

## Asymptotic integral kernel for ensembles of random normal matrices with radial potentials

Alexei M. Veneziani,<sup>1,a)</sup> Tiago Pereira,<sup>1,b)</sup> and Domingos H. U. Marchetti<sup>2,c)</sup>

<sup>1</sup>*Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Av. dos Estados 5001, Santo André, SP, Brasil*

<sup>2</sup>*Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315 São Paulo, SP, Brasil*

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The method of steepest descent is used to study the integral kernel of a family of normal random matrix ensembles with eigenvalue distribution  $P_N(z_1, \dots, z_N) = Z_N^{-1} e^{-N \sum_{i=1}^N V_\alpha(z_i)} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2$ , where  $V_\alpha(z) = |z|^\alpha$ ,  $z \in \mathbb{C}$  and  $\alpha \in ]0, \infty[$ . Asymptotic formulas with error estimate on sectors are obtained. A corollary of these expansions is a scaling limit for the  $n$ -point function in terms of the integral kernel for the classical Segal–Bargmann space. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.3688293>]

### I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The investigation of non-Hermitian random matrices, whose elements are independent complex Gaussian variables without any constraint, began with the work of Ginibre.<sup>1</sup> Applying the theory of Haar measure to the group  $GL(N, \mathbb{C})$  of  $N \times N$  complex matrices, the joint probability distribution of the eigenvalues has shown to be given by (1.2) with  $V(z) = |z|^2$  and the eigenvalue density in the complex plane, defined by

$$\int_A \rho_N(z) d^2z = \frac{1}{N} \mathbb{E} (\# \{\text{eigenvalues in } A\})$$

for any Borel set  $A \subset \mathbb{C}$ , where  $\mathbb{E}(\cdot)$  is the expectation with respect to  $P_N$ , has been shown to converges to the so-called circular law

$$\rho(z) = \begin{cases} \frac{1}{\pi} & \text{if } |z| \leq 1 \\ 0 & \text{otherwise} \end{cases}. \quad (1.1)$$

Chau and Yue<sup>2</sup> have subsequently introduced ensembles of random normal matrices in the context of the quantum Hall problem of  $N$  electrons in a strong magnetic field, opening a new front of research in the area of random matrices (see also Ref. 3 for further developments on related subjects). Since normal matrices are unitarily equivalent to a diagonal matrix, the probability distribution of eigenvalues for random normal ensembles can be achieved, exactly as in the Hermitian ensembles, by choosing an appropriated coordinate system that factorizes the eigenvalues contribution from the rest (see, respectively, Sec. 5.3 of Refs. 4 and 5 for the Hermitian and normal ensembles).

Normal ensembles differ from the Hermitian counterpart by the statistical dependence of matrix elements even for Gaussian ensembles and, most importantly, by the fact that their eigenvalues are generically complex. Among the usual questions concerning the statistics of their eigenvalues there

<sup>a)</sup>Electronic mail: [alexei.veneziani@ufabc.edu.br](mailto:alexei.veneziani@ufabc.edu.br).

<sup>b)</sup>Electronic mail: [tiago.pereira@ufabc.edu.br](mailto:tiago.pereira@ufabc.edu.br).

<sup>c)</sup>Temporary address: Mathematics Department, The University of British Columbia, Vancouver, BC, Canada V6T 1Z2. Electronic mail: [marchett@math.ubc.ca](mailto:marchett@math.ubc.ca).

are some related with universality that remain unresolved for the normal ensembles. According to the theory of random matrices, the eigenvalue correlations in Hermitian, and normal ensembles as well, are given by the determinant of an integral kernel whose asymptotic behavior for large  $N$  governs their decay. The limit integral kernel is well known to be universal for standard models of Hermitian ensembles (see Ref. 4 and references therein). The scenery for normal ensembles, despite of certain efforts in this direction (see, e.g., Refs. 5–9, for recent progress towards a Riemann–Hilbert approach), remains undisclosed.

The present work addresses the integral kernel of ensembles of normal matrices weighted by  $e^{-NV}$  with  $V$  depending only on the absolute value of eigenvalues. We apply the steepest descent method to obtain scaling limits for the integral kernel, including the first error term in power of  $1/N$ . Our results, regarding the bulk universality in the strong (or maximal) non-Hermiticity regime, can be extended for a large class of radial symmetric potentials  $V$  satisfying condition (1.3) but we shall restrict ourselves to a sub class of potentials (1.7), for simplicity. Although the  $n$ -point correlation functions are currently known to be asymptotic, in the bulk, to the Ginibre correlations for a rather large class of models (see, e.g., Refs. 5, 7, and 8), no asymptotic expansions in sectors for the integral kernel have been provided until now for models with radial symmetry.

The eigenvalue probability distribution of the ensemble of random normal matrices is given by

$$P_N(z_1, \dots, z_N) = Z_N^{-1} e^{-N \sum_{i=1}^N V(z_i)} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \quad (1.2)$$

with potentials  $V : \mathbb{C} \rightarrow \mathbb{R}$  satisfying the properties: (i)  $V$  is continuous and (ii)

$$\lim_{|z| \rightarrow \infty} \left( \frac{V(z)}{2} - \log z \right) = \infty \quad (1.3)$$

to avoid the eigenvalues escape to infinity (see, e.g., Saff and Totik<sup>10</sup>). Equation (1.2) can be written as

$$P_N(z_1, \dots, z_N) = \frac{1}{N!} \det \left( K_N(z_i, z_j) \right)_{i,j=1}^N \quad (1.4)$$

with  $K_N$  being the (Cristoffel–Darboux) integral kernel

$$K_N(z, w) = e^{-\frac{N}{2} V(z)} e^{-\frac{N}{2} \overline{V(w)}} \sum_{j=1}^N \phi_j(z) \overline{\phi_j(w)}, \quad (1.5)$$

where  $\{\phi_j\}_{j=1}^N$  is the set of the orthonormal polynomials with respect to the inner product  $(\cdot, \cdot)_{v_N}$  with weight

$$dv_N(z) = e^{-NV(z)} d^2z,$$

absolutely continuous with respect to Lebesgue measure  $d^2z$  on  $\mathbb{C} \approx \mathbb{R}^2$ , and the  $n$ -point correlation function associated to  $P_N$  can be written as

$$R_n^N(z_1, \dots, z_n) = \det \left( K_N(z_i, z_j) \right)_{i,j=1}^n. \quad (1.6)$$

The statistics of the eigenvalues are thus given by the asymptotic behavior of the integral kernel.

The main result of this paper is as follows.

**Theorem 1.1:** *Let*

$$V_\alpha(z) = |z|^\alpha, \quad \alpha > 0 \quad (1.7)$$

*be a family of radially symmetric potentials,*

$$S(\tau, K) = \left\{ \zeta \in \mathbb{C} : 0 < |\zeta| < K, \quad |\arg \zeta| < \frac{\tau}{2} \right\}$$

*be a sectorial domain (on the Riemann surface of the logarithm) of opening  $\tau$  and radius  $K$  and, for each  $0 < \delta < 1$ , let  $\gamma = \gamma(\alpha, \delta)$  be such that*

$$\alpha\gamma + \delta = 1.$$

Let  $K_N^\alpha$  stand for the integral kernel (1.5) when  $V = V_\alpha$ . Then

$$\frac{1}{N^{\delta+2\gamma}} K_N^\alpha \left( \frac{Z}{N^\gamma}, \frac{W}{N^\gamma} \right) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N^\delta \left( (Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} \left( 1 + E_N^{\alpha,\delta}(Z\bar{W}) \right) \tag{1.8}$$

holds with

$$\left| E_N^{\alpha,\delta}(\zeta) \right| \leq O(N^{-\delta/2}) \tag{1.9}$$

whenever  $\zeta \in S \left( \theta/\sqrt{N}, (2/\alpha)^{2/\alpha} N^{2(1-\delta)/\alpha} \right)$  for any  $\theta \geq 0$  and large enough  $N$ .

In particular, taking  $\delta \nearrow 1$  and, consequently,  $\gamma \searrow 0$  we obtain

$$\frac{1}{N} K_N^\alpha(Z, W) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N \left( (Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} \left( 1 + E_N^{\alpha,1}(Z\bar{W}) \right) \tag{1.10}$$

with  $O(1/\sqrt{N})$  error for  $Z\bar{W} \in S \left( \theta/\sqrt{N}, (2/\alpha)^{2/\alpha} \right)$ .

*Remark 1.2:* The parameter  $\delta < 1$  has been introduced to ensure that the eigenvalues are “sampling” in the bulk, out of any fixed compact domain containing the origin. The case of interest for applications is the limit point  $\delta = 1$ . The limit, as  $N$  goes to infinity, of any function involving the asymptotic expression (1.10) is called **bulk scaling limit** of that function. The weight  $e^{-NV_\alpha(x)}$ , with  $\alpha$  an even positive integer and  $x$  a real number, are called **Freud weight** in the literature on Hermitian ensembles and corresponding orthogonal polynomials.

*Remark 1.3:* The restriction to a sector  $S(\tau, K)$  of opening  $\tau$  that shrinks with  $1/\sqrt{N}$  is an artifact of our method. Equation (1.8) is expected to hold for  $Z\bar{W} \in S(\tau, K)$ , with  $K = K(\tau) > 0$  for  $0 \leq \tau < 4\pi/\alpha$ , but our estimates on the error for replacing a sum by an integral, giving by the Euler–Maclaurin sum formula, break down except for sectors  $S(\theta N^{-\beta}, K)$  with  $\theta \geq 0$  and  $\beta \geq 1/2$  (see (4.33) and following equations). Numerical calculations performed in Ref. 11 for  $\alpha \geq 2$  indicate that (1.8) might hold for  $Z\bar{W} \in S(4\pi/\alpha, K)$  with an error decaying faster than any power of  $N$  for some  $K < 1$  (see also the next remark for an improved and simple estimate for  $\alpha = 2$ ). There, a different error

$$\sup_{|z|, |w| < (2/\alpha)^{1/\alpha}; |\arg(z\bar{w})| < 2\pi/\alpha} \left| \frac{\alpha^2}{4\pi} (z\bar{w})^{\frac{\alpha}{2}-1} e^{N^\delta \left( (z\bar{w})^{\frac{\alpha}{2}} - \frac{|z|^\alpha}{2} - \frac{|w|^\alpha}{2} \right)} E_N^{\alpha,1}(z\bar{w}) \right|$$

denoted by  $R_N^\alpha$ , has been considered.

