕ acting on the space of 2π-periodic functions which is square integrable in the interval $[-\pi, \pi]$. For values in this range ϕ and $Y(\phi)$ do not differ from each other.

The canonical momentum associated with ϕ is given by

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\phi}. \tag{2·1}$$

Under the (false) assumption that the commutation relation

$$[\phi, L_z] = i\hbar \tag{2·2}$$

holds on the domain in which L_z and ϕ are self-adjoint operators, the HR uncertainty relation reads

$$\sigma_\phi \sigma_{L_z} \geq \frac{\hbar}{2}. \tag{2·3}$$

The product of uncertainty, however, can be made smaller than $\hbar/2$.[14)–16)]

Another apparent paradox that appears by naïve assumptions on the domain of the operators involved is as follows. Let $|lm\rangle$ denote the spherical harmonic functions. From Eq. (2·2), we have

$$\left\langle lm'\right| [\phi, L_z] \left|lm\right\rangle = i\hbar \left\langle lm'|lm\right\rangle, \tag{2·4}$$

and this leads to the (wrong) conclusion

$$\hbar(m - m') \left\langle lm'\right| \phi \left|lm\right\rangle = i\hbar\delta_{mm'},$$

that $0 = 1$ if $m = m'$. See Examples **5** and **6** of Ref. 24).

Since the operator ϕ multiplies the wave function by a bounded real number, it is Hermitian: $\langle\psi_1, \phi\psi_2\rangle = \langle\phi\psi_1, \psi_2\rangle$, and self-adjoint operator in the Hilbert space $\mathcal{H}$ of square integrable functions in $[-\pi, \pi]$. The operator L_z, on the other hand, is defined in a closed domain $D(L_z)$ of $\mathcal{H}$. It may be extended as a self-adjoint operator to the subset of 2π-periodic absolutely continuous functions $AC[-\pi,\pi]$.[27] Now, the domain $D([\phi,L_z])$ of the commutator $[\phi,L_z]$ is given by the functions $\psi\in AC[-\pi,\pi]$ such that $\psi(-\pi)=\psi(\pi)=0$. As the eigenfunctions $\psi_m(\phi)=e^{im\phi}/\sqrt{2\pi}$ of L_z do not belong to $D([\phi,L_z])$, the commutator cannot acts over $|lm\rangle$ and Eq. (2·4) does not make sense. The apparent contradiction of (2·3) rests on the same problem: the domain $D([\phi,L_z])$ of functions on the r.h.s. of (1·1) is smaller than the domain $D(L_z)\cap D(\phi)$ of the l.h.s. of (1·1) (see 24) for a detailed discussion).

An attempt to fix the domain problem in the uncertainty relation (2·3) is to abandon the commutator and introduce a sesquilinear form[16),24)] defined in $D(L_z)\cap D(\phi)$. The uncertainty relation then reads

$$\begin{aligned}\sigma_\phi\sigma_{L_z} &\geq \left|i\,\langle\phi\psi,L_z\psi\rangle - i\,\langle L_z\psi,\phi\psi\rangle\right| \\ &= \frac{\hbar}{2}\left|1-2\pi\,|\psi(\pi)|^2\right| \end{aligned} \tag{2·5}$$

which is now state-dependent (see 11), 23) and 24), for details). Note that (2·5) and (2·3) agree if $\psi\in D([\phi,L_z])$, since a state ψ in the domain of the commutator satisfies $\psi(\pi)=0$.

§3. Set up

The ground of our result is the Fourier expansions of $f_\alpha(\phi)=\sum_{n=-\infty}^{\infty}c_n(\alpha)e^{in\phi}$, where $c_n(\alpha)$ are the Fourier coefficients (frequency amplitudes) of $f_\alpha(\phi)$. Here we shall write $c_n(\alpha)=A_\alpha C_n(\alpha)$, hence, the Fourier expansions read

$$f_\alpha(\phi)=A_\alpha\sum_{n=-\infty}^{\infty}C_n(\alpha)e^{in\phi}, \tag{3·1}$$

where $C_n(\alpha)$ are unnormalized with A_α fixed by the normalization:

$$\langle f_\alpha(\phi),f_\alpha(\phi)\rangle = 2\pi|A_\alpha|^2\sum_{n=-\infty}^{\infty}|C_n(\alpha)|^2=1. \tag{3·2}$$

In view of the factor A_α, $C_n(\alpha)$ can be an arbitrary function of α independently for each n, we consider $C_n:\mathbb{R}_+\to\mathbb{C}$. We have introduced the dependence of α on $\{C_n(\alpha)\}$ in order to let the state f_α approach an arbitrary state. For notational simplicity, whenever we do not specify the sum we understand the index running from $-\infty$ to ∞. Also, whenever there is no risk of confusion, we shall omit the index α of the Fourier coefficients C_n and normalization constant $|A|^2$.

