Embedding 3-manifolds via surgery on surfaces

Kyle Larson

University of Texas at Austin
klarson@math.utexas.edu

June 12th, 2015

Embedding 3-manifolds

Embedding 3-manifolds

Every closed, orientable 3-manifold embeds in $\mathbb{R}^5$ (equivalently S^5).

Embedding 3-manifolds

Every closed, orientable 3-manifold embeds in $\mathbb{R}^5$ (equivalently S^5).

However, not every 3-manifold embeds into S^4.

Embedding 3-manifolds

Every closed, orientable 3-manifold embeds in $\mathbb{R}^5$ (equivalently S^5).

However, not every 3-manifold embeds into S^4. For example, the Rochlin invariant obstructs the Poincare homology sphere from (smoothly) embedding into S^4.

Embedding 3-manifolds

Every closed, orientable 3-manifold embeds in $\mathbb{R}^5$ (equivalently S^5).

However, not every 3-manifold embeds into S^4. For example, the Rochlin invariant obstructs the Poincare homology sphere from (smoothly) embedding into S^4.

There are other obstructions to embedding a 3-manifold in S^4 coming from: the torsion part of the first homology, Donaldson's diagonalization theorem, the Casson-Gordon invariants, the d-invariants...

Example: lens spaces

Example: lens spaces

No lens spaces embed in S^4.

Example: lens spaces

No lens spaces embed in S^4.

However, the story is more interesting if we puncture the lens space (we denote a punctured lens space by $L(p, q)^\circ$).

Example: lens spaces

No lens spaces embed in S^4.

However, the story is more interesting if we puncture the lens space (we denote a punctured lens space by $L(p, q)^\circ$).

Zeeman gave embeddings of $L(2n+1, q)^\circ$ by his twist-spinning construction.

Example: lens spaces

No lens spaces embed in S^4.

However, the story is more interesting if we puncture the lens space (we denote a punctured lens space by $L(p,q)^\circ$).

Zeeman gave embeddings of $L(2n+1,q)^\circ$ by his twist-spinning construction. On the other hand, Epstein showed that the punctured lens spaces $L(2n,q)^\circ$ do *not* embed in S^4.

Dehn surgery on knots in S^3

Dehn surgery on knots in S^3

Given a knot K in S^3, we remove a neighborhood $\nu K \cong S^1 \times D^2$ and reglue by a diffeomorphism ϕ of the boundary.

Dehn surgery on knots in S^3

Given a knot K in S^3, we remove a neighborhood $\nu K \cong S^1 \times D^2$ and reglue by a diffeomorphism ϕ of the boundary. The resulting diffeomorphism type is determined by $[\phi(pt \times \partial D^2)] = p\mu + q\lambda$ (for μ the class of a meridian and λ the class of a 0-framed longitude).

Dehn surgery on knots in S^3

Given a knot K in S^3, we remove a neighborhood $\nu K \cong S^1 \times D^2$ and reglue by a diffeomorphism ϕ of the boundary. The resulting diffeomorphism type is determined by $[\phi(pt \times \partial D^2)] = p\mu + q\lambda$ (for μ the class of a meridian and λ the class of a 0-framed longitude). We call this p/q Dehn surgery on K and denote the resulting manifold $S^3_{p/q}(K)$.

Dehn surgery on knots in S^3

Given a knot K in S^3, we remove a neighborhood $\nu K \cong S^1 \times D^2$ and reglue by a diffeomorphism ϕ of the boundary. The resulting diffeomorphism type is determined by $[\phi(pt \times \partial D^2)] = p\mu + q\lambda$ (for μ the class of a meridian and λ the class of a 0-framed longitude). We call this p/q Dehn surgery on K and denote the resulting manifold $S^3_{p/q}(K)$.

Theorem (Lickorish-Wallace)

Every closed orientable 3-manifold can be obtained by Dehn sugery on a link in S^3.

4-dimensional analogues

4-dimensional analogues

There are two 4-dimensional analogues of Dehn surgery: the Gluck twist on 2-spheres and torus surgery.

