Multivariable link invariants and renormalized quantum dimension

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Abstract

- We intend to describe a family of multivariable link invariants introduced by N. Geer and B. Patureau.
- The algebraic input will be a category of representations associated to a super Lie algebra of type one.
- The key point is to define a "renormalized quantum dimension" of a module and use it instead of the usual quantum dimension in a Reshetikhin-Turaev type construction.
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Outline

1. Renormalized Reshetikhin-Turaev type construction
   - Motivation
   - Classical Reshetikhin-Turaev invariants
   - Renormalized construction
2. Multivariable invariants
   - Geer and Patureau's Multivariable Invariants
   - Relations with other known invariants
   - Further directions

Motivation

- In 1991, Reshetikhin and Turaev defined one construction which starts with any Ribbon category and and gives colored link invariants.
- They use in the definition the notion of quantum dimension of a module.
- Usually, people apply this construction for categories which come from the representation theory of some Hopf algebras(quantum groups).
- If we start with g a super-Lie algebra of type one, and we look at the quantum enveloping algebra, this is a quantum group, but we have some issues.

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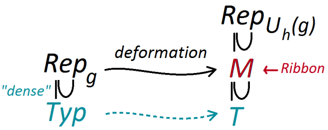
Motivation II

- We have a method to produce a Ribbon category using its representation theory.



- However, if we look at the Reshetikhin-Turaev construction for M, this leads to invariants for M-colored links that vanish on any link which has at least one strand colored with a T-color.
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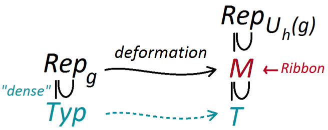

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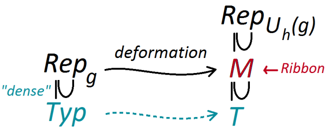

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Definition

- Let $\mathcal{C}$ a strict monoidal category.
- A braiding C is a natural set of isomorphisms $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \to W \otimes V, V, W \in \mathcal{C}\}$ such that for any $U, V, W \in \mathcal{C}$ the following relations hold:
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- If $\mathcal{C}$ has the brading C, a twist means a family of natural isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$: $\theta_{V\otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W)$.
- We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*, \quad d'_V : V \otimes V^* \to \mathbf{1}$ with the following properties: $(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_v$
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- The duality is said to be compatible with the brading and the twist if: $\forall V \in \mathcal{C}$, $(\theta_V \otimes Id_{V^*})b_V = (Id_V \otimes \theta_{V^*})b_V$. A category with a brading, a twist and a compatible duality is called a Ribbon Category.

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Category of framed colored tangles

Definition

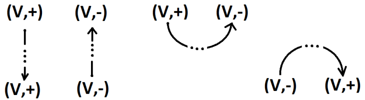
Consider $\mathcal{C}$ a category. The category of $\mathcal{C}$-colored framed tangles $\mathcal{T}_{\mathcal{C}}$ is defined as follows:

$Ob(\mathcal{T}_{\mathcal{C}}) = \{(V_1, \epsilon_1), ..., (V_m, \epsilon_m) \mid m \in \mathbb{N}, \epsilon_i \in \{\pm 1\}, V_i \in \mathcal{C}\}$.

$Morph(\mathcal{T}_{\mathcal{C}})((V_1, \epsilon_1), ..., (V_m, \epsilon_m), (W_1, \delta_1), ..., (W_n, \delta_n)) =$

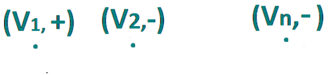
$$\frac{\mathcal{C}\,colored\ framed\ tangles\ T : (V_1, \epsilon_1), ..., (V_m, \epsilon_m) \to (W_1, \delta_1), ..., (W_n, \delta_n)}{isotopy}.$$

- *Observation* : The tangles have to respect the colors V_i.
  Once we have such a tangle, it has an induced orientation, coming from the signs ϵ_i, using the following conventions:

