Homological Stability

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Rachael Boyd

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A family of groups and inclusions:

$$G_1 \hookrightarrow G_2 \hookrightarrow \ldots \hookrightarrow G_n \hookrightarrow \ldots$$

is said to satisfy homological stability when the induced maps on homology

$$\begin{aligned} H_i(G_n) &\rightarrow H_i(G_{n+1}) \\ H_i(BG_n) &\rightarrow H_i(BG_{n+1}) \end{aligned}$$

are isomorphisms for n sufficiently large.

Examples: families of groups

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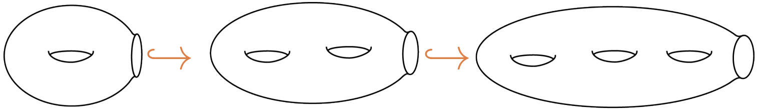
- Mapping class groups of surfaces

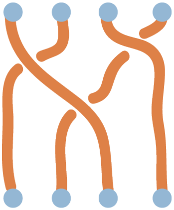
$$\Gamma_{1,1} \hookrightarrow \Gamma_{2,1} \hookrightarrow \ldots \hookrightarrow \Gamma_{g,1} \hookrightarrow \ldots$$



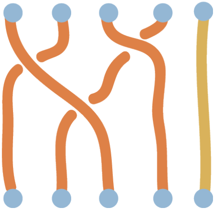
Examples: families of groups

- Braid groups

$$B_1 \hookrightarrow B_2 \hookrightarrow \ldots \hookrightarrow B_n \hookrightarrow \ldots$$



$$\longmapsto$$



Examples: families of groups

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$$B_1 \hookrightarrow B_2 \hookrightarrow \ldots \hookrightarrow B_n \hookrightarrow \ldots$$

- General linear groups

$$Gl_1(\mathbb{K}) \hookrightarrow Gl_2(\mathbb{K}) \hookrightarrow \ldots \hookrightarrow Gl_n(\mathbb{K}) \hookrightarrow \ldots$$

Examples: range of stability

- Symmetric groups (Nakaoka):

$$H_i(\Sigma_n; \mathbb{Z}) \xrightarrow{\cong} H_i(\Sigma_{n+1}; \mathbb{Z}) \qquad n \geq 2i$$

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$$H_i(Gl_n(\mathbb{K});\mathbb{Z}) \xrightarrow{\cong} H_i(Gl_{n+1}(\mathbb{K});\mathbb{Z}) \qquad n \geq i + 1$$

Stability in action

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$H_0(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

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$H_0(X;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$H_1(X;\mathbb{Z})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

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$H_2(X;\mathbb{Z})$	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Why? - Infinite homology

If we let

$$G_\infty = \lim_{n \to \infty} G_n$$

we can consider

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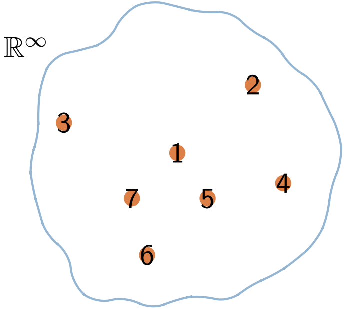
which we can often compute using completely different methods. This allows us to 'work backwards' and compute $H_*(G_n)$ in some range.

Classifying space $B\Sigma_n$

Classifying space



$Conf(\{1, \ldots, n\}, \mathbb{R}^\infty)$

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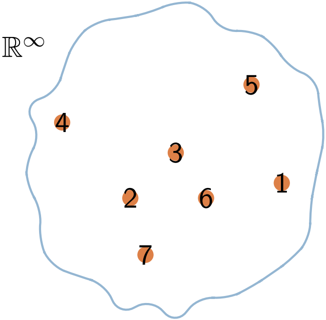

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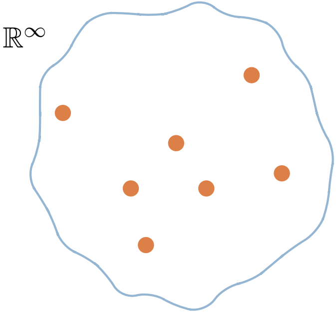

$$B\Sigma_n = Conf(\{1, \ldots, n\}, \mathbb{R}^\infty)/\Sigma_n$$

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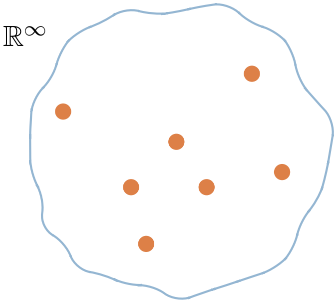
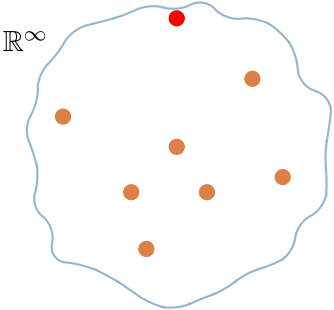


