On a Polynomial Alexander Invariant of Tangles

and its categorification

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1. Definition and properties of ∇_T^s

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Basics: What are tangles?

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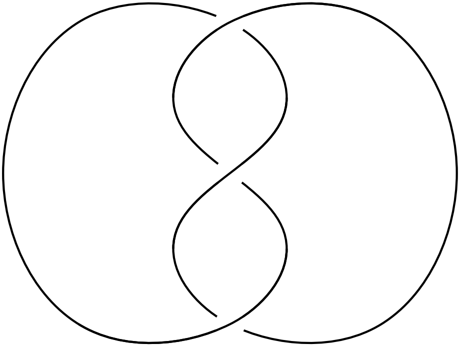

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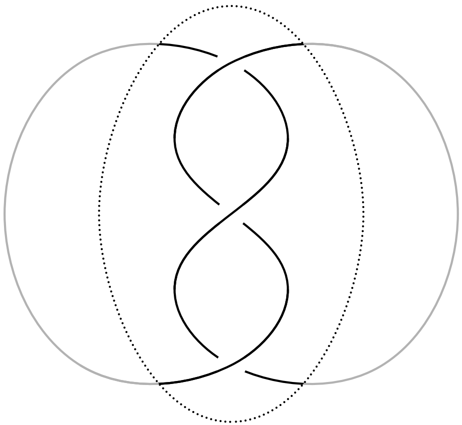

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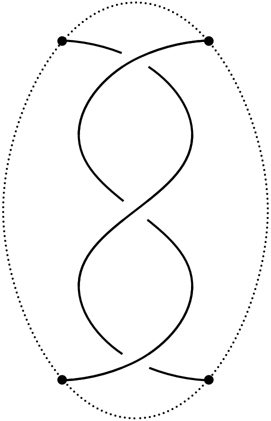

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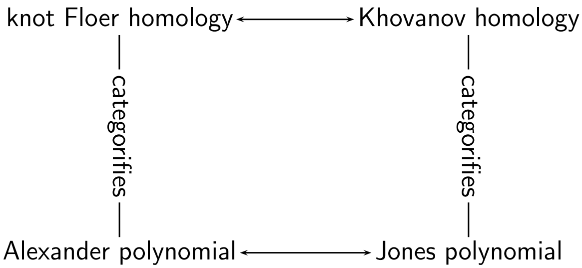
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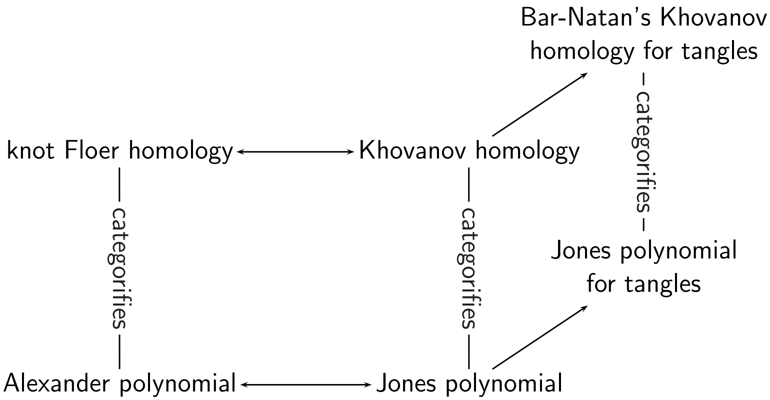

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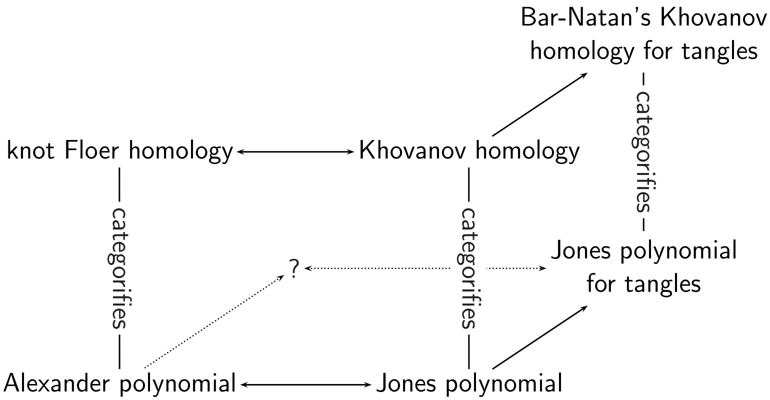
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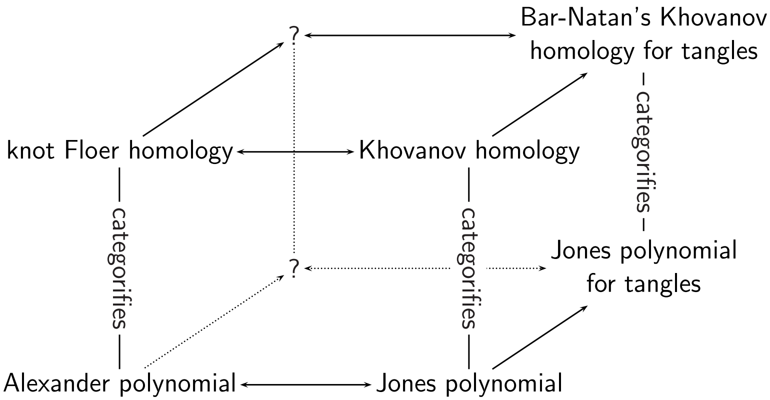
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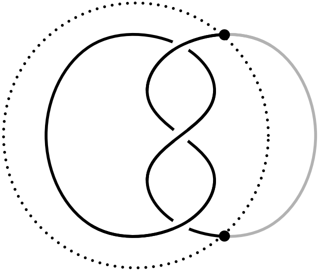