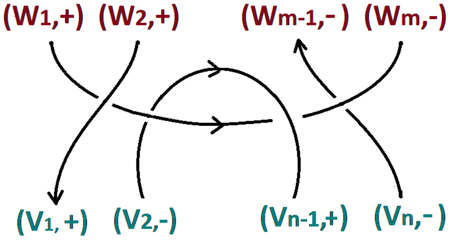


Example

$V \in Ob$



$T \in Morph(V,W)$



Reshetikhin-Turaev functor

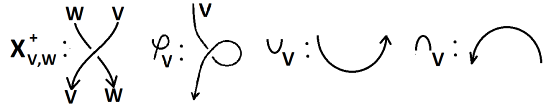
- Aim: Starting with any Ribbon Category $\mathcal{C}$, we'll define a functor from the category of framed $\mathcal{C}$-colored tangles to $\mathcal{C}$.

Theorem (Reshetikhin-Turaev)

Consider($\mathcal{C}$, C, Θ, b, d') a Ribbon category. Then there exist an unique functor $F_{\mathcal{C}} : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}$ which is monoidal and satisfies the following local relations for any $V, W \in \mathcal{C}$:

1)$F((V,+)) = V \quad F((V,-)) = (V)^*$

2)$F(X^+_{V,W}) = C_{V,W} \quad F(\varphi_V) = \theta_V \quad F(\cup_V) = b_V \quad F(\cap_V) = d'_V$, where



Super Lie algebras of type I

Definition

A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\ ,\ ] : g^{\otimes 2} \to g$ which satisfies:
1) $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$
2) Super Jacobi Identity: $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$

- There is a splitting $g = n_- \oplus \mathfrak{H} \oplus n_+$ where h is the Cartan subalgebra of g.
- Elements of $\mathfrak{H}^*$ are called weights.
- The algebra can be described by generators and relations using a Cartan matrix.
- There are two families of super Lie algebras of type I: $sl(m, n)$ and $osp(2, 2n)$.

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Reprezentation theory of g

Theorem

There is the following correspondence:

$$\{irred.\ f.\ dimensional\ g-modules\} \longleftrightarrow highest\ weights \longleftrightarrow \Lambda = \mathbb{N}^{r-1} \times \mathbb{C}$$

$$V(\lambda) \qquad \lambda \qquad ((\lambda(h_i)), \lambda(h_s))$$

$-typical$

$-atypical \qquad \hookrightarrow \mathbb{N}^{r-1} \times \mathbb{Z}$

The quantization $U_h(g)$

Definition

Let g be a super Lie algebra of type I. The quantization of g, denoted by $U_h(g)$ is the $\mathbb{C}[[h]]$-super-algebra generated by three families of elements h_i, E_i and F_i, for $i \in \{1, ..., r\}$ with the relations:

$[h_i, h_j] = 0 \qquad [E_i, F_j] = \delta ij \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$

$[h_i, E_j] = a_{ij} E_j \qquad [h_i, F_j] = -a_{ij} F_j \qquad {E_s}^2 = {F_s}^2 = 0$

and quantum Serre type relations, where $[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx$.

Definition

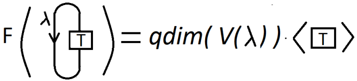
An $U_h(g)$-module W is called topologically free of finite rank if there is a finite dimensional g-module V with $W \simeq V[[h]]$ as $\mathbb{C}[[h]]$-modules.

Theorem

Denote by $\mathcal{M}$=the category of topologically free of finite rank $U_h(g)$-modules. Then this is a Ribbon category.

The modified quantum dimension

- Once we obtained the Ribbon Category $\mathcal{M}$, we might think to apply the Reshetikhin-Turaev construction for that in order to obtain $\mathcal{M}-$colored link invariants.
- From the functoriality of F, we have that:



- From an argument using Kontsevich integral, it follows that: $qdim(\tilde{V(\lambda)}) = 0$ for any typical color λ.
- As a conclusion, the Reshetikhin-Turaev invariant $F(L) = 0$ for any link L colored with at least one typical color.

