Taut Foliations and Heegard Floer L-Spaces

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Foliation Theory in Low Dimensional Topology

- in 1923, Alexander proved that a two-sphere in $\mathbb{R}^3$ bounds a three-ball mainly using a ”foliation argument”;
- during the eighties, Gabai used *taut foliations* in order to compute the genus of arborescent links;
- more recently, Ozsváth and Szabó used a mixture of foliation theory and contact geometry to prove that Heegaard Floer homology detects the Thurston norm of a three-manifold and the minimal Seifert genus of a knot.

What is a Foliation?

M oriented n dimensional manifold.
A **foliation** $\mathcal{F}$ of M is a decomposition of M

$$M = \coprod_{\alpha} \Sigma_{\alpha}$$

in a disjoint union of smoothly immersed hypersurfaces - the **leaves** of $\mathcal{F}$- locally modelled on

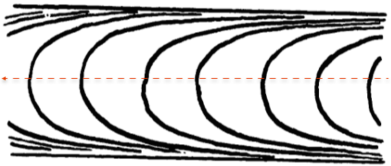
$$\left(\mathbb{R}^3, \text{ horizontal planes} \right).$$

$$S = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \mathbb{R}$$

Define a foliation $\mathcal{F}$ on S setting

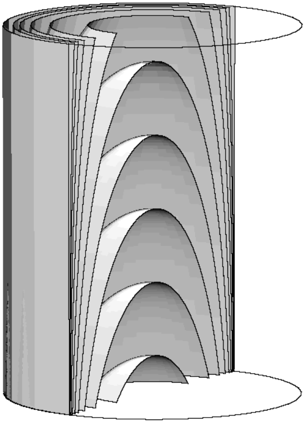
$$\mathcal{F} = \{C_t \,|\; t \in \mathbb{R}\} \cup \left\{ \pm\frac{\pi}{2} \times \mathbb{R} \right\}$$

where $C_t: \;\; y = \sec(x) + t.$

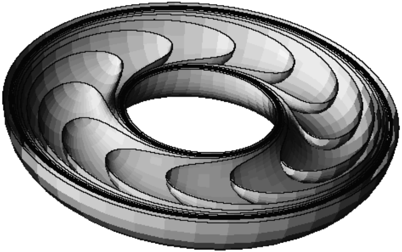


Examples

Rotating the leaves of $\mathcal{F}$ around the z-axis we obtain a foliation on $D^2 \times \mathbb{R}$. The foliated manifold $\left(D^2 \times \mathbb{R}, \mathcal{F}\right)$ is called the **Reeb tube**.



$\mathcal{F}$ induce a foliation on the solid torus $D^2 \times S^1 = D^2 \times \mathbb{R}/_{z \mapsto z+1}$ called **Reeb foliation**.



Take a genus one Heegaard splitting

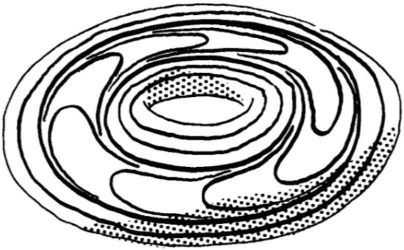
$$L(p,q) = U_0 \cup_\Sigma U_1$$

Filling U_0 and U_1 with the interior of the Reeb foliation and declaring Σ as leaf we obtain a foliation on the whole $L(p,q)$.

Theorem

Every closed oriented three-manifold has a foliation.

Reebless foliations



Reebless foliations

Definition

Y^3 connected orientable smooth 3-manifold, $\mathcal{F}$ foliation of Y.

A **Reeb component** of $\mathcal{F}$ is a smoothly embedded solid torus $T \subseteq Y$ such that

- ∂T is a compact leaf of $\mathcal{F}$,
- $\mathcal{F}|_T$ coincides with the Reeb foliation.

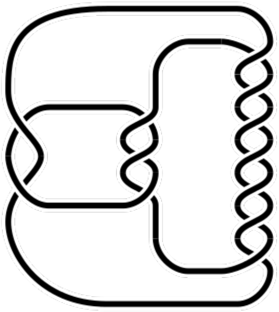
A foliation without Reeb components is said **Reebless**.

Theorem (Palmeira)

If a closed connected and orientable three-manifold Y, with $Y \neq S^2 \times S^1$, has a Reebless foliation then the universal cover of Y is homeomorphic to $\mathbb{R}^3$. Consequently: $\pi_1(Y)$ is torsion free, $\pi_k(Y) = 0$ for $k \geq 2$ and Y is irreducible.