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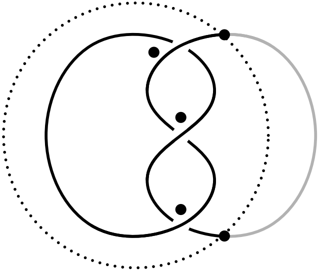

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A **Kauffman state** of a knot diagram K is an assignment of a marker to one of the four regions at each crossing such that each closed region is occupied by exactly one marker.

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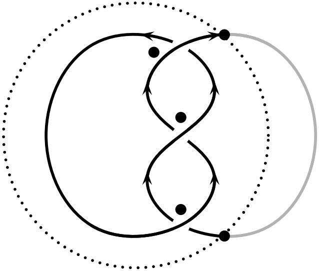

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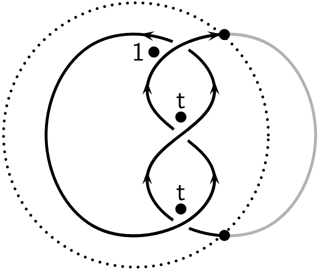

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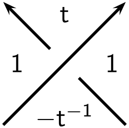

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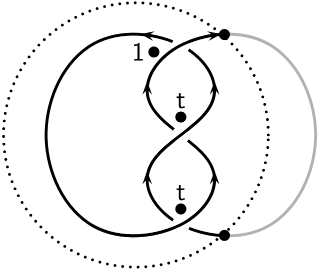

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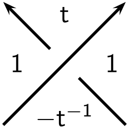

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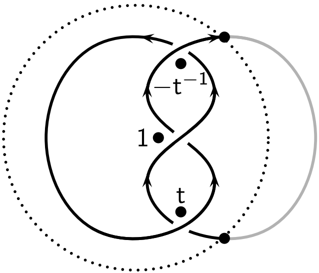

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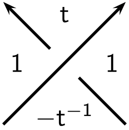

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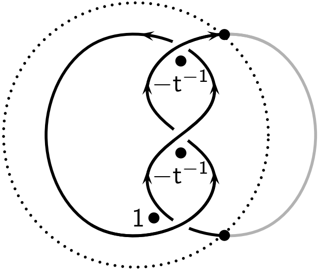

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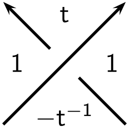

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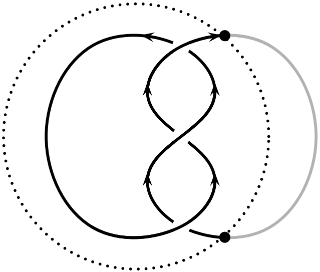

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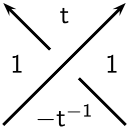

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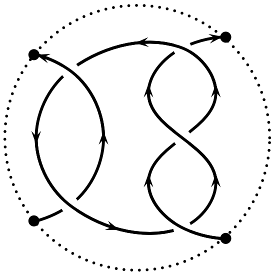

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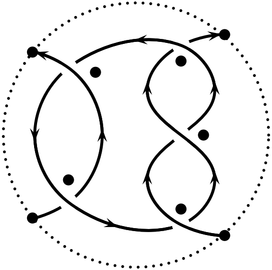

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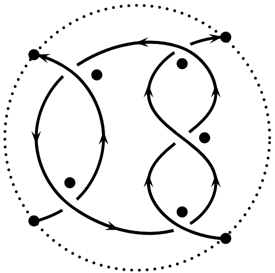

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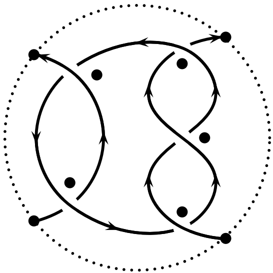

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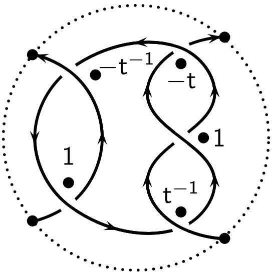

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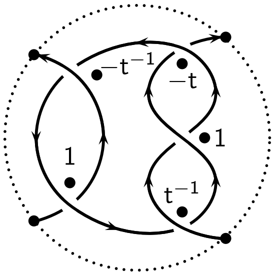

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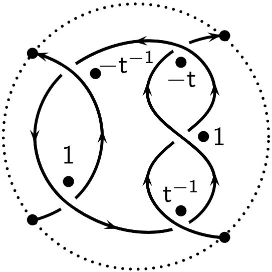

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Note: Every Kauffman state "belongs" to a site.