4-dimensional analogues

There are two 4-dimensional analogues of Dehn surgery: the Gluck twist on 2-spheres and torus surgery.

Given a 2-knot $\mathcal{S}$ in S^4 (an embedded $S^2 \subset S^4$), the *Gluck twist* on $\mathcal{S}$ is the process of removing a neighborhood $\nu\mathcal{S} \cong S^2 \times D^2$

4-dimensional analogues

There are two 4-dimensional analogues of Dehn surgery: the Gluck twist on 2-spheres and torus surgery.

Given a 2-knot $\mathcal{S}$ in S^4 (an embedded $S^2 \subset S^4$), the *Gluck twist* on $\mathcal{S}$ is the process of removing a neighborhood $\nu\mathcal{S} \cong S^2 \times D^2$ and regluing by the diffeomorphism $\rho : S^2 \times S^1 \to S^2 \times S^1$ defined by sending (x, θ) to $(rot_\theta(x), \theta)$

4-dimensional analogues

There are two 4-dimensional analogues of Dehn surgery: the Gluck twist on 2-spheres and torus surgery.

Given a 2-knot $\mathcal{S}$ in S^4 (an embedded $S^2 \subset S^4$), the *Gluck twist* on $\mathcal{S}$ is the process of removing a neighborhood $\nu\mathcal{S} \cong S^2 \times D^2$ and regluing by the diffeomorphism $\rho : S^2 \times S^1 \to S^2 \times S^1$ defined by sending (x, θ) to $(rot_\theta(x), \theta)$, where $rot_\theta : S^2 \to S^2$ rotates S^2 about a fixed axis through the angle θ.

4-dimensional analogues

There are two 4-dimensional analogues of Dehn surgery: the Gluck twist on 2-spheres and torus surgery.

Given a 2-knot $\mathcal{S}$ in S^4 (an embedded $S^2 \subset S^4$), the *Gluck twist* on $\mathcal{S}$ is the process of removing a neighborhood $\nu\mathcal{S} \cong S^2 \times D^2$ and regluing by the diffeomorphism $\rho : S^2 \times S^1 \to S^2 \times S^1$ defined by sending (x, θ) to $(rot_\theta(x), \theta)$, where $rot_\theta : S^2 \to S^2$ rotates S^2 about a fixed axis through the angle θ.

The result is a homotopy 4-sphere, although for some classes of 2-knots (for example *ribbon* 2-knots) it's known that we get the standard S^4.

4-dimensional analogues

Torus surgery:

4-dimensional analogues

Torus surgery: Given a torus $\mathcal{T} \subset S^4$, we can remove a neighborhood $\nu\mathcal{T} \cong T^2 \times D^2$ and reglue by some diffeomorphism ϕ of the boundary.

4-dimensional analogues

Torus surgery: Given a torus $\mathcal{T} \subset S^4$, we can remove a neighborhood $\nu\mathcal{T} \cong T^2 \times D^2$ and reglue by some diffeomorphism ϕ of the boundary. As with Dehn surgery, the resulting manifold is determined by $[\phi(pt \times \partial D^2)]$.

4-dimensional analogues

Torus surgery: Given a torus $\mathcal{T} \subset S^4$, we can remove a neighborhood $\nu\mathcal{T} \cong T^2 \times D^2$ and reglue by some diffeomorphism ϕ of the boundary. As with Dehn surgery, the resulting manifold is determined by $[\phi(pt \times \partial D^2)]$. There is an associated integer called the *multiplicity*, which counts how many times $pt \times \partial D^2$ wraps around in the meridinal direction.

4-dimensional analogues

Torus surgery: Given a torus $\mathcal{T} \subset S^4$, we can remove a neighborhood $\nu\mathcal{T} \cong T^2 \times D^2$ and reglue by some diffeomorphism ϕ of the boundary. As with Dehn surgery, the resulting manifold is determined by $[\phi(pt \times \partial D^2)]$. There is an associated integer called the *multiplicity*, which counts how many times $pt \times \partial D^2$ wraps around in the meridinal direction.