$B\Sigma_n = Conf(\{1, \dots, n\}, \mathbb{R}^\infty)/\Sigma_n$

Stabilization map



$B\Sigma_n \to B\Sigma_{n+1}$

Idea of proof for symmetric groups

For each n there is a space $\|B\Sigma_n^\bullet\|$ such that

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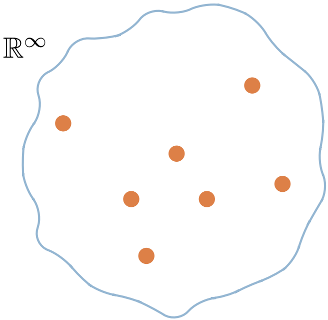
After such a space is found, the argument is to compute its homology in a range: we do this using a spectral sequence argument and induction on n.

Defining $B\Sigma_n^p$

An element of $B\Sigma_n^p$ is:

1. a choice of configuration C in $B\Sigma_n$
2.



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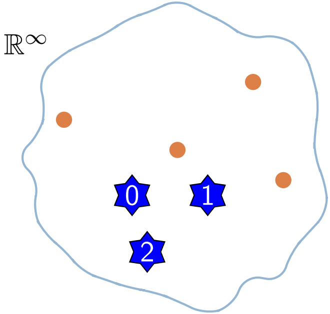

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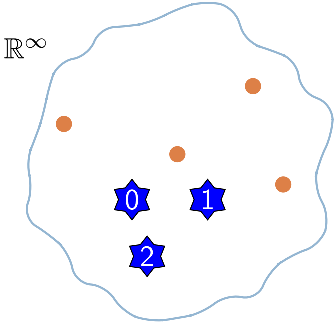

Recall we want a homotopy equivalence $B\Sigma_n^p \to B\Sigma_{n-p-1}$.

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Want to show $B\Sigma_n^p \simeq B\Sigma_{n-p-1}$.

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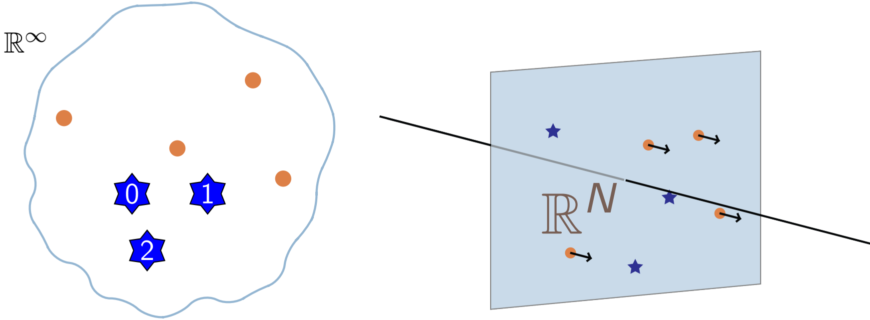

Map to classifying space

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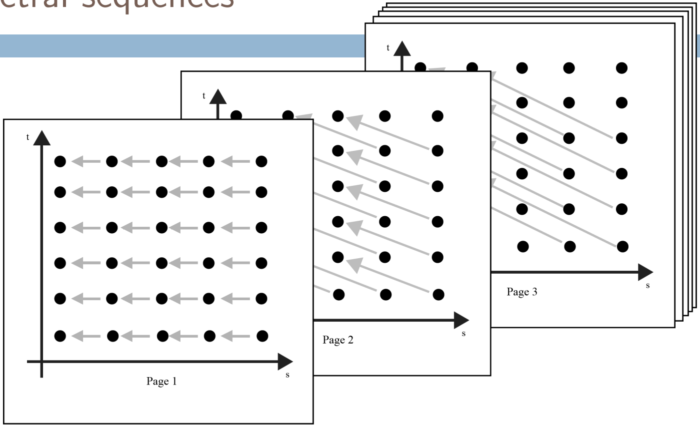
This is the obvious map which forgets the distinguished points.
We can show that the fibre of the resulting map $\|B\Sigma_n^\bullet\| \to B\Sigma_n$ is homotopy equivalent to a wedge of spheres, hence the map is highly connected.

