M3/4PA48 Dynamics of Games

Solutions to Exercises

24th December, 2020

Contents

1 Replicator Dynamics for One Population

1.1 Nash equilibrium of one population

Exercise 1.1:
1) Let K represent the density of the population of Koalas in Kangaroo Island in South Australia. On one hand, whenever koalas live in areas with an abundance of Eucalyptus leaves (areas of type A) they reproduce at a rate x, whereas they die at a rate y in every other part of the island (areas of type B). The Nash equilibrium of this "game" is simply given by always choosing to live in areas of type A.

2) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and suppose we want to study the Nash Equilibria for a game encoded by A. We will do this by looking at the best response function for A. Notice that the matrix A is a permutation matrix, therefore instead of working with entries of (Ax) we can focus on the entries of x directly. Firstly, we compute it for vectors $x \in \Delta$ for which one of the components is bigger than the other two.

Consider $x = (x_1, x_2, x_3)$ where $x_1 > x_2$ and $x_1 > x_3$, then

$$\begin{aligned} \mathcal{BR}(x) &= \arg\max_{y\in\Delta} y \cdot Ax \\ &= \arg\max_{y\in\Delta} (y_1, y_2, y_3) \cdot (x_2, x_3, x_1) \\ &= \arg\max_{y\in\Delta} y_1x_2 + y_2x_3 + y_3x_1 = \{e_3\}. \end{aligned}$$

We can proceed similarly for vectors with $x_2 > x_1, x_3$, and $x_3 > x_1, x_2$.

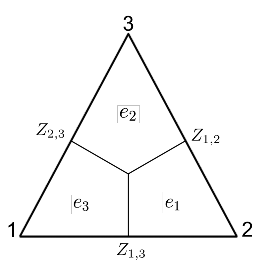

Figure 1: Diagram of $\mathcal{BR}(x)$

The next case we want to consider is the special vector for which all entries are equal, namely $x = (1/3, 1/3, 1/3)$. In this case

$$\mathcal{BR}((1/3, 1/3, 1/3)) = \arg\max_{y\in\Delta} 1/3(y_1 + y_2 + y_3) = \Delta.$$

Finally, we can consider the case where two of the entries of our vectors are equal, and the third one does not dominate (otherwise we go back to the first case we analysed). Essentially we want to study the lines $Z_{ij} = \{x \in \Delta | (Ax)_i = (Ax)_j\} = \{x = (x_1, x_2, x_3) \in \Delta | x_{i+1} = x_{j+1}\}$ for $i, j \in \mathbb{Z}/3\mathbb{Z}$. Suppose we want to find the best response along the segment $Z_{1,2} \cap \{x_1 < 1/3\} = \{x_2 =$

$x_3\} \cap \{x_1 < 1/3\}$, then we would have to maximise $y \cdot Ax$ for $y \in \Delta$, and assuming that $x = (1-a, a/2, a/2)$ and $a \in (2/3, 1)$. Clearly

$$y \cdot Ax = y_1x_2 + y_2x_3 + y_3x_1 = a(y_1 + y_2) + (1-a)y_3,$$

which is maximised whenever $y_1+y_2 = 1$ and $y_3 = 0$, or $y \in \langle e_1, e_2\rangle$. Proceeding with a similar reasoning we obtain

$$\begin{aligned}
\mathcal{BR}(Z_{1,2} \cap \{x_1 < 1/3\}) &= \langle e_1, e_2\rangle \\
\mathcal{BR}(Z_{2,3} \cap \{x_2 < 1/3\}) &= \langle e_2, e_3\rangle \\
\mathcal{BR}(Z_{1,3} \cap \{x_3 < 1/3\}) &= \langle e_1, e_3\rangle.
\end{aligned}$$