Admissible Family: Let $\mathcal{F}$ be a one parameter family of periodic functions f_α with (i) nontrivial variance, that is, $\sigma_\phi^2\geq\inf_\alpha\sigma_\phi^2=\kappa>0$; and Fourier coefficients such that: (ii) $\{nC_n(\alpha)\}\in\ell_2$ uniformly in α, that is, for every $\epsilon>0$ there is $N=N(\epsilon)$, independent of α, such that $\sum_{n=j}^{m}n^2|C_n(\alpha)|^2<\epsilon$ for all $m>j>N(\epsilon)$; A family $\mathcal{F}$ is said to be admissible if it satisfies (i) and (ii).

Condition (i) avoids a state f_α to be in a neighborhood of the Dirac delta function $\delta(\phi)$. For such states $|f_\alpha(\pi)|$ is small, so the bound given by Eq. (2·5) already prevents the uncertainty product to be close to 0. Condition (*ii*) on uniformity is of technical nature and guarantees that the limit of a sum equals to sum of the limits of a given sequence. It will be used in Eq. (8·3).

The one-parameter families under consideration fill densely the space of all admissible states (the ones for which σ_ϕ and σ_{L_z} are well defined). For an admissible family the Fourier series converges absolutely and uniformly to the state. This follows from:

Proposition 1 *The Fourier series of a continuous function $f(\phi)$ of period 2π, whose derivative (which may not exist everywhere) is square integrable, converges absolutely and uniformly to $f(\phi)$.*

For a proof see Ref. 28). An important concept throughout this works is the **Dominance Condition:** An admissible family $\mathcal{F}$ satisfies the dominance condition if given $\varepsilon > 0$ there is only one k such that $C_k(\alpha) \neq 0$ and

$$\frac{C_n(\alpha^*)}{C_k(\alpha^*)} < \varepsilon \tag{3·3}$$

holds (simultaneously) for every $n \neq k$ and for some α^*.

Note that we do not require the limit of $|C_n(\alpha)|$ and $|C_n(\alpha)|/|C_k(\alpha)|$ as α goes to infinity to exist. Condition (3·3), implies that $\liminf_{\alpha\to\infty} |C_n(\alpha)|/|C_k(\alpha)| = 0$, that is, there is at least one subsequence $(\alpha_j)_{j\geq 1}$ such that

$$\lim_{j\to\infty} \frac{|C_n(\alpha_j)|}{|C_k(\alpha_j)|} = 0 \tag{3·4}$$

for all $n \neq k$. In particular, there is an increasing sequence $(\alpha_k)_{k\geq 1}$ such that $|C_n(\alpha_j)| < |C_n(\alpha_k)|$ if $j > k$.

§4. Theorem on arbitrarily small uncertainty product

Here we state our main results. We start by introducing the following:

Definition 1 *Let the standard deviations $\sigma_\phi(\alpha)$ and $\sigma_{L_z}(\alpha)$ associated with a state $f_\alpha \in \mathcal{F}$ be given by Eqs. (5·1) and (5·3), respectively. An admissible family $\mathcal{F}$ is said to allow an arbitrarily small uncertainty product if for every $\varepsilon > 0$ there is an $\alpha^* \in (0, \infty)$ such that*

$$\sigma_\phi(\alpha^*)\sigma_{L_z}(\alpha^*) < \varepsilon.$$

Our main result is then stated as follows:

Theorem 1 *An admissible family $\mathcal{F}$ allows an arbitrarily small uncertainty product if, and only if, it satisfies the dominance condition.*

From this theorem it follows:

Corollary 1 *Any state $f_\alpha(\phi) \in \mathcal{F}$ whose Fourier coefficients are sufficiently localized in the Fourier space has uncertainty product smaller than the least value predicted by the HR relation (2·3).*

It is worthy to note that our result does not depend on the decay of the coefficients, but only on the relative decay with respect to C_k as stated in Eq. (3·3). We illustrate our result for two different decays. The proof of Theorem 1 is given in §8.