*Remark 1.4:* Taylor remainder formula can be used to estimate the difference between the Taylor polynomial  $S_N$  and the function  $f_N$ , respectively defined by (3.3) with  $\delta = 1$  (see also (4.4)) and by the infinite sum with the same summand. For  $\alpha = 2$ ,  $f_N(\zeta) = N\zeta e^{N\zeta}/\pi$ . By (3.2), together with the Lagrange remainder, one gets (1.10) with the error estimator function satisfying  $|E_N(\zeta)| = O(N^{-1/2} (e|\zeta|)^N e^{-N(1-a)\Re\zeta})$ , for some  $0 < a < 1$  and large enough  $N$  (see calculations in Appendix A). We observe that (1.10) with  $\alpha = 2$  holds with  $\sup_{\zeta \in \tilde{S}(\tau, K)} |E_N(\zeta)| = O(1/\sqrt{N})$  for  $\zeta = Z\bar{W}$  in a sectorial domain  $S(\tau, K)$  with  $K = K(\tau, a) > 0$  given by smallest solution of  $Ke^{-(1-a)K\cos(\tau/2)} + 1 = 1$ .

*Remark 1.5:* The asymptotic behavior (1.10) for  $\alpha = 2$ , without the first error term, was established in Ref. 6. See Lemma 1.2 of Ref. 8 and references therein for a local estimate on the difference between the integral kernel and its scaling limit for real-analytic potentials  $V(z)$  in a neighborhood of a point  $z_0$  inside the droplet (i.e., in the bulk).

It follows from Eqs. (1.10) and (1.6) that normal ensembles with the class of potentials  $V_\alpha$  are universal alike the Hermitian ensembles (see, e.g., Subsection 5.6.1 of Refs. 4 and 12, for recent results):

*Corollary 1.6:* Let  $r, z_1, \dots, z_n$  be  $n + 1$  complex numbers and write

$$Z_i = r + \frac{z_i}{\sqrt{\pi} K_N^\alpha(r, r)}. \quad (1.11)$$

Then, the following scaling limit for the  $n$ -point function

$$\lim_{N \rightarrow \infty} \frac{1}{\pi^n K_N^\alpha(r, r)^n} R_n^N(Z_1, \dots, Z_n) = \det(\mathbb{K}(z_i, z_j))_{i,j=1}^n \quad (1.12)$$

holds uniformly for  $r$  in any compact set of the open set  $\{z \in \mathbb{C} : 0 < |z| < (2/\alpha)^{1/\alpha}\}$ , where

$$\mathbb{K}(z, w) = \frac{1}{\pi} e^{\left(z\bar{w} - \frac{|z|^2}{2} - \frac{|w|^2}{2}\right)} \quad (1.13)$$

is the integral kernel for the classical Segal–Bargmann space of entire functions. The bulk scaling limit (1.12) is universal in the sense that it is independent of the family of potentials  $V_a$ .

We shall address this and other issues related with the conformal invariance of the integral kernel (1.5) in a forthcoming paper.<sup>13</sup> Since the cancellations involved make the implication of (1.12) far from being straightforward, a complete, although short, proof has been included in Appendix B.

For  $n = 2$ , (1.12) reads

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{\pi^2 K_N^\alpha(r, r)^2} R_n^N(Z_1, Z_2) &= (\mathbb{K}(z_1, z_1)\mathbb{K}(z_2, z_2) - \mathbb{K}(z_1, z_2)\mathbb{K}(z_2, z_1)) \\ &= \frac{1}{\pi^2} \left(1 - e^{-|z_1 - z_2|^2}\right), \end{aligned} \quad (1.14)$$

a result already obtained for certain radial potentials (see Theorem 1 of Ref. 5) and for a large class of potentials (see Theorem 1.1 of Ref. 7). For any  $n \in \mathbb{N}$ , (1.12) has been established for real-analytic potentials  $V$  (see Proposition 7.4 of Ref. 8).

Under the assumption that (1.10) can be extended to the sectorial domain  $S(4\pi/\alpha, K)$  (this actually holds for  $\alpha = 2$ . See Appendix A), a change of variables in the integral Kernel by the function  $\varphi_N(z) = (z/\sqrt{N})^{2/\alpha}$ , which maps conformally  $\{|z| < K^{\alpha/2} N^{1/2}\}$  into  $S(4\pi/\alpha, K)$ , would yields

$$\lim_{N \rightarrow \infty} \varphi_N'(z) K_N^\alpha(\varphi_N(z), \varphi_N(w)) \overline{\varphi_N'(w)} = \mathbb{K}(z, w), \quad (1.15)$$

where  $\mathbb{K}(z, w)$  is the integral kernel given by (1.13). This notion of universality has been called conformal universality in Ref. 11. The estimates in Appendix A establish the pointwise limit (1.15) in  $\mathbb{C} \times \mathbb{C}$  for  $\alpha = 2$ .

Theorem 1.1 will be proven in Sec. IV. Sections II and III contain preliminary materials. The technical part of our result concerns with the error estimation of Euler–Maclaurin formula. Different methods needs to be employed depending on the regions considered in the sum. Appendix A estimates the Taylor remainder of (3.3) for  $\delta = 1$  and  $\alpha = 2$  and Appendix B proves Corollary 1.6.

## II. ENSEMBLE OF RANDOM NORMAL MATRICES

We begin with the following:

*Definition 2.1:* By normal ensembles we mean a probability measure

$$P(M_N) dM_N = Z_N^{-1} e^{-N \text{Tr} V(M_N)} dM_N \quad (2.1)$$

on the set of  $N \times N$  complex matrices  $M_N$  supported on the variety  $[M_N, M_N^*] = 0$  and invariant by unitary conjugation  $\tilde{M}_N = U_N^* M_N U_N$ :

$$P(M_N) dM_N = P(\tilde{M}_N) d\tilde{M}_N. \quad (2.2)$$

The elements  $m_{ij} = m_{ij}^R + i m_{ij}^I$ ,  $1 \leq i \leq j \leq N$  of  $M_N$  in the normal ensemble cannot be picked independently according to any product measure, absolutely continuous with respect to the

Lebesgue measure  $\prod_{1 \leq i \leq j \leq N} dm_{ij}^R dm_{ij}^I$  in  $\mathbb{R}^{N^2+N}$ , even when the weight  $e^{-N\text{Tr}V(M_N)}$  is Gaussian, in view of the constraint on elements  $m_{ij}$  with  $i > j$ .<sup>14</sup> So, the elements of  $M_N$  when sampling on normal ensembles are always statistically dependent. Note that the set of normal matrices whose eigenvalues have multiplicity 1 is open, dense in  $\mathbb{R}^{N^2+N}$  and has full measure (see Ref. 4 for a proof in the Hermitian ensembles).

As  $M_N$  is normal,  $M_N$  is unitarily equivalent to a diagonal matrix of eigenvalues and there exist  $U_N$  satisfying  $U_N^{-1} = U_N^*$  and

$$M_N = U_N \Lambda_N U_N^* \quad (2.3)$$

with  $\Lambda_N = \text{diag}\{z_1, \dots, z_N\}$ , ordered according to their absolute value:  $|z_i| \leq |z_j|$  if  $i < j$ . Following Sec. 5.3 of Ref. 4 with few adjustments (see Refs. 5 and 17), the spectral decomposition (2.3) considered as a change of variables  $M_N \xrightarrow{\varphi} (\Lambda_N, U_N \text{ mod } \mathbb{T}^N)$  yields

$$P(M_N) dM_N = Z^{-1} e^{-N \sum_{i=1}^N V(z_i)} J(z, p) \prod_{1 \leq i \leq N} d^2 z_i \prod_{1 \leq j \leq l} d^2 p_k, \quad (2.4)$$

where  $\{p_i\}_{i=1}^l$  with  $2l + N = N^2$ , are variables associated with the eigenvectors of  $M$ ,  $d^2 z$  denotes the Lebesgue measure on  $\mathbb{C}$  and

$$J(z, p) = \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 f(p)$$

is the Jacobian of  $\varphi$ , with  $f$  a function depending only on the eigenvectors variables  $\{p_i\}_{i=1}^l$ . The eigenvalue probability distribution (2.1) of this ensemble is obtained integrating (2.4) with respect to  $\{p_i\}_{i=1}^l$ .

The  $n$ -point correlation function is defined by (see, e.g., Ref. 15)

$$R_n^N(z_1, \dots, z_n) = \frac{N!}{(N-n)!} \int P_N(z_1, \dots, z_N) \prod_{i=n+1}^N d^2 z_i \quad (2.5)$$

and it can be written as (1.6). Stochastic processes of this form are called random determinantal point fields.<sup>16</sup> The present work concerns with the asymptotic analysis of the integral kernel (1.5) and its implications to the limit of the  $n$ -point correlation function. We have seen that the limit of the 2-point correlation (1.14) can be read directly from the asymptotic formula (1.10). The eigenvalue density  $\rho^{V_\alpha}$ , associated with the normal ensemble defined by  $V_\alpha$ , is by (1.10) given by

$$\rho^{V_\alpha}(z) = \lim_{N \rightarrow \infty} \frac{1}{N} R_1^N(z) = \lim_{N \rightarrow \infty} \frac{1}{N} K_N^\alpha(z, z) = \frac{\alpha^2}{4\pi} |z|^{\alpha-2} \quad (2.6)$$

for  $|z| \leq (2/\alpha)^{1/\alpha}$  (see Remark 3.4, for more comment on this). This, together with Theorem 4.1 of Ref. 17 and the uniqueness of the equilibrium measure, implies that  $\rho^{V_\alpha}(z) d^2 z$  and the equilibrium or extremal measure  $d\sigma(z)$  (the infimum over all compactly supported Borel probability measures  $\mu$  on  $\mathbb{C}$  of the energy  $I(\mu)$  associated with a charge distribution  $\mu$ , in the presence of an external potential  $V_\alpha$ ) are the same,

$$d\sigma(z) = \frac{1}{\pi} (\Delta \hat{V}_\alpha)(z) d^2 z = \frac{1}{\pi} (\partial_z \partial_{\bar{z}} V_\alpha)(z) \mathbf{1}_{|z| \leq (2/\alpha)^{1/\alpha}}(z) d^2 z = \frac{\alpha^2}{4\pi} |z|^{\alpha-2} d^2 z,$$

where  $\hat{V}_\alpha$  is the upper envelop of  $V_\alpha$  by subharmonic functions  $f$  of (at most) logarithmic growth at infinity,  $f(z) \leq \log_+ |z|^2 + C$  for some  $C < \infty$ , and the circular domain  $\{|z| \leq (2/\alpha)^{1/\alpha}\} = \{\hat{V}_\alpha(z) = V_\alpha(z)\}$  (see, e.g., the paragraph “the droplet” of Ref. 8 and references therein) is determined by  $2r^{-1} = \alpha r^\alpha - 1$ , i.e., the tangency point  $r^*$  of  $2\log_+ r + C_\alpha$  and  $r^\alpha$ ,  $r \geq 0$ .

