CR Structures On Three Dimensional Contact Manifolds

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Presentation Outline

The subject arose from a paper of Poincaré in which he considered whether the Riemann Mapping Theorem could be generalized from $\mathbb{C}^1$ to $\mathbb{C}^2$. He showed that the analogous result does not hold in $\mathbb{C}^2$ by using two hypersurfaces of $\mathbb{C}^2$ are not equivalent.

Presentation Outline

Different authors use different CR definitions. We use the following definition.

Definition

A CR structure on an odd dimensional manifold M^{2n+1} is a germ of a complex structure J on $\mathcal{O}(M \times \{0\}) \subset M \times \mathbb{R}$

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Abstract CR manifolds:

1. (M, V) is a CR manifold if $dimM = 2n+1$, V is a subbundle of $\mathbb{C} \otimes TM$ with complex dimension $dimV = n$, $V \cap \bar{V} = 0$ and $[V, V] \subset V$.

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2. (M, H, J) is a CR manifold if $dimM = 2n+1$, H is a real subbundle of TM with $dimH = 2n$, $J : H \to H$ and $J^2 = -Id$.
   If X and Y are in H , then so is $[JX, Y] + [X, JY]$ and

$$J\{[JX, Y] + [X, JY]\} = [JX, JY] - [X, Y]$$

If (M, ξ, J) are real analytic and satisfy the equation

$$X, Y \in \xi \Rightarrow [\tilde{J}X, \tilde{J}Y] - [X, Y] = \tilde{J}([\tilde{J}X, Y] + [X, \tilde{J}Y]) \in \xi$$

where $\tilde{J} = J_{|_\xi}$ on ξ, then $\tilde{J}$ extends to an integrable complex structure on $\mathcal{O}(M \times \{0\}) \subset M \times \mathbb{R}$.

Then the weaker deinition of CR implies our definition.

Presentation Outline

Theorem (Y. Ozan, H. Coban)

Let M be a smooth orientable closed 3-manifold and ξ a contact structure on M. Then M admits a CR structure that induces a contact structure, which is isotopic to ξ.

- (M^{2n+1}, ξ) is a contact manifold if $ker(\alpha) = \xi$ and $\alpha \wedge (d\alpha)^n \neq 0$.

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- Let (N, ξ), $\xi = ker(\alpha)$ be a compact contact manifold then $(N \times \mathbb{R}, d(e^t\alpha))$ its symplectization.

In the proof, we use the fact that

1. Any closed smooth manifold is diffeomorphic to a nonsingular real algebraic set. Indeed more is true: If $M \subset \mathbb{R}^n$ is a closed smooth submnaifold then it is isotopic to a nonsingular real algebraic set in $\mathbb{R}^{n+1}$. Then we may assume that any closed smooth manifold has a real analytic structure.

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1. Any closed smooth manifold is diffeomorphic to a nonsingular real algebraic set. Indeed more is true: If $M \subset \mathbb{R}^n$ is a closed smooth submnaifold then it is isotopic to a nonsingular real algebraic set in $\mathbb{R}^{n+1}$. Then we may assume that any closed smooth manifold has a real analytic structure.
2. Tangent bundle of a nonsingular real algebraic variety is strongly algebraic. Differential forms on a real algebraic variety can be approximated by regular, in particular real analytic forms.

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From now on we identify M with X. Let $\alpha \in \Omega^1(X)$ be a smooth 1-form so that $\xi = \ker(\alpha)$. Using the coordinate chart

$$\phi : (-1/2, 1/2) \to U \subset S^1, \ \phi(t) = e^{2\pi i t}, \ t \in (-1/2, 1/2),$$

define a smooth function $f : S^1 \to \mathbb{R}$ so that $f(t) = e^t$ if $|t| < 1/4$ and $f(t) = 0$ if $|t| \geq 1/3$. Then the 2-form $\omega = d(f(t)\alpha) \in \Omega^2(X \times S^1)$ is a symplectization of the contact structure in a neighborhood of X in $X \times S^1$.

The cotangent bundle of $X \times S^1$ is strongly algebraic and therefore the smooth 1-form $f(t)\alpha \in \Omega^1(X \times S^1)$ can be approximated in the C^∞-topology by real algebraic (hence real analytic) 1-forms with arbitrary precision.

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Now, let $\beta \in \Omega^1(X \times S^1)$ be a real algebraic 1-form so close to $f(t)\alpha$ that its restriction to X is isotopic to α, as a contact form, and the 2-form $d\beta$ is still symplectic on an open neighborhood $U = X \times (-\epsilon, \epsilon)$ of X in $X \times S^1$. Moreover, the symplectic form on U is real algebraic and thus real analytic.

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Note that the restriction of the Euclidean metric of $\mathbb{R}^8$ to the submanifold $X \times S^1$ is a Riemannian metric $g = (g_{ij})$ with real analytic coefficients.

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The almost complex structure on U gives a complex line distribution on T_*X, which is nothing but the contact structure given by the 1-form $\ker(\beta_{|X})$. We know that this contact structure is isotopic to $\ker(\alpha) = \xi$. This complex line distribution on T_*X is integrable by dimension reasons.

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This finishes the proof.

THANK YOU!