Since the Fourier series is absolutely and uniformly convergent for any state of a given admissible family, Eq. (3·3) could also be formulated in terms of the states. Our preference to enunciate the criterion in terms of the Fourier series is twofold: Firstly, we shall give several examples with states characterized in terms of the Fourier series. Secondly, it is easier to visualize what is happening, when results are formulated in terms of the Fourier series.

§5. Uncertainty relations

In this section we give a formal derivation of the general formulas for the deviations σ_ϕ and σ_{L_z}, assuming that Eq. (3·1) holds. The deviation on the variable ϕ is given by

$$\sigma_\phi^2 = \langle \phi^2 \rangle - |\langle \phi \rangle|^2 , \tag{5·1}$$

and we start with the first term on the right-hand side (r.h.s):

$$\langle \phi^2 \rangle = \frac{\pi^2}{3} + 4\pi |A|^2 \xi,$$

where

$$\xi = \sum_{m \neq n} C_m^* C_n \frac{(-1)^{(n-m)}}{(n-m)^2}.$$

For the second term on the r.h.s. of (5·1), we have

$$\langle \phi \rangle = 2\pi |A|^2 \sum_{m \neq n} C_m^* C_n \frac{1}{i} \frac{(-1)^{n-m}}{n-m}.$$

Therefore the deviation is given by

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 4\pi |A|^2 \xi - \left| 2\pi |A|^2 \sum_{m \neq n} C_m^* C_n \frac{(-1)^{n-m}}{n-m} \right|^2 . \tag{5·2}$$

Next, we compute:

$$\sigma_{L_z}^2 = \langle L_z^2 \rangle - |\langle L_z \rangle|^2 \ . \tag{5·3}$$

Using condition (ii) we have $\langle L_z^2 \rangle = \langle L_z f, L_z f \rangle = 2\pi |A|^2 \hbar^2 \sum_n |C_n|^2 n^2$. For the amount $\langle L_z \rangle = \langle f, L_z f \rangle$ we have analogously $|\langle f, L_z f \rangle|^2 = 4\pi^2 |A|^4 \hbar^2 \left(\sum_n |C_n|^2 n \right)^2$. Thus, the deviation in L_z is given by

$$\sigma_{L_z}^2 = 2\pi \hbar^2 |A|^2 \sum_n |C_n|^2 n^2 - 4\pi^2 \hbar^2 |A|^4 \left(\sum_n |C_n|^2 n \right)^2 . \tag{5·4}$$

§6. Fourier coefficients with exponential decay

We restrict our attention to the case in which the frequency amplitudes C_n decay exponentially fast in $|n|$:

$$C_n = e^{-\alpha |n|}.$$

This and the next example capture most of the important features we wish to emphasize. Note that, C_n is a real even function of n: $C_n = C_{-n}$ and $C_n^* = C_n$. The sequence $\{C_n(\alpha)\}$ satisfies hypothesis (ii) but $f_\alpha(\phi)$ approaches the Dirac delta function $\delta(\phi)$ when α tends to 0.

The sequence $\{C_n(\alpha)\}$ satisfies, in addition, the dominance condition Eq. (3·3) with $k = 0$. As we shall see, the uncertainty product can be arbitrarily small despite of the noncompliance of (i).

From the properties of C_n it follows that $\langle \phi \rangle = 0$. Note that the $1/(m-n)$ is odd, while the $C_m^* C_n (-1)^{n-m}$ is even. As a result the product is odd, and a symmetric sum over an odd function is zero. Therefore, we have

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 2\frac{e^{2\alpha}-1}{e^{2\alpha}+1}\xi(\alpha), \tag{6·1}$$

where

$$\xi(\alpha) = \sum_{m \neq n} e^{-\alpha|n|} e^{-\alpha|m|} \frac{(-1)^{n-m}}{(n-m)^2}.$$

An explicit computation shows that $\sigma_\phi^2 = \dfrac{\pi^2}{3}(1 + O(e^{-\alpha}))$ holds for large α (see A). It thus follows that $\lim_{\alpha\to\infty} \sigma_\phi^2 = \frac{\pi^2}{3}$, is an upper bound for σ_ϕ^2. Note that $\sigma_\phi^2 = \pi^2/3$ is the deviation of a uniform state $\psi(\phi) = 1/\sqrt{2\pi}$, $\phi \in [-\pi, \pi]$. In A, it is proved that, for α small enough it holds $\sigma_\phi^2 = \alpha^2 + O(\alpha^3)$, hence, it yields $\lim_{\alpha\to 0} \sigma_\phi^2 = 0$.