Spectral sequences



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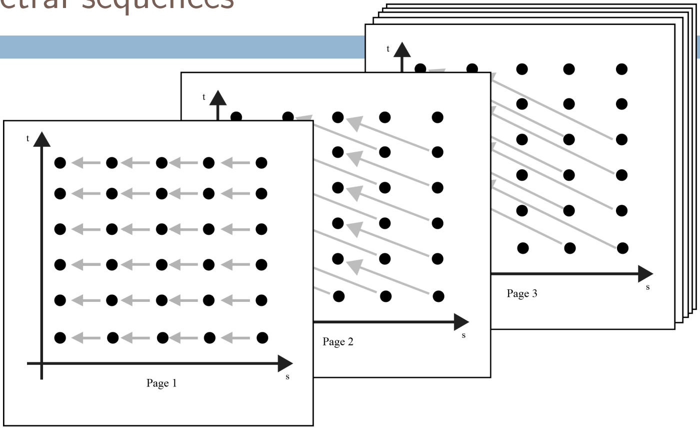
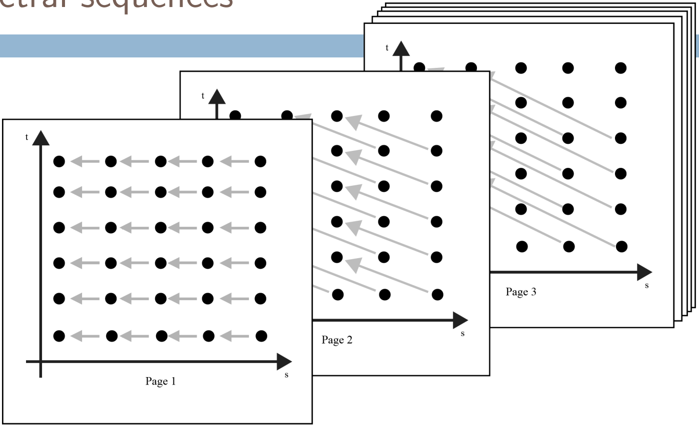


- A book where each page is a lattice of abelian groups

Spectral sequences



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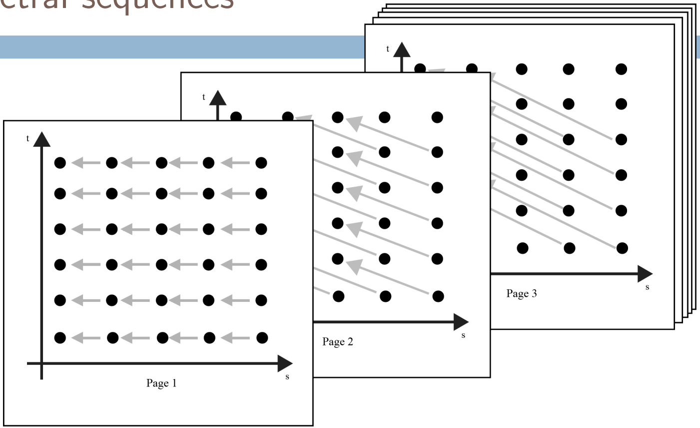

- A book where each page is a lattice of abelian groups
- Differentials move between groups and form chain complexes
- First quadrant spectral sequences result in ∞ page

Our spectral sequence

A spectral sequence can be constructed from the skeletal filtration of $\|B\Sigma^{\bullet}_{n+1}\|$ such that:

- First quadrant

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  - zero going from odd to even columns
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- On the ∞ page groups along a diagonal $s + t = k$ are a 'filtration quotient' of $H_k(\|B\Sigma^\bullet_{n+1}\|)$