Idea

Essentially, here the quantum dimension can be viewed as a function $qdim : \{weights\} \longrightarrow C[\,[\,h]\,]$.
The main point is to replace this quantum dimension with another function such that and with a similar definition to be able to obtain link invariants.

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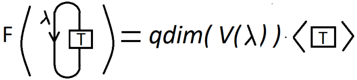

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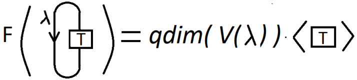

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- More specifically the definition would be in the following way:

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Let L be a $\mathcal{M}$-colored link with at least one typical color λ. The Geer and Patureau renormalized function F' is defined as:

$$F'(L) = d(\lambda) < T_\lambda >$$

where T_λ is the tangle obtained from T by cutting the λ-colored strand.

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- More specifically the definition would be in the following way:

Definition

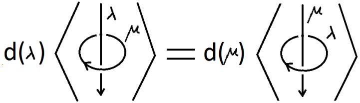
Let L be a $\mathcal{M}$-colored link with at least one typical color λ. The Geer and Patureau renormalized function F' is defined as:

$$F'(L) = d(\lambda) < T_\lambda >$$

where T_λ is the tangle obtained from T by cutting the λ-colored strand.

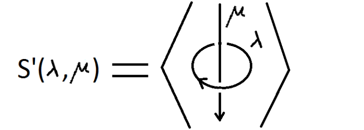
- One point that is important about that function is the fact that it should rise to link invariants. This would mean that F' should not depend on the cutting strand colored with a typical color.

- Let us look at the simplest example of a link, namely the Hopf Link. Consider it colored with two typical colors λ, μ. We would like F' to be the same either if we use the cutting strand λ or μ. This is equivalent with:



- The previous relation motivates the following notation:

Definition



- This means that a necessary condition for d would be:

$$\frac{d(\lambda)}{d(\mu)} = \frac{S'(\lambda,\mu)}{S'(\mu,\lambda)}.$$

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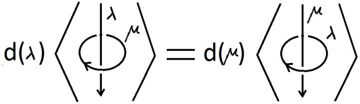

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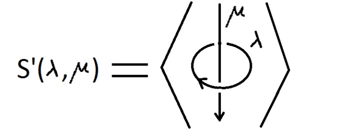

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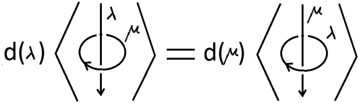

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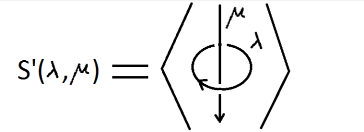

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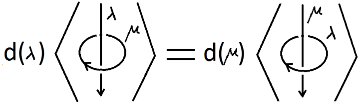

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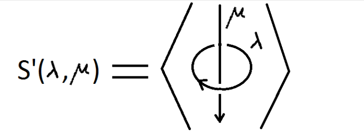

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PROPOSITION

Using the character formulas for g-modules, there is the following relation:

$$S'(\lambda,\mu)=\frac{\varphi_{\mu+\rho}(L'_1)}{\varphi_{\mu+\rho}(L'_0)}\cdot f(\lambda,\mu),$$

where f is a function which is symmetric in λ and μ.

- This means that the renormalized quantum dimension d should verify:

$$\frac{d(\mu)}{d(\lambda)}=\frac{\frac{\varphi_{\mu+\rho}(L'_0)}{\varphi_{\mu+\rho}(L'_1)}}{\frac{\varphi_{\lambda+\rho}(L'_0)}{\varphi_{\lambda+\rho}(L'_1)}}$$

Theorem Geer-Patureau 2010

Define $d : \{typical\ weights\} \to \mathbb{C}[[h]][h^{-1}]$ called the renormalized quantum dimension:

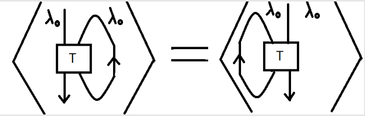
$$d(\lambda) = \frac{\varphi_{\lambda+\rho}(L'_0)}{\varphi_{\lambda+\rho}(L'_1)\varphi_{\rho}(L'_0)}.$$

Let L be a colored link with at least one typical color λ and set $F'(L) = d(\lambda) < T_\lambda >$, where T_λ is obtained from T by cutting the λ-strand. Then F' is a well defined invariant for $\mathcal{M}$-colored links colored with at least one typical color.