For the deviation σ_{L_z} (since C_n is even it implies $\langle L_z \rangle = 0$) we have $\sigma_{L_z}^2 = 2\hbar^2 \frac{e^{2\alpha}}{(e^{2\alpha}-1)^2}$. Hence, for $\alpha \to 0$ we obtain $\sigma_{L_z}^2 = \frac{\hbar^2}{2\alpha^2}(1 + O(\alpha))$, and as $\alpha \to \infty$, we have $\sigma_{L_z}^2 = 2\hbar^2 \frac{1}{e^{2\alpha}} \left(1 + O(e^{-2\alpha})\right)$. Thus, for α small enough the uncertainty product reads

$$\sigma_\phi^2 \sigma_{L_z}^2 = \frac{\hbar^2}{2}(1 + O(\alpha)),$$

asserts that the square of the uncertainty product reaches twice the smallest predicted values by the HR relation (recall $f_\alpha(\phi)$ approaches $\delta(\phi)$ in this limit and it is not affected by the boundary condition at π). For α large enough we have

$$\sigma_\phi^2 \sigma_{L_z}^2 = \frac{2\pi^2\hbar^2}{3}\frac{1}{e^{2\alpha}}(1 + O(e^{-\alpha})),$$

implying that the uncertainty product goes to zero exponentially fast with α.

In Fig. 1 we depict the uncertainty product $\sigma_\phi \sigma_{L_z}/\hbar$ as a function of α. One can see that the bound given by Eq. (2·3) holds only for $\alpha < 1.29639$ (see the dashed line).

For this case, we can obtain an explicit analytic form for the state

$$f_\alpha(\phi) = \sqrt{\frac{2\pi}{\tanh\alpha}} \left(1 - \frac{2(e^\alpha \cos\phi - 1)}{2e^\alpha \cos\phi - e^{2\alpha} - 1}\right).$$

The computation of the uncertainty product via Fourier analysis or via a probability density $\|f_\alpha(\phi)\|^2$ gives the same results for admissible families.

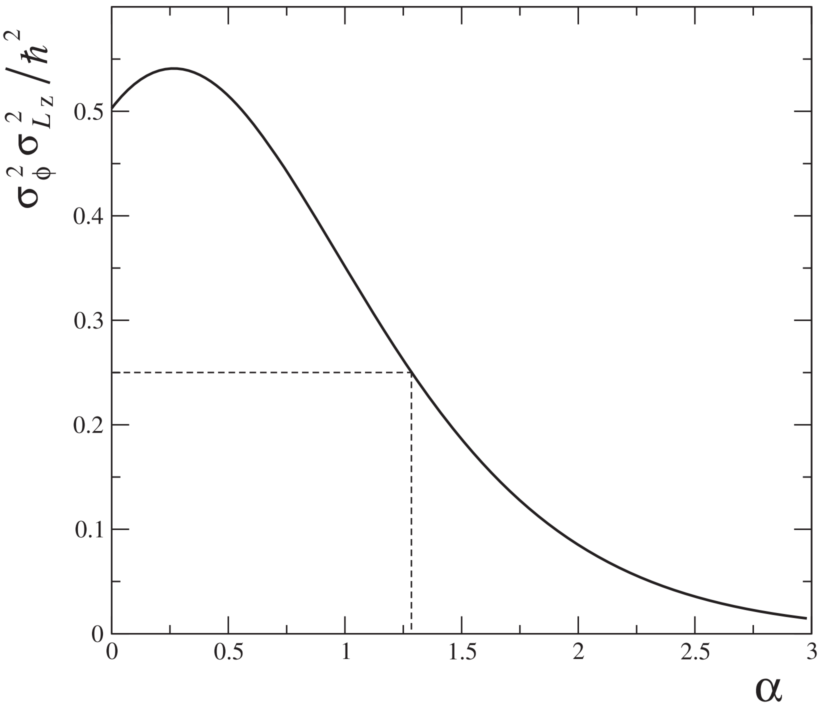

Fig. 1. The profile of $\sigma_\phi^2\sigma_{L_z}^2$ for a exponential decaying Fourier coefficients.