### III. INTEGRAL KERNEL OF NORMAL ENSEMBLES DEFINED BY $V_\alpha$ AND VARIOUS ESTIMATES

The present section is devoted to preliminary results on the integral kernel (1.5).

Let  $L^2(\mathbb{C}, \nu)$  denote the Hilbert space of square-integrable complex-valued functions

$$\|f\|_\nu^2 = \int_{\mathbb{C}} |f(z)|^2 d\nu(z) < \infty$$

with respect to a positive finite Borel measure  $\nu$  on  $\mathbb{C}$  which, in order to ensure that all analytic polynomials of Ref. 18 belong to the space is assumed to satisfy

$$\int_{\mathbb{C}} |z|^{2n} d\nu(z) < \infty, \quad n \in \mathbb{N}.$$

If  $P_N(\mathbb{C}, \nu)$  denotes the  $N$ -dimensional linear vector space of analytic polynomials of degree less than or equal  $N - 1$ , endowed with the inner product

$$(p, q)_\nu = \int_{\mathbb{C}} \overline{p(z)} q(z) d\nu(z), \quad (3.1)$$

we have

*Proposition 3.1:* For each  $N \in \mathbb{N}$ , the monomials

$$\phi_j^\alpha(z) = \sqrt{\frac{\alpha}{2\pi\Gamma(2j/\alpha)}} N^{j/\alpha} z^{j-1}$$

with  $j = 1, \dots, N$ , form an orthonormal set in  $P_N(\mathbb{C}, \nu_N^\alpha)$  with respect to

$$d\nu_N^\alpha(z) = e^{-N|z|^\alpha} d^2z, \quad \alpha > 0.$$

The integral kernel (1.5) reads in this case

$$K_N^\alpha(z, w) = e^{-\frac{N}{2}|z|^\alpha} e^{-\frac{N}{2}|w|^\alpha} \tilde{K}_N^\alpha(z, w), \quad (3.2)$$

where

$$\tilde{K}_N^\alpha(z, w) = \frac{\alpha}{2\pi} \sum_{j=1}^N \frac{N^{2j/\alpha} (z\bar{w})^{j-1}}{\Gamma(2j/\alpha)} \quad (3.3)$$

is a reproducing kernel on  $P_N(\mathbb{C}, \nu_N^\alpha)$ .

*Remark 3.2:* For the Bergman space  $A^2(\Omega)$  of square-integrable single-valued analytic functions on a compact domain  $\Omega$ , there always exist a complete set of orthonormal polynomials  $\{\phi_j(z)\}_{j=1}^\infty$

and the sum  $\sum_{j=1}^N \phi_j(z)\overline{\phi_j(w)}$  converges, as  $N$  goes to infinity, to the integral kernel

$$\tilde{K}(z, w) = \sum_{j=1}^\infty \phi_j(z)\overline{\phi_j(w)},$$

uniformly for  $z, w$  in  $\Omega$ .<sup>19</sup> This is not necessarily the case for an unbounded domain but the same properties hold for Segal–Bargmann spaces  $A^2(\mathbb{C}; \nu)$  of single-valued analytic functions in  $\mathbb{C}$ , square-integrable with respect to  $e^{-|z|^2} d^2z$ . We call the reader's attention to the  $N$  dependence on the inner product (3.1) and the fact that this dependence affects also (3.2). As one sees from (1.10), together with

$$\frac{|z|^\alpha}{2} + \frac{|w|^\alpha}{2} - \Re(z\bar{w})^{\alpha/2} = \frac{1}{2} |z^{\alpha/2} - w^{\alpha/2}|^2 \geq 0,$$

(equality if and only if  $z = w$ ) and Eq. (2.6),  $K_N^\alpha(z, w)$  goes to 0 for  $z \neq w$  and diverges for  $z = w$ , as  $N \rightarrow \infty$ .

We shall use (3.3) to obtain an asymptotic expression as stated in Theorem 1.1.

*Proof of Proposition 3.1:* We need to verify that the monomials are orthogonal with respect to the inner product (3.1). Writing

$$\phi_j(z) = \frac{z^{j-1}}{\sqrt{2\pi I_j}}$$

with  $z = re^{i\theta}$ , we have

$$\begin{aligned} (\phi_k(z), \phi_j(z))_{v_N^\alpha} &= \frac{1}{2\pi \sqrt{I_k I_j}} \int \overline{z^{k-1}} z^{j-1} e^{-N|z|^\alpha} d^2z \\ &= \frac{1}{\sqrt{I_k I_j}} \int_0^\infty r^{k+j-1} e^{-Nr^\alpha} dr \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(j-k)} d\theta = \delta_{k,j}, \end{aligned}$$

with the Kronecker delta function  $\delta_{k,j} = 1$  if  $k = j$  and 0 otherwise, provided

$$I_j = \int_0^\infty r^{2j-1} e^{-Nr^\alpha} dr = \frac{N^{-2j/\alpha}}{\alpha} \Gamma\left(\frac{2j}{\alpha}\right).$$

Consequently, any analytic polynomial  $p(z)$  in  $P_N(\mathbb{C}, v_N^\alpha)$  can be written as

$$p(z) = \sum_{j=1}^N c_j \phi_j(z) \tag{3.4}$$

with Fourier coefficients

$$c_j = (\phi_j, p)_{v_N^\alpha} = \int_{\mathbb{C}} \overline{\phi_j(w)} p(w) e^{-N|w|^\alpha} d^2w. \tag{3.5}$$

Inserting (3.5) into (3.4), gives  $p(z) = \left(\tilde{K}_N^\alpha(z, \cdot), p\right)_{v_N^\alpha}$  where

$$\tilde{K}_N^\alpha(z, w) = \sum_{j=1}^N \phi_j(z) \overline{\phi_j(w)} = \frac{\alpha}{2\pi} \sum_{j=1}^N \frac{N^{2j/\alpha} (z\bar{w})^{j-1}}{\Gamma(2j/\alpha)}. \tag{3.6}$$

□

Looking for an asymptotic expansion of (3.2), a complex valued function is defined on the positive real line  $\mathbb{R}_+ = (0, \infty)$  coinciding with the summand of the integral kernel (3.6) on  $\mathbb{N}$ . For fixed numbers  $\alpha > 0, 0 < \delta < 1, \zeta \in \mathbb{C} \setminus \{0\}$  and  $N$  a positive integer, let  $g_\zeta : \mathbb{R}_+ \rightarrow \mathbb{C}$  be given by

$$g_\zeta(x) = \frac{\left(N^{\frac{2\delta}{\alpha}} \zeta\right)^x}{\Gamma(2x/\alpha)}, \tag{3.7}$$

where  $\zeta^x = e^{x \log \zeta}$  is determined by the logarithm  $\log \zeta = \log r + i\theta, \zeta = re^{i\theta}, r \geq 0$  and  $-\pi < \theta < \pi$ , defined in the slit plane cut along the negative real axis from the origin to infinity. Note that  $|g_\zeta(x)| = g_{|\zeta|}(x)$ .

*Lemma 3.3:* Under the above conditions on  $\alpha, \delta, \zeta$ , and  $N$ , the real valued function  $g_{|\zeta|} : \mathbb{R}_+ \rightarrow \mathbb{R}$  has a global maximum

$$g_{|\zeta|}(x) \leq \max_{x \geq 0} g_{|\zeta|}(x) = g_{|\zeta|}(x^*)$$

at  $x^* = x^*(\alpha, \delta, |\zeta|, N) > 0$ . For  $N$  large enough so that  $N > N_0$ ,

$$N_0 = \max \left( \left(\frac{k}{|\zeta|}\right)^{\frac{\alpha}{2\delta}}, \left(\frac{\alpha}{2} |\zeta|^{\frac{\alpha}{2}}\right)^{\frac{1}{1-\delta}} \right) \tag{3.8}$$

with  $k$  a large universal constant, the inequality

$$0 < x^* < N$$

holds and

$$g_{|\zeta|}(x^*) = \frac{1}{\sqrt{2\pi}} |\zeta|^{\frac{\alpha}{2}} N^{\frac{\delta}{2}} \exp(|\zeta|^{\frac{\alpha}{2}} N^{\delta}) \left(1 + O\left(\frac{1}{N^{\delta}}\right)\right) \quad (3.9)$$

$$x^* = \frac{\alpha}{2} |\zeta|^{\frac{\alpha}{2}} N^{\delta} - \frac{\alpha}{4} + O\left(\frac{1}{N^{\delta}}\right) \quad (3.10)$$

*Proof:* Differentiating  $g_{|\zeta|}(x)$  with respect to  $x$ , we have

$$g'_{|\zeta|}(x) = g_{|\zeta|}(x) \left( \log\left(N^{\frac{2\delta}{\alpha}} |\zeta|\right) - \frac{2}{\alpha} \psi\left(\frac{2}{\alpha}x\right) \right), \quad (3.11)$$

where  $\psi(x) = \Gamma'(x)/\Gamma(x)$  is the digamma function. Since  $g_{|\zeta|}(x)$  does not vanish and  $\psi(x)$  belongs to a Pick class of functions that can be analytically continued through  $\mathbb{R}_+$  (see, e.g., Ref. 20), as  $x$  varies in the semi-line  $\psi(x)$  increases monotonously from  $-\infty$  to  $\infty$  – alternatively,  $\psi(x)$  is monotone increasing since the trigamma function  $\psi_1(z) = \psi'(z) = \sum_{n=0}^{\infty} 1/(z+n)^2$  is strictly positive (see, e.g., Ref. 21) – and the maximum of  $g_{|\zeta|}$  is attained at the unique solution  $x = x^*$  of

$$\log\left(N^{\frac{2\delta}{\alpha}} |\zeta|\right) - \frac{2}{\alpha} \psi\left(\frac{2}{\alpha}x\right) = 0. \quad (3.12)$$