Case n=5

$$H_i(B\Sigma_5) \xrightarrow{\cong} H_i(B\Sigma_6) \qquad i \leq \frac{5}{2}.$$

1st page: (s, t) entry is $H_t(B\Sigma_6^s)$

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$$\begin{array}{|ccccccccc}
H_3(B\Sigma_6^0) & \overset{0}{\leftarrow} & H_3(B\Sigma_6^1) & \leftarrow & H_3(B\Sigma_6^2) & \overset{0}{\leftarrow} & H_3(B\Sigma_6^3) & \leftarrow & H_3(B\Sigma_6^4) \\
H_2(B\Sigma_6^0) & \overset{0}{\leftarrow} & H_2(B\Sigma_6^1) & \leftarrow & H_2(B\Sigma_6^2) & \overset{0}{\leftarrow} & H_2(B\Sigma_6^3) & \leftarrow & H_2(B\Sigma_6^4) \\
H_1(B\Sigma_6^0) & \overset{0}{\leftarrow} & H_1(B\Sigma_6^1) & \leftarrow & H_1(B\Sigma_6^2) & \overset{0}{\leftarrow} & H_1(B\Sigma_6^3) & \leftarrow & H_1(B\Sigma_6^4) \\
H_0(B\Sigma_6^0) & \overset{0}{\leftarrow} & H_0(B\Sigma_6^1) & \leftarrow & H_0(B\Sigma_6^2) & \overset{0}{\leftarrow} & H_0(B\Sigma_6^3) & \leftarrow & H_0(B\Sigma_6^4) \\
\hline
\end{array}$$

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1st page: (s,t) entry is $H_t(B\Sigma_{5-s})$

$$\begin{array}{ccccccccc}
H_3(B\Sigma_5) & \overset{0}{\leftarrow} & H_3(B\Sigma_4) & \leftarrow & H_3(B\Sigma_3) & \overset{0}{\leftarrow} & H_3(B\Sigma_2) & \leftarrow & H_3(B\Sigma_1) \\
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$$\begin{array}{ccccccccc}
H_3(B\Sigma_5) & \xleftarrow{0} & H_3(B\Sigma_4) & \leftarrow & H_3(B\Sigma_3) & \xleftarrow{0} & H_3(B\Sigma_2) & \leftarrow & H_3(B\Sigma_1) \\
H_2(B\Sigma_5) & \xleftarrow{0} & H_2(B\Sigma_4) & \leftarrow & H_2(B\Sigma_3) & \xleftarrow{0} & H_2(B\Sigma_2) & \leftarrow & H_2(B\Sigma_1) \\
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\end{array}$$

Case n=5

$$H_i(B\Sigma_5) \xrightarrow{\cong} H_i(B\Sigma_6) \qquad i \leq \frac{5}{2}.$$

2*nd* page

$H_3(B\Sigma_5)$	?	?	?	?
$H_2(B\Sigma_5)$	?	?	?	?
$H_1(B\Sigma_5)$	0	0	?	?
$H_0(B\Sigma_5)$	0	0	0	0

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$$H_i(B\Sigma_5) \xrightarrow{\cong} H_i(B\Sigma_6) \qquad i \leq \frac{5}{2}.$$

∞ page

$?'$	$?'$	$?'$	$?'$	$?'$
$H_2(B\Sigma_5)$	$?'$	$?'$	$?'$	$?'$
$H_1(B\Sigma_5)$	0	0	$?'$	$?'$
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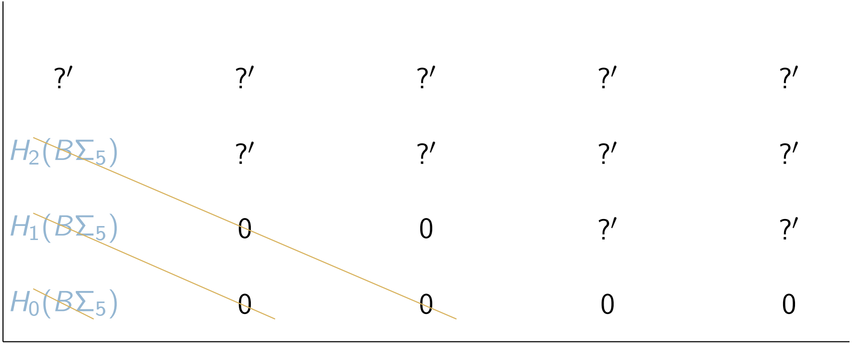

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∞ page

$?'$	$?'$	$?'$	$?'$	$?'$
$H_2(B\Sigma_5) \cong H_2(\lVert B\Sigma_6^\bullet \rVert)$		$?'$	$?'$	$?'$
$H_1(B\Sigma_5) \cong H_1(\lVert B\Sigma_6^\bullet \rVert)$		0	$?'$	$?'$
$H_0(B\Sigma_5) \cong H_0(\lVert B\Sigma_6^\bullet \rVert)$		0	0	0

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∞ page

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$H_2(B\Sigma_5)$	$\cong H_2(B\Sigma_6)$	$?'$	$?'$	$?'$
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$H_0(B\Sigma_5)$	$\cong H_0(B\Sigma_6)$	0	0	0

Current work

- Coefficient systems - Djament - Vespa, Wahl ...

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- Configurations of manifolds - Palmer
- Automating parts of proof - Wahl
- Representation stability - Church, Ellenberg, Farb …

Thank you