§7. Polynomial decay of fourier coefficients

The fact that the Fourier coefficients with exponential decay have an arbitrarily small lower bound is not a privilege of this particular decay. Any other decay which fulfills the hypotheses will also do so.

In our next example we want to illustrate that if the hypothesis of a unique C_k in Eq. (3·3) is not fulfilled, the uncertainty product is bounded from below as predicted by the HR uncertainty relation (2·3). We consider a symmetric family of Fourier coefficients but we set C_0 to zero. As a consequence, there are two coefficients with the same decay as a function of α, and the dominance condition is no longer fulfilled by the family. So, according to Theorem 1, the uncertainty product cannot be made arbitrarily small.

In the following, we shall consider

$$C_n = |n|^{-\alpha} \,, \qquad n \neq 0$$

and $C_0 = 0$. If $\alpha \gg 1$ and $n \neq 0$ the polynomial decay gives an upper bound for the exponential decay. Note that in such limit $|n|^{-\alpha} > \alpha^{-|n|}$.

In this case, the normalization constant is given by

$$|A|^2 = \frac{1}{2\pi \sum_n |n|^{-2\alpha}}.$$

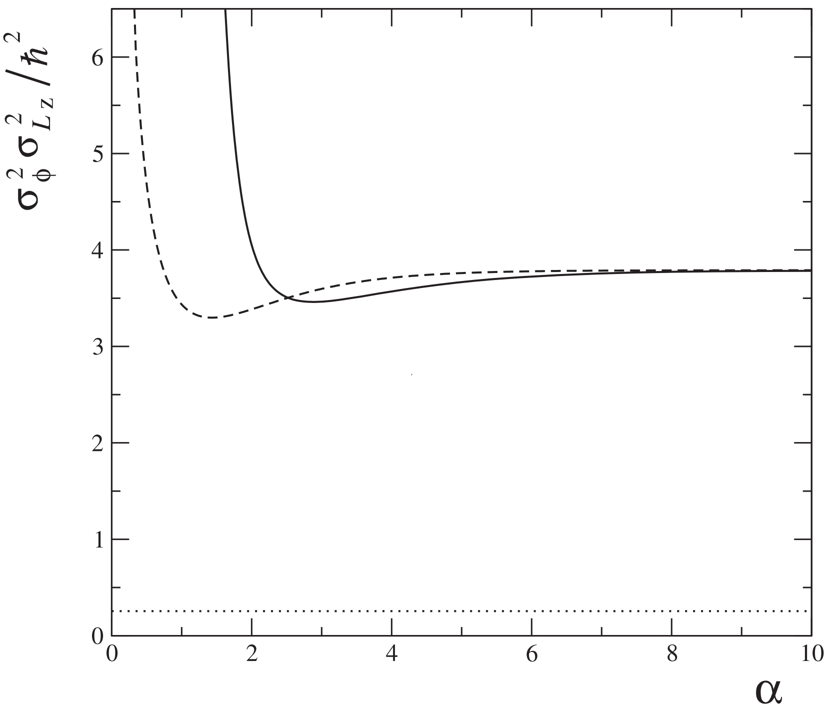

Fig. 2. The profile of $\sigma_\phi^2\sigma_{L_z}^2$ for polynomial (solid line) and exponential (short dashed line) decaying Fourier coefficients. Dashed line is the least prediction of the HR uncertainty relation.

The deviations now take the form

$$\sigma_\phi^2 = \frac{\pi^2}{3} + \frac{1}{\sum_{n\geq 1} n^{-2\alpha}} \sum_{m\neq n} |n|^{-\alpha}|m|^{-\alpha}\frac{(-1)^{(n-m)}}{(n-m)^2},$$

$$\sigma_{L_z}^2 = \frac{\hbar^2}{\sum_{n\geq 1} n^{-2\alpha}} \sum_{n\geq 1} n^{-2(\alpha-1)}.$$

In order to have σ_{L_z} finite α must be bigger than $3/2$, which guarantees that $|A|^2$ is larger than 0. In the limit $\alpha \to 3/2$ the deviation σ_{L_z} diverges, while σ_ϕ remains finite. The opposite situation yields:

$$\lim_{\alpha\to\infty} \sigma_\phi^2\sigma_{L_z}^2 = \left(\frac{\pi^2}{3} + \frac{1}{2}\right)\hbar^2 \approx 3.78986\hbar^2 \tag{7·1}$$

an uncertainty product larger than the least predicted value given by Eq. (2·3).