For  $N$  so large that the asymptotic expansion<sup>21</sup>

$$\psi(y) \sim \log y + \frac{1}{2y} - \sum_{j=1}^{\infty} B_{2j} \frac{1}{2^j y^{2j}} \quad (3.13)$$

of digamma function at  $y = N^{\frac{2\delta}{\alpha}} |\zeta|$  can be applied (i.e.,  $y > k$ , where  $k$  is the constant mention in (3.8)), we have by (3.12)

$$\log\left(N^{\delta} |\zeta|^{\frac{\alpha}{2}}\right) = \log \frac{2}{\alpha} x^* + \frac{\alpha}{4x^*} + O\left(\frac{1}{x^{*2}}\right)$$

or equivalently,

$$\frac{\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2} = x^* + \frac{\alpha}{4} + O\left(\frac{1}{x^*}\right)$$

which establishes (3.10). The coefficients  $B_{2j}$  in (3.13) are the Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

For (3.8), it suffices to solve  $\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}} / 2 \leq N$  for  $N$ . For (3.9), we plug (3.10) into  $g_{|\zeta|}(x^*)$ . As  $x^*$  is order  $N^{\delta}$  therefore is large enough for applying Stirling formula,

$$\begin{aligned} g_{|\zeta|}(x^*) &= \frac{\left(N^{\frac{2\delta}{\alpha}} |\zeta|\right)^{x^*}}{\Gamma\left(\frac{2}{\alpha}x^*\right)} \\ &= \sqrt{\frac{x^*}{\alpha\pi}} \left(\frac{\alpha e}{2x^*}\right)^{\frac{2}{\alpha}x^*} \left(|\zeta| N^{\frac{2\delta}{\alpha}}\right)^{x^*} \left(1 + O\left(\frac{1}{N^{\delta}}\right)\right) \\ &= \frac{|\zeta|^{\frac{\alpha}{2}}}{\sqrt{2\pi}} N^{\frac{\delta}{2}} e^{N^{\delta} |\zeta|^{\frac{\alpha}{2}}} \left(1 + O\left(\frac{1}{N^{\delta}}\right)\right). \end{aligned} \quad (3.14)$$

□

*Remark 3.4:* Lemma 3.3 still holds for  $\delta = 1$  provided  $0 < |\zeta| \leq (2/\alpha)^{2/\alpha}$ . Note that  $x^* = N - \alpha/4 + O(1/N) < N$  for  $|\zeta| = (2/\alpha)^{2/\alpha}$ , which defines the domain boundary of the equilibrium density



(2.6) (by (3.2), (3.6), (3.7), and (1.8))  $\zeta = Z\bar{W}$  with  $|Z|, |W| \leq (2/\alpha)^{1/\alpha}$ , and  $|\zeta|$  is less or equal the square of the this bound).

The limit  $\lim_{N \rightarrow \infty} \tilde{K}_N^\alpha(z, w)/N$  calculated at  $z\bar{w} = \zeta/N^{2/\alpha}$ , given by the series

$$(\alpha/2\pi) \sum_{j=1}^{\infty} \zeta^{j-1} / \Gamma(2j/\alpha),$$

converges uniformly in compact sets of  $\mathbb{C}$  to an entire function of  $\zeta$  of order  $\alpha/2$ , whose maximum is determined, essentially, by a single term of the series, the so-called central index  $j^* = j^*(|\zeta|)$  (see, e.g., Ref. 22). The next result estimates the range of indices  $j$  in (3.6) that contribute for its asymptotic expansion for large  $N$ .

*Lemma 3.5:* Let  $x$  be a point that is at least  $N^{\frac{\delta}{2}} \log N$  away from the global maximum (3.10) of  $g_{|\zeta|}(x)$ , that is,

$$|x - x^*| \geq N^{\frac{\delta}{2}} \log N. \tag{3.15}$$

Then,

$$g_{|\zeta|}(x) \leq \max(g_{|\zeta|}(x_+), g_{|\zeta|}(x_-)), \tag{3.16}$$

where  $x_{\pm} = x^* \pm N^{\frac{\delta}{2}} \log N$  and

$$g_{|\zeta|}(x_{\pm}) = \frac{1}{N^{2 \log N / (\alpha^2 |\zeta|^{\alpha/2})}} g_{|\zeta|}(x^*) \left(1 + O\left(\frac{\log^3 N}{N^{\delta/2}}\right)\right). \tag{3.17}$$

*Proof:* Equation (3.16) follows by uniqueness of the maximum value. For (3.17), we repeat the estimates that led to (3.14) with  $x_{\pm}$  in the place of  $x^*$ :

$$g_{|\zeta|}(x_{\pm}) = \sqrt{\frac{x_{\pm}}{\alpha\pi}} \left(\frac{e\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2x_{\pm}}\right)^{2x_{\pm}/\alpha} \left(1 + O\left(\frac{1}{N^{\delta}}\right)\right). \tag{3.18}$$

Plugging

$$x_{\pm} = \frac{\alpha}{2} |\zeta|^{\frac{\alpha}{2}} N^{\delta} \pm N^{\frac{\delta}{2}} \log N - \frac{\alpha}{4} + O\left(\frac{1}{N^{\delta}}\right)$$

into each term that appears in (3.18), yields

$$\sqrt{\frac{x_{\pm}}{\alpha\pi}} = \sqrt{\frac{N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2\pi}} \left(1 + O\left(\frac{\log N}{N^{\frac{\delta}{2}}}\right)\right),$$

$$\begin{aligned} \frac{e\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2x_{\pm}} &= e \left(1 \pm \frac{2}{\alpha |\zeta|^{\alpha/2}} \frac{\log N}{N^{\delta/2}} - \frac{1}{2 |\zeta|^{\alpha/2}} \frac{1}{N^{\delta}} + O\left(\frac{1}{N^{2\delta}}\right)\right)^{-1} \\ &= \exp\left(1 \mp \frac{2}{\alpha |\zeta|^{\alpha/2}} \frac{\log N}{N^{\delta/2}} + \frac{2}{\alpha^2 |\zeta|^{\alpha}} \frac{\log^2 N}{N^{\delta}} + \frac{1}{2 |\zeta|^{\alpha/2}} \frac{1}{N^{\delta}} + O\left(\frac{\log N}{N^{3\delta/2}}\right)\right), \end{aligned}$$

where we have used

$$\frac{e}{1 + \kappa} = \exp(1 - \log(1 + \kappa)) = \exp\left(1 - \kappa + \frac{\kappa^2}{2} + O(\kappa^3)\right)$$

and, therefore,

$$\left(\frac{e\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2x_{\pm}}\right)^{2x_{\pm}/\alpha} = \exp\left(|\zeta|^{\alpha/2} N^{\delta} - \frac{2}{\alpha^2 |\zeta|^{\alpha/2}} \log^2 N\right) \left(1 + O\left(\frac{\log^3 N}{N^{\delta/2}}\right)\right).$$

Replacing in (3.18), together with (3.9), results (3.17). □

We need one more ingredient.

*Lemma 3.6:* Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for any  $x, y \in [a, b]$  and  $0 < \lambda < 1$  and let

$$P : a = x_0 < \dots < x_K = b$$

be the partition of  $[a, b]$  into  $K$  equally spacing subintervals of length  $\Delta$ :

$$x_j = a + j\Delta, \quad j \in \{0, \dots, K\}.$$

Define  $t_j \in [x_j, x_{j+1}]$  by the mean value theorem for integration,

$$\int_{x_j}^{x_{j+1}} f(x) dx = f(t_j) \Delta. \quad (3.19)$$

Then, the error in the trapezoidal approximation to the integral

$$\Sigma(f; P) := \sum_{j=0}^{K-1} \left( \int_{x_j}^{x_{j+1}} f(x) dx - \frac{1}{2} (f(x_j) + f(x_{j+1})) \Delta \right) \quad (3.20)$$

is bounded by

$$0 \geq \Sigma(f; P) \geq \left( -\frac{f(t_0)}{2} + \frac{f(x_1)}{2} + \frac{f(x_K)}{2} - \frac{f(t_K)}{2} \right) \Delta. \quad (3.21)$$

*Proof:* Without loss of generality, we suppose that  $f$  is a positive convex function. Let  $\{k_j\}_{j=0}^{2K}$  be a numerical sequence defined by

$$\begin{aligned} k_{2j} &= \int_{x_j}^{x_{j+1}} f(x) dx, \\ k_{2j+1} &= f(x_{j+1}) \Delta \end{aligned} \quad (3.22)$$

for  $j \in \{0, \dots, K-1\}$  and note that, by the mean value theorem (3.19),

$$k_{2j} = f(t_j) \Delta \quad (3.23)$$

for some  $t_j \in [x_j, x_{j+1}]$ . We shall prove, by a geometric argument together with the convexity of  $f$ , that the following inequality:

$$k_i \leq \frac{k_{i+1} + k_{i-1}}{2} \quad (3.24)$$

holds for each  $i \in \{1, \dots, 2K-1\}$ .

Since  $f$  is convex, the inequality (3.24) for  $i = 2j$ :

$$\int_{x_j}^{x_{j+1}} f(x) dx = k_{2j} \leq \frac{k_{2j+1} + k_{2j-1}}{2} = \frac{f(x_{j+1}) + f(x_j)}{2} \Delta$$

is verified comparing the area under the function  $f$  in the interval  $[x_j, x_{j+1}]$  (left side of (3.24)) with the area of a trapezoid formed by the points  $(x_j, 0)$ ,  $(x_{j+1}, 0)$ ,  $(x_j, f(x_j))$ , and  $(x_{j+1}, f(x_{j+1}))$  (right side of (3.24) see Figure 1).

Once again, by convexity of  $f$ , the inequality (3.24) for  $i = 2j + 1$ :

$$f(x_{j+1}) \Delta = k_{2j+1} \leq \frac{k_{2j} + k_{2j+2}}{2} = \frac{1}{2} \int_{x_j}^{x_{j+2}} f(t) dt \quad (3.25)$$

can be verified comparing the area under the function  $f$  in the interval  $[x_j, x_{j+2}]$  ( $2 \times$  the right side of (3.25)) with the area of a rectangle of base in the interval  $[x_j, x_{j+2}]$  and height  $f(x_{j+1})$  ( $2 \times$  the left side of (3.25) see Figure 1).