There is a unique torus that bounds a solid torus $S^1 \times D^2$ in S^4; we call it the *unknotted* torus.

4-dimensional analogues

Torus surgery: Given a torus $\mathcal{T} \subset S^4$, we can remove a neighborhood $\nu\mathcal{T} \cong T^2 \times D^2$ and reglue by some diffeomorphism ϕ of the boundary. As with Dehn surgery, the resulting manifold is determined by $[\phi(pt \times \partial D^2)]$. There is an associated integer called the *multiplicity*, which counts how many times $pt \times \partial D^2$ wraps around in the meridinal direction.

There is a unique torus that bounds a solid torus $S^1 \times D^2$ in S^4; we call it the *unknotted* torus. Some facts: multiplicity 1 surgery on the unknotted torus results in S^4

4-dimensional analogues

Torus surgery: Given a torus $\mathcal{T} \subset S^4$, we can remove a neighborhood $\nu\mathcal{T} \cong T^2 \times D^2$ and reglue by some diffeomorphism ϕ of the boundary. As with Dehn surgery, the resulting manifold is determined by $[\phi(pt \times \partial D^2)]$. There is an associated integer called the *multiplicity*, which counts how many times $pt \times \partial D^2$ wraps around in the meridinal direction.

There is a unique torus that bounds a solid torus $S^1 \times D^2$ in S^4; we call it the *unknotted* torus. Some facts: multiplicity 1 surgery on the unknotted torus results in S^4, and multiplicity 0 surgery on the unknotted torus results in either $S^1 \times S^3 \# S^2 \times S^2$ or $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$.

Embedding theorems

Embedding theorems

Theorem (L)

If L is a ribbon link in S^3, and M_L is a 3-manifold obtained by Dehn surgery on L with all coeficients belonging to the set $\{1/n\}_{n\in\mathbb{Z}}$, then M_L smoothly embeds in S^4.

Embedding theorems

Theorem (L)

If L is a ribbon link in S^3, and M_L is a 3-manifold obtained by Dehn surgery on L with all coeficients belonging to the set $\{1/n\}_{n\in\mathbb{Z}}$, then M_L smoothly embeds in S^4.

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$.

Embedding theorems

Theorem (L)

If L is a ribbon link in S^3, and M_L is a 3-manifold obtained by Dehn surgery on L with all coeficients belonging to the set $\{1/n\}_{n\in\mathbb{Z}}$, then M_L smoothly embeds in S^4.

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$.

If pq is odd, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \tilde{\times} S^2$.

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \times S^2.$$

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \times S^2.$$

If pq is odd, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \tilde{\times} S^2$

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \times S^2.$$

If pq is odd, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \widetilde{\times} S^2.$$

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \times S^2.$$

If pq is odd, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \widetilde{\times} S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \widetilde{\times} S^2.$$

Theorem (Gompf)

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \times S^2.$$

If pq is odd, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \tilde{\times} S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \tilde{\times} S^2.$$

Theorem (Gompf)

If M is an integral homology sphere that is surgery on a knot ($M \cong S^3_{1/n}(K)$), then M° smoothly embeds in S^4.

Strategy

Strategy

We realize Dehn surgery as a cross section of a:

Strategy

We realize Dehn surgery as a cross section of a:

1. Gluck twist on a (ribbon) 2-knot

Strategy

We realize Dehn surgery as a cross section of a:

1. Gluck twist on a (ribbon) 2-knot
2. multiplicity 0 surgery on the unknotted torus

Strategy

We realize Dehn surgery as a cross section of a:

1. Gluck twist on a (ribbon) 2-knot
2. multiplicity 0 surgery on the unknotted torus
3. multiplicity 1 surgery on the unknotted torus

Proof of the first theorem

Proof of the first theorem

First suppose that K is a ribbon knot. We want to show that $S^3_{1/n}(K)$ smoothly embeds in S^4.

