Introduction to the Berge Conjecture

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Outline

Introduction

Dehn Surgery
- Definition
- Example

Lens Spaces and the Berge conjecture
- Lens Spaces
- Berge Knots

Martelli and Petronio

Baker

Families of Berge Knots

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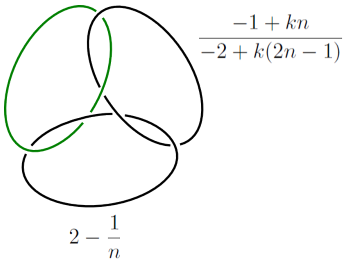
Introduction

In their 2008 paper, “Dehn Surgery and the magic 3-manifold”, Martelli and Pertronio ended with the following statement:

Introduction

It is not yet known whether [the partial filling on the 3-chain link]...



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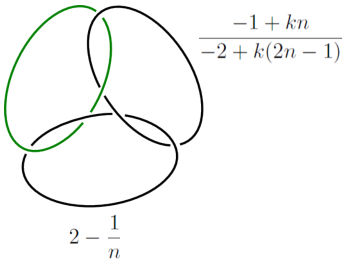

gives rise to Berge knots.

In this talk I will aim to answer this question and discuss how this relates to the Berge conjecture and future work.

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Dehn Surgery

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- A knot L.

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- a specifed simple closed curve J in ∂N.

Then we can construct the 3-manifold:

$$M = (S^3 - \mathring{N}) \bigcup_h N$$

where $\mathring{N}$ denotes the interior of N, and h is a homeomorphism which takes the meridian, μ, of N to the specifed J.

Dehn Surgery

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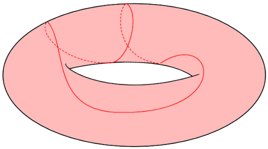

Dehn Surgery Example

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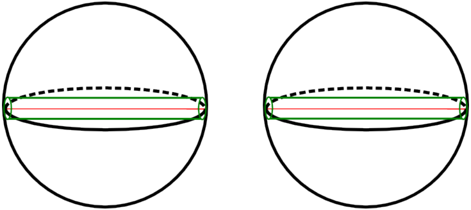
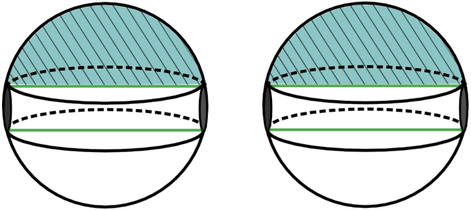
Consider surgery on the unknot with surgery coefficent $\frac{0}{1}$. Embedded in S^3, we can depict the unknot with closed tubular neighbourhood, N, as:



With the two solid balls identified at their boundary, as in a standard representation of S^3.

Dehn Surgery Example

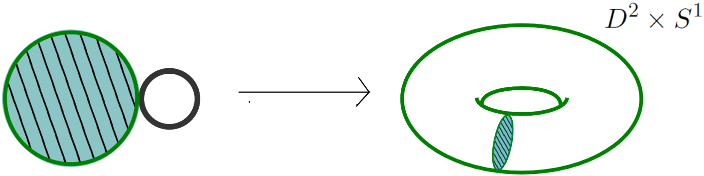
Now we remove the interior of N from S^3 and consider the cross section shown in blue:



We can see that when the two solid balls are glued together by their boundaries now, the two blue cross sections will form a disk.

Dehn Surgery Example

We have such disks all the way round the green boundary, i.e.



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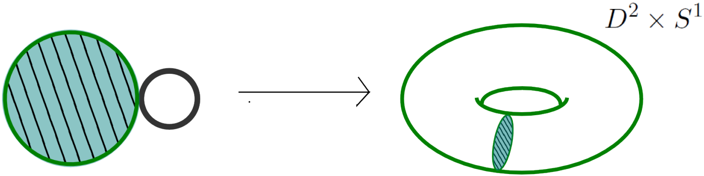

This gives us a solid torus, $D^2 \times S^1$.

- Note: This is a special case, in general we will just have a 3-manifold, M, with a boundary component at this stage.