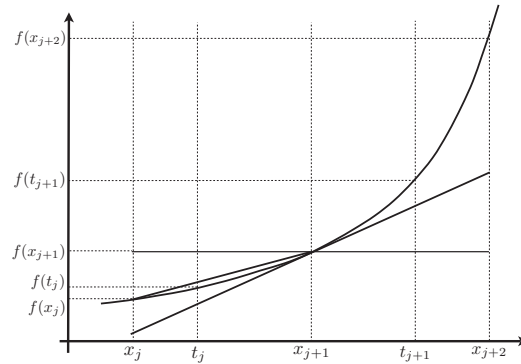


FIG. 1. Geometric representation of the argument leading to inequality (3.24).

The later assertion is facilitated if the rectangle is replaced by a trapezoid of same area obtained by rotating the horizontal segment at the top around the point  $(x_{j+1}, f(x_{j+1}))$  until it becomes tangent to the graph of  $f$  at that point.

Now let us consider the sum

$$\begin{aligned} \Sigma_1 &= \sum_{j=0}^{2K} (-1)^j k_j = k_0 - k_1 + \dots - k_{2K-1} + k_{2K} \\ &= \frac{k_0}{2} - \sum_{j=0}^{K-1} \left( k_{2j+1} - \frac{k_{2j} + k_{2j+2}}{2} \right) + \frac{k_{2K}}{2} \end{aligned} \tag{3.26}$$

$$= k_0 - \frac{k_1}{2} + \sum_{j=1}^{K-1} \left( k_{2j} - \frac{k_{2j-1} + k_{2j+1}}{2} \right) - \frac{k_{2K-1}}{2} + k_{2K}. \tag{3.27}$$

From (3.24) and (3.26), we have

$$\Sigma_1 = \frac{k_0}{2} - \sum_{j=0}^{K-1} \left( k_{2j+1} - \frac{k_{2j} + k_{2j+2}}{2} \right) + \frac{k_{2K}}{2} \geq \frac{k_0}{2} + \frac{k_{2K}}{2} \tag{3.28}$$

and from (3.24) and (3.27), we have

$$\Sigma_1 = k_0 - \frac{k_1}{2} + \sum_{j=1}^{K-1} \left( k_{2j} - \frac{k_{2j-1} + k_{2j+1}}{2} \right) - \frac{k_{2K-1}}{2} + k_{2K} \leq k_0 - \frac{k_1}{2} - \frac{k_{2K-1}}{2} + k_{2K}. \tag{3.29}$$

Since Eqs. (3.20) and (3.27) are related by the definition of  $\{k_j\}_{j=0}^{2K}$  as

$$\Sigma_1 = k_0 - \frac{k_1}{2} + \Sigma(f; P) - \frac{k_{2K-1}}{2} + k_{2K},$$

the lower (3.28) and the upper (3.29) bounds yields

$$\frac{k_0}{2} + \frac{k_{2K}}{2} \leq k_0 - \frac{k_1}{2} + \Sigma(f; P) - \frac{k_{2K-1}}{2} + k_{2K} \leq k_0 - \frac{k_1}{2} - \frac{k_{2K-1}}{2} + k_{2K}$$

or, equivalently,

$$-\frac{k_0}{2} + \frac{k_1}{2} + \frac{k_{2K-1}}{2} - \frac{k_{2K}}{2} \leq \Sigma(f; P) \leq 0$$

which, in view of definitions (3.22) and (3.23), concludes the proof of lemma.  $\square$

*Remark 3.7:* The ideas of this proof are based on an argument used to establish the phenomenon of Fresnel diffraction (see, e.g., Ref. 24).

*Corollary 3.8:* Let  $f : [a, b] \rightarrow \mathbb{R}$  be a concave function:

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x, y \in [a, b]$  and  $0 < \lambda < 1$  and let  $P, (t_j)_{j=1}^K$  and  $\Sigma(f; P)$  be as in the previous lemma. Then

$$0 \leq \Sigma(f; P) \leq \left( -\frac{f(t_0)}{2} + \frac{f(x_1)}{2} + \frac{f(x_K)}{2} - \frac{f(t_K)}{2} \right) \Delta x.$$

We are now in position to prove Theorem 1.1.  $\square$

#### IV. PROOF OF THEOREM 1.1

We shall proceed the asymptotic analysis applying the steepest descent method to the integral kernel (3.2). For this we assume  $N$  to be large in comparison to all other variables which, from now on, are kept fixed.

It is convenient rewrite  $z, w$  and the difference of their argument using scale parameters  $\gamma$  and  $\beta$ :

$$\begin{aligned} Z &= zN^\gamma, \\ W &= wN^\gamma, \quad \gamma > 0, \end{aligned} \quad (4.1)$$

and

$$\theta = N^\beta (\arg z - \arg w), \quad \beta > 0. \quad (4.2)$$

Equation (1.5) can thus be written as

$$K_N^\alpha \left( \frac{Z}{N^\gamma}, \frac{W}{N^\gamma} \right) = \frac{\alpha}{2\pi} e^{-N^{1-\alpha\gamma}|Z|^\alpha/2} e^{-N^{1-\alpha\gamma}|W|^\alpha/2} \frac{N^{2\gamma}}{Z\bar{W}} S_N, \quad (4.3)$$

where

$$S_N := \sum_{j=1}^N \frac{(N^{2(1-\alpha\gamma)/\alpha} Z\bar{W})^j}{\Gamma(2j/\alpha)}. \quad (4.4)$$

We introduce another auxiliary scale parameter  $\delta$  satisfying  $0 < \delta < 1$  and

$$\alpha\gamma + \delta = 1, \quad (4.5)$$

in order to adjust the spacing in the label that indexes the sum. Note that  $\gamma$  and  $\delta$  are not independent. Equation (4.4) can be written as

$$S_N = \sum_{j=0}^{N-1} \frac{(N^{2\delta/\alpha} Z\bar{W})^{y_j N^\delta}}{\Gamma(2y_j N^\delta/\alpha)}, \quad (4.6)$$

where

$$y_j = N^{-\delta} + jN^{-\delta}, \quad j = 0, \dots, N-1. \quad (4.7)$$

Given a function  $f$  of the class  $C^{(p)}$  in  $[a, b]$ , the Euler–Maclaurin sum formula (see, e.g., Ref. 21 with  $\omega = 0$  and  $p = 1$ ),

$$\sum_{j=0}^{N-1} f(y_j) = \frac{1}{h} \int_a^b f(x) dx + R_1 + R_2, \quad (4.8)$$

associated with the uniform partition  $P: a = y_0 < y_1 < \dots < y_N = b$ ,

$$y_j = a + jh,$$

for  $j \in \{0, \dots, N - 1\}$ , can be employed to estimate the errors

$$R_1 = \frac{1}{2} (f(b) - f(a)),$$

and

$$R_2 = -h \int_0^1 \left( \frac{1}{2} - t \right) \left( \sum_{j=0}^{N-1} f'(a + (j+t)h) \right) dt$$

in replacing the Riemann sum of  $f$  by its integral.

We take

$$f(y) = g_{Z\bar{W}}(yN^\delta) = \frac{(N^{2\delta/\alpha} Z\bar{W})^{yN^\delta}}{\Gamma(2yN^\delta/\alpha)}, \quad (4.9)$$

in (4.8) with  $g_\zeta(x)$  defined by (3.7). The partition  $N^{-\delta} = y_0 < y_1 < \dots < y_{N-1} = N^{1-\delta}$  of  $[N^{-\delta}, N^{1-\delta}]$  is chosen with the  $y_j$ 's given by (4.7). In order to simplify the notation in (4.9), from now on we fix  $\zeta = Z\bar{W} = |\zeta| e^{i\theta/N^\beta}$ .

Equation (4.4) can thus be written as

$$S_N = N^\delta \int_{N^{-\delta}}^{N^{1-\delta}} g_\zeta(N^\delta y) dy + r_1 + r_2, \quad (4.10)$$

where

$$r_1 = \frac{1}{2} (g_\zeta(N) - g_\zeta(1)), \quad (4.11)$$

and

$$\begin{aligned} r_2 &= -N^{-\delta} \int_0^1 \left( \frac{1}{2} - t \right) \left( \sum_{j=1}^N f'(N^{-\delta} + (j+t)N^{-\delta}) \right) dt \\ &= -\sum_{j=1}^N \int_0^1 \left( \frac{1}{2} - t \right) df((j+t)N^{-\delta}) \\ &= -\sum_{j=1}^N \int_0^1 \left( \frac{1}{2} - t \right) dg_\zeta(j+t). \end{aligned} \quad (4.12)$$

The proof now proceeds in two parts. The longest one, Part **A**, concerns with the estimates of  $r_1$  and  $r_2$ . Part **B** applies the method of steepest descent to the integral term of the representation (4.8).

### A. Estimate of $r_1$ and $r_2$

By the Stirling formula (see (3.14)),

$$g_\zeta(N) = \frac{(N^{2\delta/\alpha} \zeta)^N}{\Gamma(2N/\alpha)} = \sqrt{\frac{N}{\alpha\pi}} \left( \frac{\alpha e}{2N} \right)^{2N/\alpha} \left( N^{2\delta/\alpha} \zeta \right)^N (1 + O(1/N)) = O(N^{-k}),$$

holds for any power  $k$  of  $1/N$ , in view of  $2N(1 - \delta)/\alpha > 0$ . Since

$$g_\zeta(1) = \frac{N^{2\delta/\alpha} \zeta}{\Gamma(2/\alpha)}$$

we conclude, by (4.11),

$$r_1 = O(N^{2\delta/\alpha}). \quad (4.13)$$

According to the second mean value theorem (see, e.g., Ref. 23), for each  $j \in \{0, \dots, N-1\}$  there exists  $x_j \in [0, 1]$  such that

$$\int_0^1 \left(\frac{1}{2} - t\right) dg_\zeta(j+t) = \frac{1}{2} (g_\zeta(j+x_j) - g_\zeta(j)) - \frac{1}{2} (g_\zeta(j+1) - g_\zeta(j+x_j)).$$

Taking this into consideration, (4.12) can thus be written as

$$r_2 = \frac{-1}{2} \sum_{j=1}^N (2g_\zeta(j+x_j) - (g_\zeta(j) + g_\zeta(j+1))). \quad (4.14)$$

Some considerations about (4.14) are required. We have to avoid to take the absolute value inside the sum since any estimate that disregards the change of sign in (4.14), leads  $r_2$  to be of the leading order of the integral (4.8) given by  $O(N^\delta e^{N^\delta |\zeta|^{1/\alpha/2}})$ .<sup>25</sup> This follows from (3.9) and the fact that there are  $O(N^{\delta/2})$  terms contributing to the sum (4.14), in view of Lemma 3.5. One needs to be careful and exploits the change of sign in a clever way in order to reduce the dependence on  $N$  of the number of terms of this sum. Because the estimates involve exponential growth, it is convenient to divide  $r_2$  by the maximum value of  $N^{\delta/2} g_{|\zeta|}(x)$  (see (3.9)). We set

$$\hat{r}_i = \frac{r_i}{N^{\delta/2} g_{|\zeta|}(x^*)} \quad (4.15)$$

for  $i = 1, 2$ , and note by (4.13) that  $\hat{r}_1$  is exponentially small in  $N^\delta$ .