Proof of the first theorem

First suppose that K is a ribbon knot. We want to show that $S^3_{1/n}(K)$ smoothly embeds in S^4.

K bounds a ribbon disk $\mathcal{D}$ in B^4.

Proof of the first theorem

First suppose that K is a ribbon knot. We want to show that $S^3_{1/n}(K)$ smoothly embeds in S^4.

K bounds a ribbon disk $\mathcal{D}$ in B^4. We can double the disk to get a ribbon 2-knot $\mathcal{S}$ in S^4, with K as the equator.

Proof of the first theorem

First suppose that K is a ribbon knot. We want to show that $S^3_{1/n}(K)$ smoothly embeds in S^4.

K bounds a ribbon disk $\mathcal{D}$ in B^4. We can double the disk to get a ribbon 2-knot $\mathcal{S}$ in S^4, with K as the equator. We perform a Gluck twist on $\mathcal{S}$ (cut out $\nu\mathcal{S} \cong S^2 \times D^2$ and reglue by ρ), which returns S^4, and analyze what happens to a neighborhood of K.

Proof of the first theorem

First suppose that K is a ribbon knot. We want to show that $S^3_{1/n}(K)$ smoothly embeds in S^4.

K bounds a ribbon disk $\mathcal{D}$ in B^4. We can double the disk to get a ribbon 2-knot $\mathcal{S}$ in S^4, with K as the equator. We perform a Gluck twist on $\mathcal{S}$ (cut out $\nu\mathcal{S} \cong S^2 \times D^2$ and reglue by ρ), which returns S^4, and analyze what happens to a neighborhood of K. We can choose our axis for S^2 so that ρ preserves the equator K, and hence the surgery removes $K \times D^2$ and reglues by sending (x, θ) to $(rot_\theta(x), \theta)$.

Proof of the first theorem

First suppose that K is a ribbon knot. We want to show that $S^3_{1/n}(K)$ smoothly embeds in S^4.

K bounds a ribbon disk $\mathcal{D}$ in B^4. We can double the disk to get a ribbon 2-knot $\mathcal{S}$ in S^4, with K as the equator. We perform a Gluck twist on $\mathcal{S}$ (cut out $\nu\mathcal{S} \cong S^2 \times D^2$ and reglue by ρ), which returns S^4, and analyze what happens to a neighborhood of K. We can choose our axis for S^2 so that ρ preserves the equator K, and hence the surgery removes $K \times D^2$ and reglues by sending (x, θ) to $(rot_\theta(x), \theta)$. For a fixed $x \in K$, we see that this sends a meridian to a (1,1) curve.

Proof of the first theorem

First suppose that K is a ribbon knot. We want to show that $S^3_{1/n}(K)$ smoothly embeds in S^4.

K bounds a ribbon disk $\mathcal{D}$ in B^4. We can double the disk to get a ribbon 2-knot $\mathcal{S}$ in S^4, with K as the equator. We perform a Gluck twist on $\mathcal{S}$ (cut out $\nu\mathcal{S} \cong S^2 \times D^2$ and reglue by ρ), which returns S^4, and analyze what happens to a neighborhood of K. We can choose our axis for S^2 so that ρ preserves the equator K, and hence the surgery removes $K \times D^2$ and reglues by sending (x, θ) to $(rot_\theta(x), \theta)$. For a fixed $x \in K$, we see that this sends a meridian to a (1,1) curve. In other words, this is just $+1$ Dehn surgery on K. To get 1/n surgery we just perform the surgery using a power ρ^n of ρ.

Theorem (L)

If pq is even, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \times S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \times S^2.$$

If pq is odd, $S^3_{p/q}(K)$ embeds in $S^1 \times S^3 \# S^2 \tilde{\times} S^2$, and

$$S^3_{p/q}(K)^\circ \text{ embeds in } S^2 \tilde{\times} S^2.$$

Theorem (Gompf)

If M is an integral homology sphere that is surgery on a knot ($M \cong S^3_{1/n}(K)$), then M° smoothly embeds in S^4.

Thank you!