This means that if we look at the best response over all of Δ we obtain the following

$$\mathcal{BR}(x) = \begin{cases} \{e_3\} & \text{if } x_1 > x_2, x_3 \\ \{e_1\} & \text{if } x_2 > x_1, x_3 \\ \{e_2\} & \text{if } x_3 > x_1, x_2 \\ \langle e_1, e_2\rangle & \text{if } x \in Z_{1,2} \cap \{x_1 < 1/3\} \\ \langle e_2, e_3\rangle & \text{if } x \in Z_{2,3} \cap \{x_2 < 1/3\} \\ \langle e_1, e_3\rangle & \text{if } x \in Z_{1,3} \cap \{x_3 < 1/3\} \\ \Delta & \text{if } x_1 = x_2 = x_3. \end{cases}$$

Now recall that $\hat{x}$ is a Nash Equilibrium if and only if $\hat{x} \in \mathcal{BR}(\hat{x})$. Figure 1 summarises all the information contained in the function above, except for the behaviour on the boundary of every region. It tells us that if there were to Nash Equilbria they would have to belong to either $\partial\Delta$ or $(Z_{1,2} \cap \{x_1 < 1/3\}) \cup (Z_{2,3} \cap \{x_2 < 1/3\}) \cup (Z_{1,3} \cap \{x_3 < 1/3\})$.No vectors in $\langle e_i, e_j\rangle$ are contained in $Z_{i,j} \cap \{x_i < 3\}$, where $i < j \pmod 3$. However, we do have that $(1/3, 1/3, 1/3) \in \Delta = \mathcal{BR}((1/3, 1/3, 1/3))$, hence the vector $(1/3, 1/3, 1/3)$ is the only Nash Equilibrium for A in Δ. It is important to notice that $(1/3, 1/3, 1/3)$ is the meeting point of the three indifference lines $Z_{1,2}, Z_{2,3}$, and $Z_{1,3}$.

1.2 Evolutionary stable strategies

Exercise 1.2:
1) We want to determine the Nash Equilibria for

$$A = \begin{pmatrix} 0 & 2 & -1 \\ -1 & 0 & 2 \\ 2 & -1 & 0 \end{pmatrix}$$

It is quite immediate to see that the point $\hat{x} = (1/3, 1/3, 1/3)$ is a Nash Equilibrium:

$$\mathcal{BR}(\hat{x}) = \arg\max_{y\in\Delta} y \cdot A\hat{x} = \arg\max_{y\in\Delta} y \cdot (1/3, 1/3, 1/3) = \Delta \ni \hat{x}.$$

Notice that $A\hat{x} = \hat{x} = (1/3, 1/3, 1/3)$ in accordance with Lemma 1.2 (here $c = 1/3$). We now want to show $\hat{x}$ is an Evolutionary Stable Strategy, and in order to do that we are going to use the second part of Lemma 1.3 in the notes. For any $y \in \Delta$ we have

$$y \cdot Ay = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \cdot \begin{pmatrix} 2y_2 - y_3 \\ -y_1 + 2y_3 \\ 2y_1 - y_2 \end{pmatrix} = y_1y_2 + y_2y_3 + y_1y_3$$
$$\hat{x} \cdot Ay = \frac{2}{3}y_2 - \frac{1}{3}y_3 - \frac{1}{3}y_1 + \frac{2}{3}y_3 + \frac{2}{3}y_1 - \frac{1}{3}y_2 = \frac{1}{3}$$

We now want to show that the function $f(y_1, y_2, y_3) = y_1y_2 + y_2y_3 + y_1y_3$ is maximised at $(1/3, 1/3, 1/3)$ in Δ. In order to do so we will work with Lagrange multipliers. The only constraint we have is that we want to maximum to be in Δ, so the constraint function we will work with is given by $g(y_1, y_2, y_3) = y_1+y_2+y_3-1$. Now we can consider the following $\mathcal{L}(y_1, y_2, y_3, \lambda) = f(y_1, y_2, y_3) - \lambda g(y_1, y_2, y_3) = y_1y_2 + y_2y_3 + y_1y_3 - \lambda y_1 - \lambda y_2 - \lambda y_3 + \lambda$ for $\lambda \in \mathbb{R}$. Therefore, the point of maximum $(\tilde{y_1}, \tilde{y_2}, \tilde{y_3})$ is found by solving the following