Writing  $\zeta = |\zeta| e^{i\theta/N^\beta}$  with  $\theta \in \mathbb{R}$ , we have by definition (3.7)

$$g_\zeta(x) = g_{|\zeta|}(x) \cos(\theta N^{-\beta} x) + i g_{|\zeta|}(x) \sin(\theta N^{-\beta} x). \quad (4.16)$$

As  $r_2$  is a linear function of  $g_\zeta$ , it suffices to estimate its real part  $\Re(r_2)$ , since the estimate of  $\Im(r_2)$  can be done in analogous manner.

The estimation of the real and imaginary parts of (4.16) depends on the period

$$p = \frac{2\pi}{|\theta|} N^\beta \quad (4.17)$$

of oscillation of  $g_\zeta(x)$ . For this, let  $n_N(\theta)$  be the cardinality of the set

$$A_N(\theta) = \left\{ l \in \mathbb{N} : \frac{|\theta|}{\pi} N^{-\beta} < l \leq \frac{|\theta|}{\pi} N^{1-\beta} \right\}. \quad (4.18)$$

The number  $n_N(\theta)$  counts how many oscillations between the maximum and minimum value of  $\cos \theta N^{-\beta} x$  there are as  $x$  varies in the interval  $[1, N]$ . For pedagogical reason, we divide the estimate in two cases (i)  $n_N(\theta) = O(1)$  and (ii)  $n_N(\theta) = O(N^\varepsilon)$  for some  $0 < \varepsilon \leq 1 - \beta$ .<sup>26</sup> The estimate for the first case can be done with less effort. In the second case, which may also include the previous one, the estimate is more subtle and leads to sharper result.

(i) If  $n_N(\theta) = n = O(1)$ , we write (4.14) as

$$r_2 = r_2^{(1)} + r_2^{(2)},$$

where the real part of  $r_2^{(i)}$ , with  $i = 1, 2$ , is given by

$$\Re r_2^{(i)} = - \sum_{j \in A_N^{(i)}} \left( \Re g_\zeta(j+x_j) - \left( \frac{\Re g_\zeta(j) + \Re g_\zeta(j+1)}{2} \right) \right) \quad (4.19)$$

with  $A_N^{(i)}$  being the set of points  $j \in \{1, \dots, N\}$  such that

$$\Re g_\zeta(j+1) - \Re g_\zeta(j+x_j) \begin{cases} \geq 0 & \text{if } i = 1 \\ < 0 & \text{if } i = 2 \end{cases}.$$

Let  $(j_k)_{k=1}^L$  denote a sequence of points right before  $\Re g_\zeta(j+1) - \Re g_\zeta(j+x_j)$ , as a function of  $j \in \{1, \dots, N\}$ , changes its sign

$$A_N^{(1)} = \{1, \dots, j_1\} \cup \{j_2 + 1, \dots, j_3\} \cup \dots \cup \{j_{L-1} + 1, \dots, j_L\},$$

$$A_N^{(2)} = \{j_1 + 1, \dots, j_2\} \cup \{j_3 + 1, \dots, j_4\} \cup \dots \cup \{j_L + 1, \dots, N\}.$$

Since  $0 \leq x_j \leq 1$  and  $g_{|\zeta|}(x)$  is increasing in  $[1, x^*]$  and decreasing in  $(x^*, N]$ , the points  $(j_k)_{k=1}^L$  are essentially determined by the oscillations of the function  $\cos \theta N^{-\beta} x$  in  $\Re g_\zeta(x) = g_{|\zeta|}(x) \cos(\theta N^{-\beta} x)$  and  $L = O(n_N(\theta)) = O(1)$ , by hypothesis.

By definition, we have

$$\begin{aligned} |\Re r_2^{(1)}| &\leq \frac{1}{2} \left| \sum_{j \in A_N^{(1)}} (\Re g_\zeta(j+1) - \Re g_\zeta(j)) \right| \\ &= \frac{1}{2} |\Re g_\zeta(j_1 + 1) - \Re g_\zeta(1) + \dots + \Re g_\zeta(j_L + 1) - \Re g_\zeta(j_{L-1} + 1)| \end{aligned}$$

and

$$\begin{aligned} |\Re r_2^{(2)}| &< \frac{1}{2} \left| \sum_{j \in A_N^{(2)}} (\Re g_\zeta(j) - \Re g_\zeta(j+1)) \right| \\ &= \frac{1}{2} |\Re g_\zeta(j_1 + 1) - \Re g_\zeta(j_2 + 1) + \dots + \Re g_\zeta(j_L + 1) - \Re g_\zeta(N + 1)| \end{aligned}$$

so that

$$|\Re r_2| \leq \sum_{k=1}^L g_{|\zeta|}(j_k + 1) + \frac{g_{|\zeta|}(1) + g_{|\zeta|}(N + 1)}{2}$$

yields, together with (4.15) and (4.13), Lemma 3.3 and the fact that the same holds for  $\Im m(r_2)$ ,

$$|\hat{r}_2| \leq O\left(\frac{1}{N^{\delta/2}}\right).$$

(ii) Let  $n_N(\theta) = O(N^\varepsilon)$  for some  $0 < \varepsilon \leq 1 - \beta$ . Integrating (4.12) by parts gives

$$\begin{aligned} r_2 &= \sum_{j=1}^N \int_0^1 \left(\frac{1}{2} - t\right) dg_\zeta(j+t) \\ &= \sum_{j=1}^N \left( \left(\frac{1}{2} - t\right) g_\zeta(j+t) \Big|_0^1 + \int_0^1 g_\zeta(j+t) dt \right) \\ &= \sum_{j=1}^N \left( \int_0^1 g_\zeta(j+t) dt - \frac{1}{2} (g_\zeta(j) + g_\zeta(j+1)) \right). \end{aligned} \tag{4.20}$$

We now split the above sum into

$$r_2 = r_2^\sqcup + r_2^\sqcap, \tag{4.21}$$

where the real part of  $r_2^{\sqcup(\Gamma)}$  is given by

$$\Re r_2^{\sqcup(\Gamma)} = \sum_{j \in A_N^{\sqcup(\Gamma)}} \left( \int_0^1 \Re g_\zeta(j+t) dt - \frac{1}{2} (\Re g_\zeta(j) + \Re g_\zeta(j+1)) \right)$$

with  $A_N^{\sqcup(\Gamma)}$  being the set of points  $j \in \{1, \dots, N\}$  such that  $(\Re g_\zeta)''(j) \geq 0 (< 0)$ .

Let us note that the function  $\Re g_\zeta(x) = g_{|\zeta|}(x) \cos(\theta x)$  always has a well defined concavity and the cardinality of inflection points is of same order in  $N$  of the cardinality of critical points, since the main function responsible for both, the number of oscillations and changes of concavity, is the cosine.

Let  $(k_i)_{i=1}^L$  denote a sequence of points in  $\{1, \dots, N\}$  right before  $(\Re g_\zeta)''(j)$  changes sign. Analogously, we have

$$\begin{aligned} A_N^\square &= \{1, \dots, k_1\} \cup \{k_2 + 1, \dots, k_3\} \cup \dots \cup \{k_{L-1} + 1, \dots, k_L\}, \\ A_N^\sqcup &= \{k_1 + 1, \dots, k_2\} \cup \{k_3 + 1, \dots, k_4\} \cup \dots \cup \{k_L + 1, \dots, N\}, \end{aligned}$$

where, by the same reason as in item (i),  $L = O(n_N(\theta)) = O(N^\epsilon)$  and, consequently,

$$k_{i+1} - k_i = O(N^{1-\epsilon}) \tag{4.22}$$

holds for  $i = 1, \dots, L - 1$ . Note also that, by (4.17),

$$\theta = O(N^{\epsilon+\beta-1}). \tag{4.23}$$

Applying Lemma 3.6 (and Corollary 3.8) to each interval  $I_i = \{k_i + 1, \dots, k_{i+1}\}$ ,  $i = 0, \dots, L$  ( $k_0 \equiv 0$  and  $k_{L+1} = N$ ) of size  $K = O(N^{1-\epsilon})$  with  $f(x)$  replaced by  $\Re g_\zeta(x)$  and  $\Delta = 1$ , yields

$$|\Re r_2| \leq \sum_{i=1}^L \left| \Re g_\zeta(t_i) - \frac{\Re g_\zeta(k_i) + \Re g_\zeta(k_i + 1)}{2} \right| + |\Re g_\zeta(1)| + |\Re g_\zeta(N + 1)|, \tag{4.24}$$

with  $t_k$  defined by the mean value theorem  $\Re g_\zeta(t_i) = \int_{k_i}^{k_i+1} \Re g_\zeta(x) dx$ . Note that the points  $(k_i)_{i=1}^L$  are closed to the inflection points  $(x_i)_{i=1}^L$  of  $\Re g_\zeta(x)$  and, moreover, the value of  $\Re g_\zeta(x)$  at these points are small compared with the maximum value  $g_{|\zeta|}(x^*)$ . We shall estimate the order of  $\Re g_\zeta(x_i)$  and use Lemma 3.5 to reduce the number of terms involved in the sum (4.21).