Similar results hold for the exponential decay if we set $C_0 = 0$. The profile of the uncertainty product for polynomial (solid line) and exponential (short dashed line) decays, as a function of α, are shown in Fig. 2.

§8. Proof of the main results

For convenience, and pedagogic purposes, we consider the case of symmetric Fourier coefficients $|C_n(\alpha)| = |C_{-n}(\alpha)|$. Theorem 1 states that the uncertainty product is arbitrarily small if, and only if, there is only one coefficient $C_k(\alpha)$ such that the rate $C_n(\alpha)/C_k(\alpha)$ converges to zero as α grows (dominance condition). For the symmetric case this coefficient must be

$$C_0(\alpha) = \frac{1}{2\pi A_\alpha} \int_{-\pi}^{\pi} f_\alpha(\phi) d\phi$$

which is proportional to the spacial average of f_α. $C_0(\alpha)$ is the only possibility because otherwise it would always exist at least two terms which, as a function of α, decay slower than the other coefficients. Thus, if a family of Fourier coefficient is symmetric and the spacial average of the wave function is zero, our result implies, in particular, that it is impossible to make $\sigma_\phi \sigma_{L_z}$ as small as one wishes.

We start by showing that if the assumptions in Theorem 1 are fulfilled then $\sigma_\phi \sigma_{L_z}$ can be made arbitrarily small. The uncertainty of angular momentum is given by

$$\sigma_{L_z}^2 = 2\pi\hbar^2 |A|^2 \sum_n |C_n(\alpha)|^2 n^2 \ . \tag{8·1}$$

Given $\varepsilon > 0$, we show that

$$2\pi^3\hbar^2 |A|^2 \sum_n |C_n(\alpha)|^2 n^2 < \varepsilon \tag{8·2}$$

holds for some $\alpha = \alpha(\varepsilon)$. Introducing $|d_n(\alpha)|^2 = |C_n(\alpha)|^2/|C_0(\alpha)|^2$, Eq. (8·2) is equivalent to

$$\frac{\pi^2\hbar^2}{\sum_n |d_n(\alpha)|^2} \sum_n |d_n(\alpha)|^2 n^2 < \varepsilon.$$

But since $\lim_{j\to\infty} |d_n(\alpha_j)| = 0$, for all $n \neq 0$, and the series $\sum_n |d_n(\alpha)|^2 n^2$ is uniformly convergent, by (3·4) and condition (ii), respectively, we have

$$\lim_{j\to\infty} \sum_n |d_n(\alpha_j)|^2 n^2 = \sum_n \lim_{j\to\infty} |d_n(\alpha_j)|^2 n^2 = 0. \tag{8·3}$$

Note that $\sum_n |d_n(\alpha)|^2 \geq 1$. Thus, by condition (3·3), for any $\varepsilon > 0$ there is a α^* such that

$$\frac{\pi^2\hbar^2}{\sum_n |d_n(\alpha^*)|^2} \sum_n |d_n(\alpha^*)|^2 n^2 < \varepsilon/\pi^2.$$

It follows that $\sigma_\phi^2 = \int_{-\pi}^{\pi} \phi^2 |f_\alpha(\phi)|^2 d\phi \leq \pi^2 \int_{-\pi}^{\pi} |f_\alpha(\phi)|^2 d\phi$. This implies $\sigma_\phi^2 \leq \pi^2$. Hence, from (8·1) and (8·2) we have

$$\sigma_\phi^2 \sigma_{L_z}^2 < \varepsilon,$$

and we finish the first part of the proof.

Next, we show the opposite implication. We want to show that outside our hypothesis there exists $\varepsilon > 0$ such that for all $\alpha \in (0, \infty)$

$$\sigma_\phi^2 \sigma_{L_z}^2 > \varepsilon\hbar^2$$

and the uncertainty product cannot be made arbitrarily small.

