Invariants for transverse knots from Khovanov-type homologies

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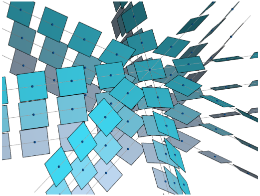
Università degli studi di Firenze

Definition

Let $\mathcal{M}$ be an odd-dimensional manifold. A **contact structure** ξ (on $\mathcal{M}$) is a totally non-integrable hyperplane field.

The **symmetric structure** on $\mathbb{R}^3$ is given by

$$\xi_{sym} = Ker(dz + xdy - ydx);$$



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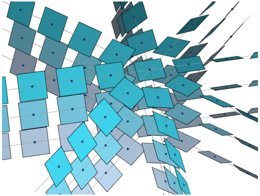

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Let $(\mathcal{M}^3, \xi)$ be a contact (3-)manifold. A (smooth) link in $\mathcal{M}$ is called

1. **Legendrian** if it is everywhere tangent to the contact structure;
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Braids

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The **braid group** on i-strands, denoted by B_i, is the group generated by σ_j, for $j \in \{1, \ldots, i\}$, and subject to the following relations:

$$\sigma_k \sigma_j = \sigma_j \sigma_k, \quad |k-j| > 1,$$

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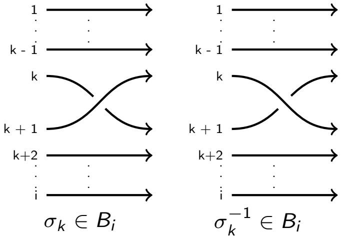

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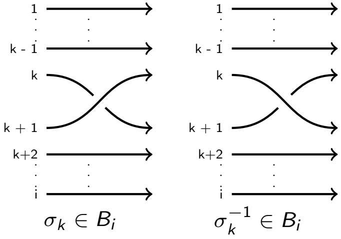

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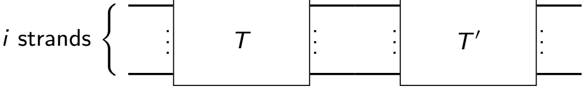

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Operations on braids

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A **braid** T is an element of any B_i, and the integer $i = i(T)$ will be called **braid index** of T.

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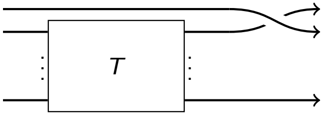

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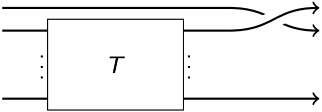

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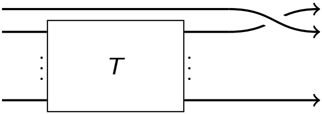

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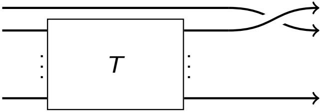

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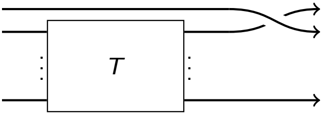

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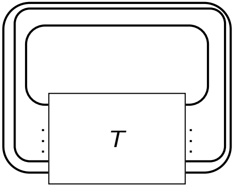

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Transverse links and Braids

Theorem (Bennequin, '83)

Any transverse link in $(\mathbb{R}^3, \xi_{sym})$ is transversely isotopic to the Alexander closure of a braid.

Theorem (Orevkov and Shevchishin, Wrinkle, '03)

Two braids represent the same transverse link type if, and only if, they are related by a finite sequence of conjugations in the braid group, positive stabilizations and positive destabilizations.

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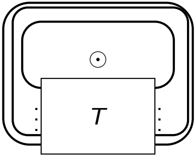

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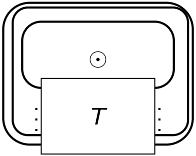

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Classical invariants

There are two classical invariants for transverse links:

1. the **link-type**;
2. the **self-linking number**;

The latter could be defined, in the case of a braid T, as

$$sl(T) = n_+(T) - n_-(T) - i(T).$$

Any invariant which is strictly more powerful than sl and the link-type is called **effective**. A family of transverse link whose elements are told apart one from the other by the two classical invariants is called **simple** (e.g. the unknot, torus knots, figure eight).

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Frobenius Algebras

Definition

A **Frobenius Algebra** $\mathcal{F}$ (over R) is a finitely generated, commutative, free R-algebra A, together with two maps

$$\Delta : A \to A \otimes_R A, \quad \varepsilon : A \to R,$$

such that:

1. Δ is an A-bi-module isomorphism (i.e. commutes with the left and right action of A over $A \otimes A$);
2. ε is R-linear;
3. Δ is co-associative and co-commutative;
4. $(id_A \otimes \varepsilon) \circ \Delta = id_A$.

The map Δ is called **co-multiplication**, while ε is the **co-unit** relative to Δ.

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Example (Khovanov Theory)

$$R_{Kh} = \mathbb{F},\ A_{Kh} = \frac{\mathbb{F}[X]}{(X^2)}.$$

$$\Delta(1) = 1 \otimes X + X \otimes 1,$$
$$\Delta(X) = X \otimes X.$$

$$\varepsilon(1) = 0, \quad \varepsilon(X) = 1.$$

Example (Bar-Natan Theory)

$$R_{BN} = \mathbb{F}[U],\ A_{BN} = \frac{(\mathbb{F}[U])[X]}{(X^2 - U)}.$$

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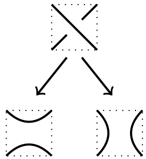
Resolutions

Let D be an oriented link diagram. A **local resolution** of D in a crossing is its replacement with either , a 0-resolution, or with , a 1-resolution.