Taking the second derivative of the real part of (4.16), we obtain

$$(\Re g)''(x) = (g_{|\zeta|}''(x) - \theta^2 N^{-2\beta} g_{|\zeta|}(x)) \cos(\theta N^{-\beta} x) - 2\theta N^{-\beta} g_{|\zeta|}'(x) \sin(\theta N^{-\beta} x)$$

Since derivatives of  $g_{|\zeta|}(x)$  increases its value by a logarithm of  $N$  factor (see equation (3.11)), combined with (4.23), it gives

$$\frac{(\Re g)''(x)}{g_{|\zeta|}(x)} = (O(\log^2 N) + O(N^{2(\epsilon-1)})) \cos(\theta N^{-\beta} x) + O(N^{\epsilon-1} \log N) \sin(\theta N^{-\beta} x). \tag{4.25}$$

But we have, on the other hand,

$$(\Re g)''(x_i) = (g_{|\zeta|}''(x_i) - \theta^2 N^{-2\beta} g_{|\zeta|}(x_i)) \cos(\theta N^{-\beta} x_i) - 2\theta N^{-\beta} g_{|\zeta|}'(x_i) \sin(\theta N^{-\beta} x_i) = 0$$

holds at each inflection point  $x_i$ . This together with (4.25) implies that the inflection point  $x_i$  must be at  $O(1/\log N)$  distance from the  $k$ th zero of  $\cos \theta N^{-\beta} x$ . Indeed, defining  $\Delta_i = O(1/\log N)$  by

$$x_i = \frac{(2i - 1)\pi}{2|\theta| N^{-\beta}} + \Delta_i$$

we have

$$\begin{aligned} \cos(\theta N^{-\beta} x_i) &= \cos(\pm(i - 1/2)\pi + \theta N^{-\beta} \Delta_i) = \pm(-1)^i \sin(\theta N^{-\beta} \Delta_i) = O\left(\frac{N^{\epsilon-1}}{\log N}\right), \\ \sin(\theta N^{-\beta} x_i) &= \sin(\pm(i - 1/2)\pi + \theta N^{-\beta} \Delta_i) = \mp(-1)^i \cos(\theta N^{-\beta} \Delta_i) = O(1), \end{aligned}$$



and, together with (4.25), one sees that  $(\Re g)''(x_i) = 0$  holds in the leading order. Since the points  $k_i, t_i$ , and  $k_i + 1$  are not distant from the inflection point  $x_i$  ( $g_\zeta(x)$  varies slowly for each interval  $k_i \leq x \leq k_i + 1$ ),

$$\frac{|\Re g_\zeta(x)|}{g_{|\zeta|}(x)} = |\cos(\theta N^{-\beta} x)| \leq O(N^{-1+\varepsilon} / \log N) \tag{4.26}$$

holds for  $x$  at the values  $\{k_i, t_i, k_i + 1\}_{i=1}^L$ .

The number of terms that contribute to (4.21), as well as to the sum (4.24), can be estimated using Lemma 3.5. Instead of an interval  $I$  of size  $N$  we shall consider an interval  $I'$  containing  $x^*$  with  $O(N^{\delta/2} \log N)$  points. By (4.22), a number of order  $N^{\delta/2} \log N / N^{1-\varepsilon}$  of terms give an appreciable contribution to (4.24) and, together with (4.26), the fact that the same estimate holds for  $\Im m g_\zeta(x)$  and (4.15), we conclude

$$|\hat{f}_2| = O(N^{-2(1-\varepsilon)})$$

uniformly in every closed interval of  $0 < \varepsilon \leq 1 - \beta$ .

**B. The method of steepest descent**

Equation (4.10) can be written as

$$S_N = N^{\delta/2} g_{|\zeta|}(x^*) \left( N^{\delta/2} \int_{N^{-\delta}}^{N^{1-\delta}} f(y) dy + \hat{r}_1 + \hat{r}_2 \right), \tag{4.27}$$

where, by the Stirling formula (see (3.14)),

$$f(y) = \frac{g_\zeta(N^\delta y)}{g_{|\zeta|}(x^*)} = \sqrt{\frac{2y}{\alpha |\zeta|^{\alpha/2}}} e^{N^\delta h(y)} (1 + O(1/N^\delta)) \tag{4.28}$$

with

$$h(y) = \frac{2y}{\alpha} \log \frac{\alpha e \zeta^{\alpha/2}}{2y} - |\zeta|^{\alpha/2}. \tag{4.29}$$

Note that  $\Re h(y) \leq 0$  holds for all  $y > 0$  and attains to its maximum  $\Re h(y^*) = 0$  at  $y^* = \alpha |\zeta|^{\alpha/2} / 2$  inside the domain of integration  $[N^{-\delta}, N^{1-\delta}]$ , by condition  $0 < \delta < 1$  and  $N$  large enough.

We now use the steepest descent technique to estimate the integral that appears in (4.27). This technique uses the Cauchy theorem to deform the interval of integration  $[N^{-\delta}, N^{1-\delta}]$  into a curve  $\mathcal{C}$ :

$$I = \sqrt{\frac{2N^\delta}{\alpha |\zeta|^{\alpha/2}}} \int_{N^{-\delta}}^{N^{1-\delta}} \sqrt{y} e^{N^\delta h(y)} dy = \sqrt{\frac{2N^\delta}{\alpha |\zeta|^{\alpha/2}}} \int_{\mathcal{C}} \sqrt{\eta} e^{N^\delta h(\eta)} d\eta, \tag{4.30}$$

where  $h : \mathbb{R}_+ \rightarrow \mathbb{C}$  is extended analytically to the cut complex plane  $\mathbb{C} \setminus (-\infty, 0]$ ,  $\eta = y + iw$  and  $\mathcal{C}$  is a smooth curve with extreme points  $\eta_1 = N^{-\delta}$  and  $\eta_2 = N^{1-\delta}$  chosen in such a way that (a) it passes by the saddle point  $\eta_0 = \alpha \zeta^{\alpha/2} / 2$  ( $|\eta_0| = y^*$ ) defined implicitly by

$$h'(\eta_0) = \frac{2}{\alpha} \log \frac{\alpha \zeta^{\alpha/2}}{2\eta_0} = 0 \tag{4.31}$$

and (b) it maximizes the function  $\Re h(y, w)$  along a level curve

$$\Im h(y, w) = c$$

in a neighborhood  $U_0$  of  $\eta_0$ . If, in addition,

$$\Re h(y, w) \geq \max\{\Re h(N^{-\delta}, 0), \Re h(N^{1-\delta}, 0)\} \tag{4.32}$$

holds along  $\mathcal{C}$ , then the main contribution to (4.30) will be given by the saddle point  $\eta_0$ ; if, on the other hand, Eq. (4.32) cannot be satisfied by any such curve  $\mathcal{C}$ , the main contribution to the integral (4.30) will be given by the extreme points.

At the extreme points, neither  $\eta_1$  nor  $\eta_2$  plays an important role, since both leave the integral (4.30) exponentially small with  $N$ . So, the contribution to (4.30) is given by the vicinity of the saddle point.

Expanding  $h$  in Taylor series about  $\eta_0 = \alpha \zeta^{\alpha/2}/2 = \alpha |\zeta|^{\alpha/2} e^{i\alpha N^{-\beta}\theta/2}/2$ , gives

$$\begin{aligned} h(\eta) &= h(\eta_0) + \frac{1}{2}h''(\eta_0)(\eta - \eta_0)^2 + O((\eta - \eta_0)^3) \\ &= \zeta^{\alpha/2} - |\zeta|^{\alpha/2} - \frac{2}{\alpha^2 |\zeta|^{\alpha/2}} \rho^2 e^{i(2\varphi - \alpha\theta N^{-\beta}/2)} + O((\eta - \eta_0)^3) \end{aligned}$$

with  $\eta - \eta_0 = \rho e^{i\psi} \in U_0$ . We choose  $\mathcal{C}$  so that  $2\varphi - \alpha\theta N^{-\beta}/2 = 0$  at the saddle point. Applying the steepest descent technique, the integral (4.30) can be approximate by a Gaussian integral in the vicinity  $U_0$  of  $\eta_0$ , resulting (see, e.g., Ref. 27, for details)

$$\begin{aligned} S_N &= N^{\delta/2} g_{|\zeta|}(x^*) \left( e^{N^\delta(\zeta^{\alpha/2} - |\zeta|^{\alpha/2})} \sqrt{\frac{2N^\delta}{\alpha |\zeta|^{\alpha/2}}} \cdot \frac{2\pi \eta_0}{-N^\delta h''(\eta_0)} \left( 1 + O\left(\frac{1}{N^\delta}\right) \right) + \hat{r}_1 + \hat{r}_2 \right) \\ &= N^{\delta/2} g_{|\zeta|}(x^*) \left( e^{N^\delta(\zeta^{\alpha/2} - |\zeta|^{\alpha/2})} \sqrt{\frac{2\pi}{|\zeta|^{\alpha/2}}} \eta_0 \left( 1 + O\left(\frac{1}{N^\delta}\right) \right) + \hat{r}_1 + \hat{r}_2 \right). \end{aligned} \quad (4.33)$$

Now, since by (3.8)  $\alpha N^\delta |\zeta|^{\alpha/2}/2 < N$ ,

$$\begin{aligned} |\exp(N^\delta(\zeta^{\alpha/2} - |\zeta|^{\alpha/2}))| &= \exp(N^\delta |\zeta|^{\alpha/2} (\cos \alpha N^{-\beta}\theta/2 - 1)) \\ &\geq \exp(-\alpha\theta^2 N^{1-2\beta}) \end{aligned} \quad (4.34)$$

and, provided  $\beta \geq 1/2$ , it follows from the estimates of  $r_1$  and  $r_2$  in Sec. IV A that

$$S_N = \frac{\alpha}{2} \zeta^{\alpha/2} N^\delta \exp\left(\zeta^{\frac{\alpha}{2}} N^\delta\right) \left(1 + E_N^{\alpha,\delta}(\zeta)\right)$$

with

$$\left| E_N^{\alpha,\delta}(\zeta) \right| \leq O(N^{-\delta/2})$$

whenever  $\zeta \in S(\theta N^{-1/2}, K^{\alpha,\delta})$ , where  $K^{\alpha,\delta} = (2N^{1-\delta}/\alpha)^{2/\alpha}$ . Therefore, we obtain from (4.3)

$$\frac{1}{N^{\delta+2\gamma}} K_N^\alpha \left( \frac{Z}{N^\gamma}, \frac{W}{N^\gamma} \right) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N^\delta \left( (Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} \left( 1 + E_N^{\alpha,\delta}(Z\bar{W}) \right), \quad (4.35)$$

where we have used (4.5) with  $0 < \delta < 1$ . In particular, taking  $\delta \nearrow 1$ ,

$$\frac{1}{N} K_N^\alpha(Z, W) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N \left( (Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} \left( 1 + E_N^{\alpha,1}(Z\bar{W}) \right). \quad (4.36)$$

□

*Remark 4.1:* Equation (4.34) prevents  $\zeta = Z\bar{W}$  to be defined in a sector  $S(\theta N^{-\beta}, K^{\alpha,\delta})$  of opening wider than  $O(N^{-1/2})$ . The introduction of the scale  $\delta < 1$  guarantees that the main contribution to (4.30) comes from the saddle point for any  $\zeta \in \mathbb{C}$  fixed. Note that  $K^{\alpha,\delta} = O(N^{2(1-\delta)/\alpha})$  and for  $\delta = 1$  we need  $|\zeta| \leq K^{\alpha,1} = (2/\alpha)^{2/\alpha}$  (see Remark 3.11). As the calculation in the appendixes below indicates,  $|\zeta|$  may be even smaller than that, depending on the sector opening  $\tau$ .