Dehn Surgery Example

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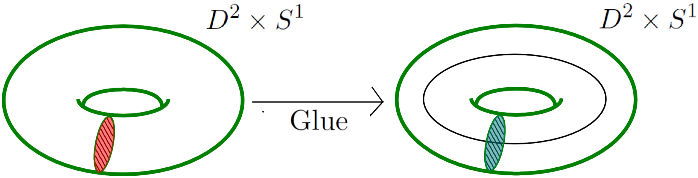

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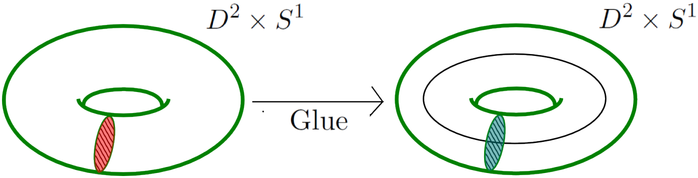
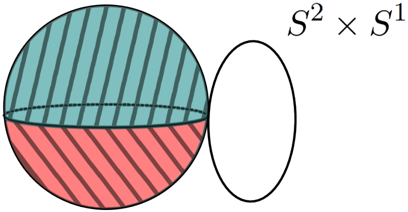


This gives the space, $S^2 \times S^1$.



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- Then the space $T_1 \bigcup_h T_2$ is denoted $L(b, a)$ and is called a Lens space.

Alternatively, as we saw in the last example, we can think of the Lens space, $L(p, q)$ as a surgery on the unknot with surgery coefficient $\frac{p}{q}$.

Berge Knots

A Berge knot (also called a double primitive knot) is a particular type of knot classified by John Berge.

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Conjecture

The Berge Conjecture states that the only knots which emit lens space surgeries are the Berge knots.

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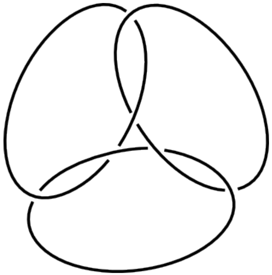
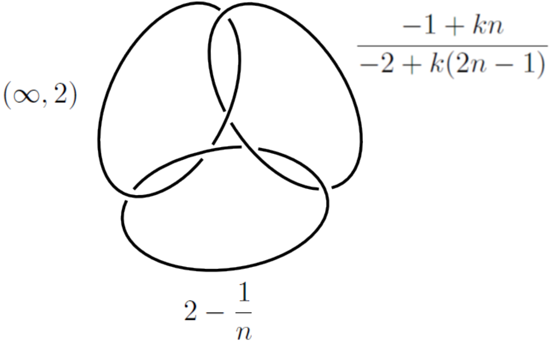
Martelli and Petronio

In their 2008 paper, Martelli and Petronio enumerate all lens space surgeries on the 3-chain link.



Martelli and Petronio

In particular, they show that Dehn surgery on the 3-chain link according to the instructions shown below, obtains a family of knot exteriors with lens space fillings.



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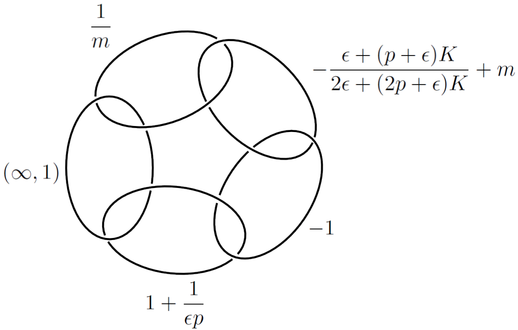
In his 2005 paper, “Surgery descriptions and volumes of Berge knots II”, Kenneth Baker provides a classification of a subset of type IV Berge knots, as a surgery on the minimally twisted 5-chain link.