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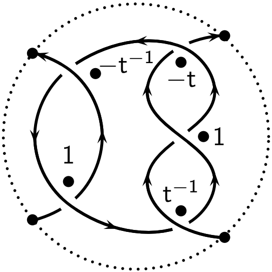

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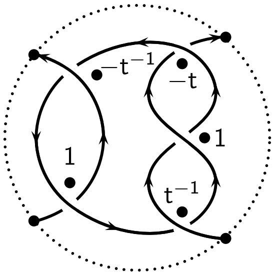

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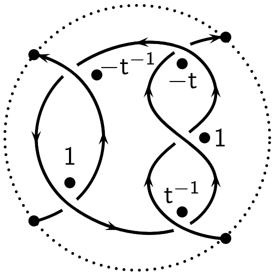

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Properties of ∇_T^s

Theorem (CZ'14)

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  $\rightsquigarrow$ *Cor: The* multivariate *Alexander polynomial is a mutation invariant.*

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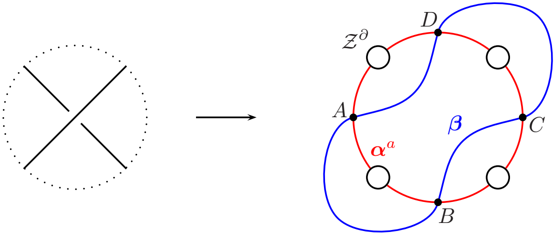

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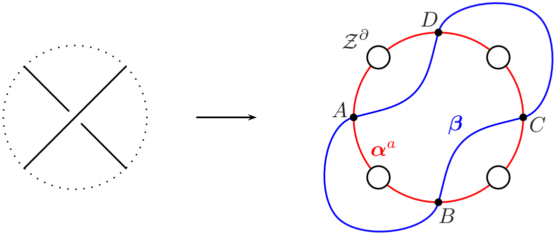

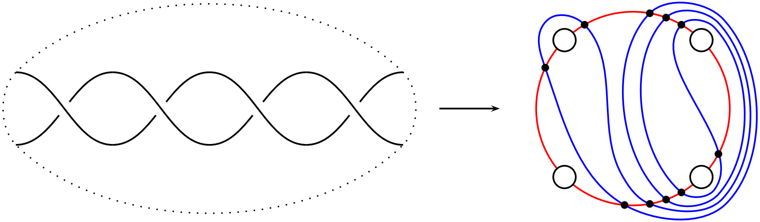

A categorification of ∇_T^s

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Let T be a tangle without any closed components, $\mathcal{H}_T$ a Heegaard diagram for T and s a site of T. Then, we can define a graded chain complex $\widehat{CFT}(\mathcal{H}_T, s)$, whose graded chain homotopy type is an invariant of the tangle T. Its graded Euler characteristic is equal to ∇_T^s. Furthermore, if T represents a knot K, $\widehat{HFT}(T, s)$ coincides with the knot Floer homology $\widehat{HFK}(K)$ of K.

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- The differential only counts holomorphic disks away from punctures.

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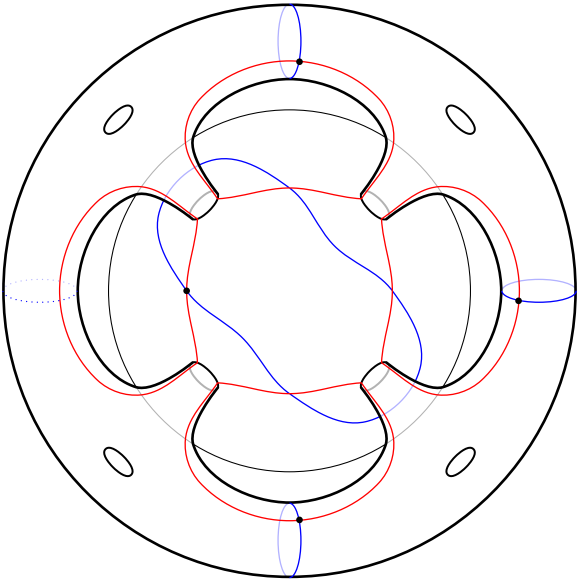
  ...or some alternative approach (work in progress ☺).



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2. A tangle Floer homology $\widehat{HFT}$

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Existing approaches

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