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### APPENDIX A: TAYLOR REMAINDER

Let  $f_N(\zeta) = N\zeta e^{N\zeta}$  be a function defined for  $\zeta = |\zeta| e^{i\theta} \in \mathbb{C}$  and  $N$  a fixed natural number. Its Taylor remainder with respect to the polynomial  $S_N(\zeta) = N\zeta + \dots + \frac{1}{(N-1)!}(N\zeta)^N$  of order  $N$  can be expressed by the Lagrange formula (see, e.g., Ref. 28)

$$R_N(\zeta) = f_N(\zeta) - S_N(\zeta) = \frac{1}{(N+1)!} g_N^{(N+1)}(a)$$

for some  $0 < a < 1$ , where  $g_N(x) = f_N(x\zeta)$ ,  $x \in [0, 1]$ , satisfies

$$g_N^{(r)}(x) = (rN^r + N^{r+1}x\zeta) \zeta^r e^{Nx\zeta} \quad (\text{A1})$$

for every  $r \in \mathbb{N}$ , by induction.

Writing

$$S_N(\zeta) = f_N(\zeta)(1 + E_N(\zeta))$$

the error estimator function  $E_N(\zeta) = R_N(\zeta)/f_N(\zeta)$  is estimated for  $\zeta$  in a sectorial domain  $S(\tau, K) = \{\zeta \in \mathbb{C} : |\arg(\zeta)| < \tau/2, |\zeta| < K\}$  using (A1) together with the Stirling formula  $r! = \sqrt{2\pi r}(r/e)^r(1 + O(1/r))$ :

$$|E_N(\zeta)| = \frac{1}{\sqrt{2\pi N}} |1 + a\zeta| e^N |\zeta|^N e^{-N(1-a)|\zeta|\cos\theta} (1 + O(1/N))$$

so  $\sup_{\zeta \in S(\tau, K)} |E_N(\zeta)| = O(1/\sqrt{N})$  where  $K = K(a, \tau) > 0$  is given by the smallest solutions of

$$K e^{-(1-a)K \cos(\tau/2)+1} = 1, \quad (\text{A2})$$

which exists and is continuous for all  $0 < a < 1$  and  $\tau \in [0, 2\pi]$ . The implicit solutions of (A2) for  $K = K(a, \tau)$  are depicted in Figure 2 for  $a = 1/4, 1/8$ , and  $1/16$ .

### APPENDIX B: PROOF OF COROLLARY 1.6

Assuming temporarily that (1.10) holds with  $Z = W = r$ , we observe that by (1.11)

$$\begin{aligned} Z_i &= r + \frac{1}{\sqrt{N}} \frac{2z_i}{\alpha |r|^{\alpha/2-1}} + O(1/N) \\ &= r \exp\left(\frac{1}{\sqrt{N}} \frac{2z_i}{\alpha r |r|^{\alpha/2-1}} + O(1/N)\right) \end{aligned}$$

and

$$\arg(Z_i \bar{Z}_j) < \theta/\sqrt{N},$$

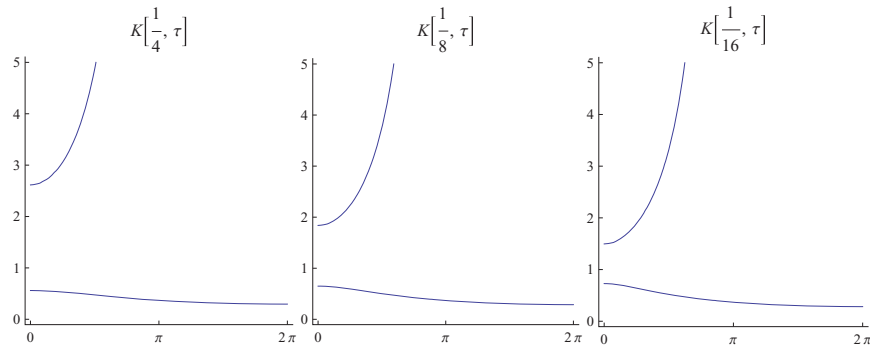


FIG. 2. (Color online) Plot of  $K(a, \tau)$  as a function of  $\tau$  for  $a = 1/4, 1/8$ , and  $1/16$ .

for some  $\theta \geq 0$  and any  $i, j$ , if  $N$  is large enough, say  $N > N_1$ . We take, in addition,  $N > N_0$  where  $N_0$  is given by (3.8) with  $1/|\zeta|$  and  $|\zeta|$  replaced by  $1/\min_{i,j} (|Z_i \bar{Z}_j|)$  and  $\max_{i,j} (|Z_i \bar{Z}_j|)$ , respectively. So, for  $N > \max(N_0, N_1)$  Eq. (1.10) holds with  $(r, r)$  and  $(Z_i, Z_j)$ , for any  $i, j$ , in the place of  $(Z, W)$ . From Eq. (1.9) and (1.11), it holds for  $r \in \mathbb{C}$  with  $0 < |r| < (2/\alpha)^{1/\alpha}$ , whose closure is the support of the eigenvalues density (see Eq. (2.6)).

Now, applying the Taylor expansion

$$(1+w)^{\alpha/2} = 1 + \frac{\alpha}{2}w + \frac{\alpha}{4}\left(\frac{\alpha}{2} - 1\right)w^2 + O(w^3)$$

to the exponent of  $K_N^\alpha(Z_i, Z_j)$ , yields

$$N \left( (Z_i \bar{Z}_j)^{\alpha/2} - \frac{1}{2}|Z_i|^\alpha - \frac{1}{2}|Z_j|^\alpha \right) = A_{ij} + i\sqrt{N}B_{ij} + O(1/\sqrt{N}), \quad (\text{B1})$$

where

$$A_{ij} = z_i \bar{z}_j - \frac{1}{2}|z_i|^2 - \frac{1}{2}|z_j|^2, \\ B_{ij} = \lambda_i - \lambda_j$$

and

$$\lambda_i = |r|^{\alpha/2+1} \Im \frac{z_i}{r} + \frac{1}{2\sqrt{N}} |r|^2 \left( 1 - \frac{2}{\alpha} \right) \Im \frac{z_i^2}{r^2}$$

is a real number. Let  $C_N$  and  $D_N$  denote  $n \times n$  matrices with respective entries  $(C_N)_{ij} = \frac{1}{\pi} \exp(A_{ij} + i\sqrt{N}B_{ij}) (1 + O(1/\sqrt{N}))$  and  $D_{ij} = \frac{1}{\pi} \exp(A_{ij}) (1 + O(1/\sqrt{N}))$  ( $= C_{ij}$  with  $B_{ij} = 0$ ). If we write  $\Lambda_N = \text{diag}(\exp(i\sqrt{N}\lambda_i))$ , then  $C_N = \Lambda_N D_N \bar{\Lambda}_N$ ,  $\Lambda_N \bar{\Lambda}_N = I$  ( $\bar{\Lambda}_N$  and  $I$  are the complex conjugate of  $\Lambda_N$  and the identity matrix) and

$$\det C_N = \det \Lambda_N D_N \bar{\Lambda}_N = \det D_N \bar{\Lambda}_N \Lambda_N = \det D_N.$$

by the Cauchy-Binet formula. This concludes the proof since, by (1.6), (1.10), and (B1), the lhs of (1.12) is the determinant of a matrix whose asymptotic expansion is given by  $C_N$  and

$$\lim_{N \rightarrow \infty} \det C_N = \lim_{N \rightarrow \infty} \det D_N = \det \left( \mathbb{K}(z_i, z_j) \right)_{i,j=1}^n$$

by continuity. □

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<sup>17</sup>P. Elbau and G. Felder, *Commun. Math. Phys.* **259**, 433 (2005).

<sup>18</sup>Analytic polynomials are of the form  $P_n(z) = a_0 + a_1z + \dots + a_nz^n$  with  $a_k \in \mathbb{C}$ . This makes sense in the context since our weight  $e^{-NV_\alpha(z)}$  depends on  $|z|^2 = z\bar{z}$  and harmonic polynomials:  $T_n(z) = P_n(z) + \overline{Q_n(z)}$ , where  $P_n(z)$  and  $Q_n(z)$  are analytic polynomials, could be considered instead.

- <sup>19</sup>S. Bergman, *The Kernel Function and Conformal Mapping*, Mathematical Surveys and Monographs Vol. **5**, 2nd ed. (American Mathematical Society, Providence, 1970).
- <sup>20</sup>W. F. Donoghue Jr., *Monotone Matrix Function and Analytic Continuation*, Die Grundlehren der mathematischen Wissenschaften Vol. 207 (Springer-Verlag, Berlin, 1974).
- <sup>21</sup>M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).
- <sup>22</sup>B. J. Levin, *Distribution of Zeros of Entire Functions* (American Mathematical Society, Providence, 1964).
- <sup>23</sup>W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, New York, 1964).
- <sup>24</sup>M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1959).
- <sup>25</sup>We have  $g_{|\zeta|}(x^*) = O\left(N^{\delta/2} e^{N^\delta |\zeta|^{\alpha/2}}\right)$ , the Euler–Maclaurin formula (4.10) gives an extra  $N^\delta$  and  $N^{-\delta/2}$  results from the Gaussian integration in the steepest-descent method. See Part **B** for more details.
- <sup>26</sup>We set  $\varepsilon = 0$  when  $\beta \geq 1$ . In this case  $n_N(\theta)$  is always  $O(1)$ . If  $\beta < 1$ ,  $n_N(\theta) = O(1)$  when  $\theta = O(N^{-1 + \beta})$ .
- <sup>27</sup>J. D. Murray, *Asymptotic Analysis* (Clarendon, Oxford, 1974).
- <sup>28</sup>W. F. Trench, *Introduction to Real Analysis* (Free Edition 1.08, January 2012).