Let $k \neq 0$ be the smallest integer such that Eq. (3·3) holds, and introduce $d_n(\alpha) = C_n(\alpha)/|C_k(\alpha)|$. Here, for sake of simplicity, we assume that k is unique, in the sense that only $|d_{-k}|$ and $|d_k|$ are different from zero as $\alpha \to \infty$.

By (i) we have $\sigma_\phi^2 > \kappa$. Thus it suffices to demonstrate that $\sigma_{L_z}^2$ is bounded away from zero. To this end, we write

$$\sigma_{L_z}^2 = \frac{\hbar^2}{\sum_n |d_n(\alpha)|^2} \sum_n |d_n(\alpha)|^2 n^2.$$

We split the sum in the numerator and denominator as

$$\sum_n |d_n|^2 = 2 + \sum_{|n|\neq k} |d_n|^2$$

and note that, by condition (ii), there is $K < \infty$ independent of α such that $\sum_{|n|\neq k} |d_n|^2 \leq 2K$. Hence,

$$\sigma_{L_z}^2 \geq \frac{\hbar^2}{1+K} \left(k^2 + \sum_{n\neq k} |d_n|^2 n^2 \right) \geq \frac{\hbar^2}{1+K}$$

in view of $\sum_{n\neq k} |d_n|^2 n^2 \geq 0$ and $k \geq 1$. The uncertainty product can be bounded from below by

$$\sigma_\phi^2 \sigma_{L_z}^2 > \kappa \frac{\hbar^2}{1+K}.$$

Since K does not depend on α and $\kappa > 0$ is fixed, we can take $\varepsilon > 0$ so that $\kappa/(1+K) > \varepsilon$, concluding

$$\sigma_\phi^2 \sigma_{L_z}^2 > \varepsilon \hbar^2 \ .$$

Our result also holds for asymmetric Fourier coefficients. We do not consider it here since the arguments are the same as for the symmetric case with further technicalities.

§9. Conclusions

In conclusion, we have analyzed the uncertainty product for the azimuthal angle ϕ and its canonical conjugate moment L_z. We have provided necessary and sufficient conditions for a state to have an arbitrary small uncertainty product. These conditions are related to the existence of a Fourier coefficient of f_α which decays slower than the others Fourier modes. More precisely, a state allows for an arbitrary small uncertainty product if, and only if, there is only one coefficient $C_k(\alpha)$, such that $|C_n(\alpha)|/|C_k(\alpha)| = 0$ for some α (the dominance condition).

Our results concern the behavior of the uncertainty product for large α, that is, when the uncertainty product becomes arbitrarily small. It would be interesting to analyze for certain parametrized families the whole profile of the uncertainty product, either by means of the Fourier analysis, as we have done, or by an explicit probability density.

Acknowledgements

We would like to thank W. F. Wreszinski for enlightening discussions concerning the problem. We also thank F. Gieres, S. Tanimura and R. Jackiw for pointing out some useful references on the topic. We are in debt to Dr. A. Veneziani for his careful and critical reading of the manuscript. We are grateful to the peer referees whose criticism and pertinent suggestions helped to improve the manuscript. T. P. was partially supported by FAPESP, Project 07/04579-2.

Appendix A
—— *Estimation of σ_ϕ^2 for $\alpha \ll 1$* ——

Proceeding the variable change $k = n - m$ in $\xi(\alpha)$ we have

$$\xi(\alpha) = \sum_{k \neq 0} \frac{(-1)^k}{k^2} \sum_n e^{-\alpha|n|} e^{-\alpha|n-k|}.$$

This can also be written as

$$\xi(\alpha) = 2 \sum_{k \geq 1} \frac{(-1)^k}{k^2} \left[\left(\sum_{n>0} e^{-2\alpha n} + k \right) e^{-\alpha k} + \sum_{n \geq k} e^{-2\alpha n} e^{\alpha k} \right],$$

noting that $\sum_{n \geq k} e^{-2\alpha n} = e^{2\alpha} e^{-2\alpha k}/(e^{2\alpha} - 1)$, then

$$\xi(\alpha) = 2 \sum_{k \geq 1} \frac{(-1)^k}{k^2} \left(\frac{e^{2\alpha}+1}{e^{2\alpha}-1} + k \right) e^{-\alpha k}.$$

Thus, the deviation takes the form:

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 4 \sum_{k \geq 1} \frac{(-1)^k}{k^2} e^{-\alpha k} + 4 \frac{e^{2\alpha}-1}{e^{2\alpha}+1} \sum_{k \geq 1} \frac{(-1)^k}{k} e^{-\alpha k} \,. \tag{A·1}$$