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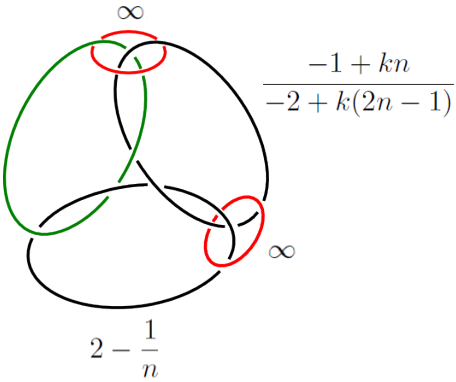
Results so far

By inserting two unknotted components with ∞-fillings, and performing left handed twists around those components, we can transform the description given by Martelli and Petronio into a surgery instruction on the minimally twisted 5-chain link.



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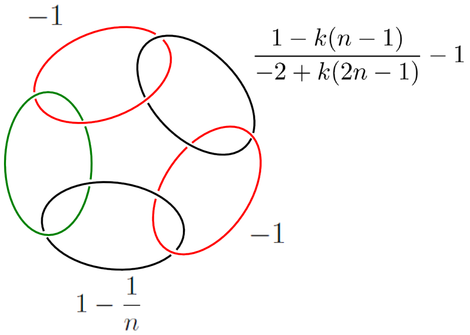

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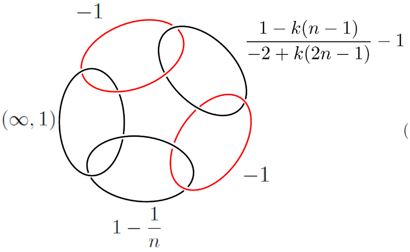
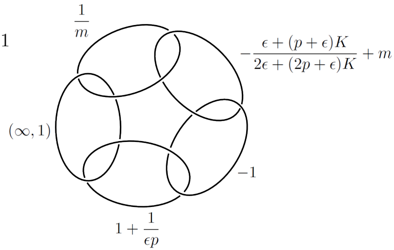
This description can then be shown to correspond to the classification of Berge knots given by Baker (by setting $m = -1$, $\epsilon = -1$ and $p = n$).





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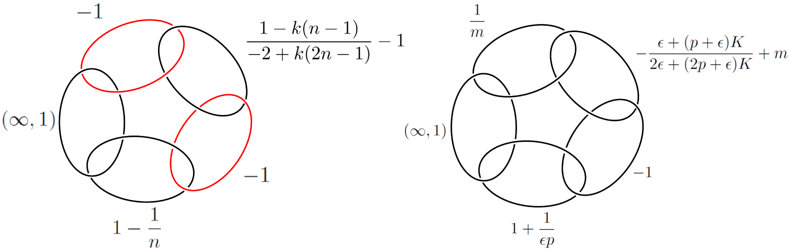

Thus showing that the family of knots described by Martelli and Petronio are in fact Berge knots.

Other interesting families

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$$\left(M_3\left(1-\frac{1}{n},1+\frac{1}{n}\right),\infty,3,0\right)$$

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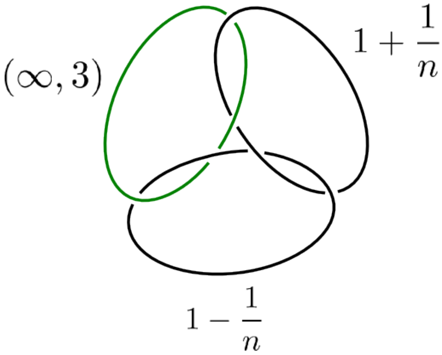

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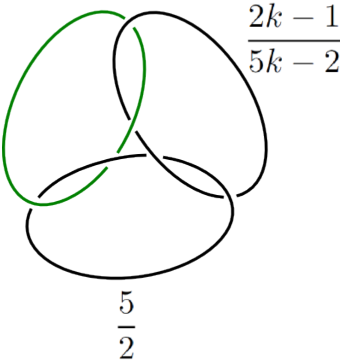

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A family of knot exteriors with two lens space fillings, not from the Berge manifold:

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A family of knot exteriors with a lens space filling and a toroidal filling:

$$\left(M_3\left(1-\frac{1}{n}, 1+\frac{1}{n-2}\right), \infty, 3, 0\right)$$

Other interesting families

With the use of Baker's classification of Berge knots types I-VI, we aim to check whether these families of knots are also Berge knots or possible counter examples to the Berge conjecture.