A **resolution** $\underline{r}$ of D is the set of circles obtained by performing a local resolution at each crossing; We will denote by $|\underline{r}|$ the number of 1-resolution in $\underline{r}$.
A resolution $\underline{s}$ is an **immediate successor** of a resolution $\underline{r}$, if and only if, the following conditions hold.

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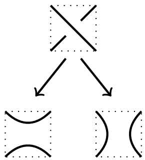

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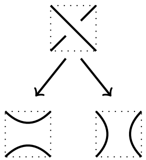

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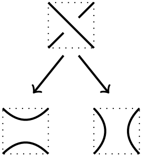

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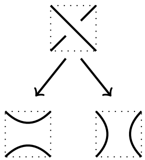

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The $\mathcal{F}$-chain module

Let $\mathcal{F}$ be a Frobenius algebra. The $\mathcal{F}$-chain module $C_{\mathcal{F}}$ is the free $R_{\mathcal{F}}$-module generated by the **states**, i.e. resolutions with circles labelled with a fixed $R_{\mathcal{F}}$-basis for $A_{\mathcal{F}}$.

Figure: A state the Bar-Natan chain module.

The **homological degree** of a state is given by

$$\text{number of 1-resolutions in the underlying resolution} - n_-$$

Moreover, if $A_{\mathcal{F}}$ is a graded $R_{\mathcal{F}}$-algebra, then is it possible to define a **quantum degree** as follows.

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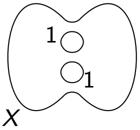

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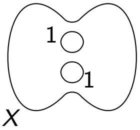

Figure: A state the Bar-Natan chain module.

The **homological degree** of a state is given by

$$\text{number of 1-resolutions in the underlying resolution} - n_{-}$$

Moreover, if $A_{\mathcal{F}}$ is a graded $R_{\mathcal{F}}$-algebra, then is it possible to define a **quantum degree** as follows.

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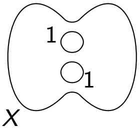

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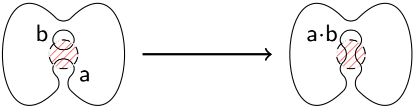
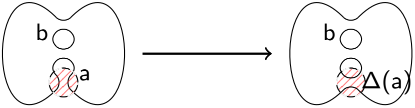
Differential

Given a state $\mathfrak{s}$, its differential is given (up to signs) by summing all the states $\mathfrak{r}$, whose underlying resolution is an immediate successor of the underlying resolution in $\mathfrak{s}$, and whose labels are obtained from those of $\mathfrak{s}$ by multiplying the labels of the circles merged, or co-multiplying* the label of the circle split.





Theorem (Khovanov)

The complex $(C^\bullet_{\mathcal{F}}, d^\bullet_{\mathcal{F}})$ is a (co)chain complex.

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If $\mathcal{F}$ is a Frobenius Algebra of rank two (i.e. $A_{\mathcal{F}}$ has rank 2 as $R_{\mathcal{F}}$-module), then the isomorphism-type of the homology $H^\bullet(C_{\mathcal{F}}(D))$ is a link invariant.

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Plamenevskaya Invariant

1. $\psi \in CKh^{0,sl(T)}(T)$ is a cycle;
2. ψ is a transverse braid invariant;
3. if T' is the negative stabilization of T, then
$$[\psi(T')] = 0;$$
4. if T is a quasi-positive braid, then $[\psi(T)] \neq 0$
5. If T is a braid such that it contains at least a factor σ_i^{-1}, but no factors of the form σ_i, then $[\psi(T)] = 0$;
6. $sl(T) \leq s(\mathring{T}) - 1$, and if the equality holds, then $[\psi(T)] \neq 0$;
7. if $\mathring{T}$ represents a quasi-alternating link and $[\psi(T)] \neq 0$, then $sl(T) = s(\mathring{T}) - 1$. (Plamenevskaya-Baldwin)

Recently, Lenhard Ng, Robert Lipshitz and Sucharit Sarkar introduced a family transverse invariants, $\psi_{p,q}$.

Each $\psi_{p,q}$ belongs to a quotient complex, obtained from a twisted version of the filtered Lee theory.

These invariants recover all the information contained in the Plamenevskaya invariant, in fact $\psi = \psi_{0,1}$.

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The invariant β

1. $\beta \in CBN^{0,sl(T)}(T)$ is a cycle;
2. β is a transverse braid invariant;
3. $[\beta]$ is a non-trivial non-torsion element of $HBN^{0,sl}$;
4. $[\psi] = 0$ if, and only if, $[\beta] = U[\gamma]$, for a certain $[\gamma] \in HBN^{0,sl}$;
5. recovers the all information contained in the invariants $\psi_{p,q}$;
6. the number
$$c(\beta) = max\ \{k \in \mathbb{N} \mid \exists\ [\eta] :\ U^k[\eta] = [\beta]\},$$
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Some problems

1. is β (or ψ) effective?
2. is $c(\beta)$ effective?
3. In which cases it holds

$$sl(T) + 2c(\beta(T)) = s(\mathring{T}) - 1.$$

4. If we consider a reduced version of Bar-Natan theory, then a reduced version of β, say β_{red}, could be defined. How much "transverse information" is stored in β_{red}? Is it effective?