Let us note that

$$\sum_{k \geq 1} \frac{(-1)^k}{k} e^{-\alpha k} = -\ln\left(1 + e^{-\alpha}\right) \tag{A·2}$$

since the series converges absolutely for $\alpha > 0$ and the sum can be performed before the integral. Also the second term in Eq. (A·1) can be written in a closed form by using the dilogarithm function $\mathrm{Li}_2(z) = \sum_{k>0} z^k/k^2$. Hence, the deviation can be written as

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 4\mathrm{Li}_2(-e^{-\alpha}) + g(\alpha), \tag{A·3}$$

where

$$g(\alpha) = -4 \frac{e^{2\alpha}-1}{e^{2\alpha}+1} \ln\left(1 + e^{-\alpha}\right).$$

To obtain the asymptotic behavior for $\alpha \ll 1$, note that the expansion in power of $\alpha \ll 1$ up to third order gives

$$g(\alpha) = -4\alpha \ln 2 + 2\alpha^2 + O(\alpha^3). \tag{A·4}$$

The expansion of the dilogarithm function in power of α up to order 3 is given by

$$\mathrm{Li}_2(-e^{-\alpha}) = \frac{-\pi^2}{12} + \alpha \ln 2 - \frac{\alpha^2}{4} + O(\alpha^3)\ . \tag{A·5}$$

Replacing Eqs. (A·4) and (A·5) in Eq. (A·3) it yields

$$\sigma_\phi^2 = \alpha^2 + O\left(\alpha^3\right),$$

which dictates the behavior of the product $\sigma_\phi \sigma_{L_z}$ as $\alpha \to 0$, as can be seen in Fig. 1.

References

1) W. Heisenberg, Z. Phys. **43** (1927), 172.
2) H. P. Robertson, Phys. Rev. **34** (1929), 163.
3) A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47** (1935), 777.
4) S. Massar and P. Spindel, Phys. Rev. Lett. **100** (2008), 190401.
5) M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Science **269** (1995), 198.
6) A. L. Schawlow and C. H. Townes, Phys. Rev. **112** (1958), 1940.
7) M. Salerno, M. R. Samuelsen and A. V. Yulin, Phys. Rev. Lett. **86** (2001), 5397.
8) P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40** (1968), 411.
9) R. Jackiw, J. Math. Phys. **9** (1968), 339.
10) P. Jordan, Z. Phys. **44** (1927), 1.
11) J. Unffink, "Measures of Uncertainty and the Uncertainty Principle", Ph.D. thesis, Utrecht University, Netherlands, 1990.
12) S. Tanimura, Prog. Theor. Phys. **90** (1993), 271.
13) G. B. Folland and A. Sitaram, J. Four. Anal. Appl. **3** (1997), 207.
14) S. Franke-Arnold, S. M. Barnett, E. Yao, J. Leach, J. Courtial and M. J. Padgett, New J. Phys. **6** (2004), 103.
15) D. Pegg, S. M. Barnett, R. Zambrini, S. Franke-Arnold and M. Padgett, New J. Phys. **7** (2005), 62.
16) K. Kraus, Z. Phys. **188** (1965), 374.
17) D. Deutsch, Phys. Rev. Lett. **50** (1983), 631.
18) H. Massen and J. B. M. Uffink, Phys. Rev. Lett. **60** (1988), 1103.
19) R. W. Finkel, Phys. Rev. A **35** (1987), 1486.
20) I. Bialynicki-Birula, Phys. Lett. A **103** (1984), 253.
21) J-M. Lévy-Leblond, Ann. of Phys. **101** (1976), 319.
22) A. S. Holevo, Rep. Math. Phys. **16** (1979), 385.
23) S. Barnett and D. T. Pegg, Phys. Rev. A **41** (1990), 3427.
24) F. Gieres, Rep. Prog. Phys. **63** (2000), 1893.
25) W. H. Louisell, Phys. Lett. **7** (1963), 60.
26) D. Judge and J. T. Lewis, Phys. Lett. **5** (1963), 190.
27) B. Simon and M. Read, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness* (Academic Press, 1975).
28) G. P. Tolstov, *Fourier Series* (Dover Publications Inc., 1972).