An example of Reebless foliation

$K = (-2, 3, 7)$ pretzel knot.



An example of Reebless foliation

Proposition

$Y = S^3_{+\frac{37}{2}}(K)$ *contains a Reebless foliation.*

Sketch of proof: $Y = X_1 \cup_\psi X_2$ where

$$X_1 = S^3 \setminus N(3_1) \qquad X_2 = S^3 \setminus N(3_1^m)$$

$\phi_i : X_i \to S^1$ fibration, set

$$\mathcal{F}_i = \left\{ \phi_i^{-1}(\theta) \subset X_i \;\middle|\; \theta \in S^1 \right\}$$

Spinning around the boundary both $\mathcal{F}_1$ and $\mathcal{F}_2$ we obtain two foliations $\mathcal{F}_1^o$ and $\mathcal{F}_2^o$ filling $\mathring{X}_1$ and $\mathring{X}_2$ respectively. A Reebless foliation on Y is now given by

$$\mathcal{F} = \mathcal{F}_1^o \cup \mathcal{F}_2^o \cup \text{ separating torus of } Y.$$

Taut foliations

Definition

A foliation $\mathcal{F}$ is called **taut** if: for each leaf Σ there exist a simple closed curve $\gamma \subseteq Y$ intersecting Σ and $\gamma \pitchfork \mathcal{F}$.

Example: $Y = S^2 \times S^1$, $\mathcal{F} = \left\{ S^2 \times \theta | \theta \in S^1 \right\}$. A transverse curve intersecting all the leaves is given by $\gamma = \text{pt.} \times S^1$.

Theorem

Taut foliations are Reebless.

What about existence of taut foliations?

Theorem (Gabai)

A prime, compact, connected and orientable three-manifold with $b_1 > 0$ has a taut foliation. Furthermore, if Y is such a manifold and $\Sigma \subset Y$ is a properly embedded surface minimizing the Thurston norm of its homology class, then Y has a taut foliation having Σ as leaf.

Question

When does an irreducible rational homology sphere have a taut foliation?

Ozsváth and Szabó introduced a package of invariants

$$HF^+, HF^-, HF^\infty, \widehat{HF}$$

called **Heegaard Floer homology**.

$$\widehat{HF} : Y \text{ rational homology 3-sphere} \rightsquigarrow \text{f.d. } \mathbb{F}_2 \text{ vector space}$$

$$\widehat{HF}(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c} \widehat{HF}(Y, \mathfrak{s})$$

Proposition

If Y is a rational homology sphere

$$dim\widehat{HF}(Y,\mathfrak{s}) \geq 1$$

for all $Spin^c$ structures.

If the equality hold for all Spin^c structures we say that Y is an **Heegaard Floer lens space**, or an **L-space** for short.

$$\dim\widehat{HF}(Y) = \#\text{Spin}^c(Y) = |H_1(Y,\mathbb{Z})|.$$

Examples of L-Spaces

Exaples of L-Spaces are:

- S^3
- Lens spaces (whence the name)
- All manifolds with finite fundamental group
- Branched double covers of quasi-alternating links

Question

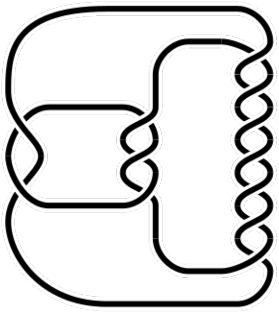
Is there a topological characterization (i.e. that does not refer to Heegard Floer homology) of L-spaces?

Theorem (Ozsváth & Szabó)

*A rational homology three-sphere that has a taut foliation is **not** an L-space.*

An application

$K =$ $(-2, 3, 7)$ pretzel knot.



Proposition

*Then $Y = S^3_{+\frac{37}{2}}(K)$ does **not** contains a taut foliation.*

Sketch of proof:

The $+18$ surgery on K produces the lens space $L(18,5)$.

HF surgery exact sequence $\Rightarrow$ if $S^3_r(K)$ is an L-space for some $r > 0$ then so is $S^3_s(K)$ for all $s > r$.

Conjectural characterizations L-spaces

Conjecture (L-space Conjecture)

An irreducible $\mathbb{Q}H$ sphere is an L-space if and only if has **not** a taut foliation.

Conjecture (Hedden & Levine)

If Y is an irreducible $\mathbb{Z}H$ sphere that is an L-space, then Y is homeomorphic to either S^3 or the Poincaré homology sphere.