- We will outline a sketch of the proof:

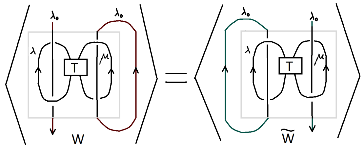
Lemma 1

There exist a special color λ_0 such that $\forall T \in \mathcal{T}((\tilde{V}(\lambda_0), \tilde{V}(\lambda_0)))$:



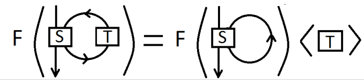
Lemma 2

As an immediate consequence of *Lemma*1, we have:

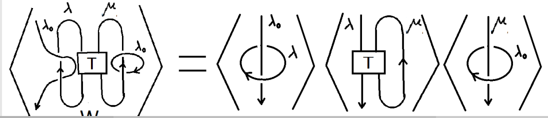


Observation

From the monoidality of the Reshetikhin-Turaev functor, it follows that:



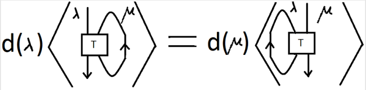
Lemma 3



End of the proof

Final Lemma

For any two typical weights λ and μ we have:



- This previous relation shows that F' does not depends on the cutting strand so it concludes the well definition of the renormalized construction.

- We just defined invariants for links, but which have values almost in $\mathbb{C}[[h]]$. The next theorem shows that in fact they have in some sense one polynomial behavior once we fix the semicolors parametrized by $\mathbb{N}^{r-1}$ and we allow the last complex numbers to vary.

Theorem (Geer and Patureau)

Consider L a link with k components which are ordered and colored with elements $\bar{c}_i \in \mathbb{N}^{r-1}$. Denote by $\bar{c} = (\bar{c}_1, ..., \bar{c}_k)$. Then there is a Laurent polynomial in many variables $M(L, \bar{c})$ such that:
1)

$$M(L,\bar{c}) \in \begin{cases} M_1^{\bar{c}_1}(q,q_1)^{-1}\mathbb{Z}[q^{\pm 1}, {q_1}^{\pm 1}] & \text{if } k=1 \\ \mathbb{Z}[q^{\pm 1}, {q_1}^{\pm 1}, ..., {q_k}^{\pm 1}] & \text{if } k \geq 2 \end{cases}$$

2)For L' a framing on L and $(\xi_1, ..., \xi_k) \in \mathbb{T}_{\bar{c}_1} \times ... \times \mathbb{T}_{\bar{c}_k}$, if we color the $i'th$ knot from L' with $\tilde{V}_{\xi_i}^{\bar{c}_i}$ then:

$$F'(L') = e^{\sum lk_{i,j} < \lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} + 2\rho > \frac{h}{2}} M(L,\bar{c}) \mid_{q_i = e^{\frac{\xi_i h}{2}}} .$$

Relations with other known invariants

- The importance of these multivariable polynomial invariants can be seen from the fact that they are strongly related with other previously known invariants of polynomial type.
- First of all, one specialization of the renormalized invariants $M^{(0,\ldots,0)}_{sl(m|1)}$ recovers the multivariable Alexander polynomials.
- Moreover, the multivariable invariants recover the ADO (Akutsu, Deguchi and Ohtsuki) invariants and they are a generalization of the invariants defined by Links and Gould.
- Also, $\{M^{(0,\ldots,0)}_{sl(m|1)}\}_{m\geq 2}$ have non-trivial intersection with the HOMFLY-PT polynomials and this intersection contains the Kashaev invariants.

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Further directions

- As we have seen, the renormalized construction has as an input an algebraic data, namely a super Lie algebra of type I and leads to multivariable link invariants.
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THANK YOU!