$$\nabla\mathcal{L}(\tilde{y_1}, \tilde{y_2}, \tilde{y_3}, \tilde{\lambda}) = \begin{pmatrix} \tilde{y_2} + \tilde{y_3} - \tilde{\lambda} \\ \tilde{y_1} + \tilde{y_3} - \tilde{\lambda} \\ \tilde{y_1} + \tilde{y_2} - \tilde{\lambda} \\ -\tilde{y_1} - \tilde{y_2} - \tilde{y_3} + 1 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} \tilde{y_1} \\ \tilde{y_2} \\ \tilde{y_3} \\ \tilde{\lambda} \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

Therefore we know that the maximum of $f(y_1, y_2, y_3)$ in Δ is achieved at $\hat{x}$, and it is precisely $1/3$. So we can conclude that $\hat{x}$ is and ESS, since $y \cdot Ay < \hat{x} \cdot Ay$ for all $y \in \Delta \setminus \{\hat{x}\}$. Since $\hat{x} \in \operatorname{int} \Delta$ is an ESS we can conclude there are no other Nash Equilibria.

Finally $\hat{x}$ is not a strict Nash Equilibrium since for any $y \in \Delta \setminus \{\hat{x}\}$

$$\frac{1}{3} = y \cdot A\hat{x} = \hat{x} \cdot A\hat{x} = \frac{1}{3.}$$

If there was a strict Nash Equilibrium then such a point would be automatically a Nash Equilibrium, contradicting Lemma 1.2.

2) We want to show that e_1, e_2 and e_3 are ESS for

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define the regions

$$\Xi_i := \{x \in \Delta \,|\, x_i > x_{i+1} \text{ and } x_i > x_{i+2}\}$$

for $i \in \mathbb{Z}/3\mathbb{Z}$. Notice that these regions are the same as the ones over which the Best Response function is single-valued (look at the particular shape of A...).

Since Ξ_i represents a set of points close to e_i, in order to prove that e_i is an ESS we will show that for all $y \in \Xi_i \setminus \{e_i\}$ the following holds

$$y \cdot Ay < e_i \cdot Ay.$$

Fix $i \in \mathbb{Z}/3\mathbb{Z}$. There are two main observation to make here: if $y \in \Xi_i$ then $y_i > 0$ and $\frac{y_{i+1}}{y_i} < 1$ and $\frac{y_{i+2}}{y_i} < 1$. With this in mind we now have for all $y \in \Xi_i \setminus \{e_i\}$

$$\begin{aligned} y \cdot Ay = |y|^2 &= y_1^2 + y_2^2 + y_3^2 \\ &= y_i(y_i + \frac{y_{i+1}}{y_i} y_{i+1} + \frac{y_{i+2}}{y_i} y_{i+2}) \\ &< y_i(y_i + y_{i+1} + y_{i+2}) \\ &= y_i = e_i \cdot Ay, \end{aligned}$$

as we wanted. We can conclude that e_1, e_2, and e_3 are all ESS.

1.3 Replicator dynamics

Exercise 1.3:
1) We want to study the replicator dynamics described by the matrix

$$A = \begin{pmatrix} 0 & 10 & 1 \\ 10 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

To start with, let us compute the lines

$$Z_{i,j} = \{x \in \Delta | (Ax)_i = (Ax)_j\}$$

for $i, j \in \{1, 2, 3\}$:

$$\begin{aligned} Z_{1,2} &= \{10x_2 + x_3 = 10x_1 + x_3\} \\ &= \{x_1 = x_2\} \\ Z_{2,3} &= \{9x_1 = x_2\} \\ Z_{1,3} &= \{9x_2 = x_1\}. \end{aligned}$$

See Figure 2 for a representation of such indifference lines in Δ. In order to establish the ESS for this system, we will firstly understand its Nash Equilbria, given that every ESS is a NE.

Recall that the intersection of all the indifference lines is a Nash Equilibrium. Given Figure 2 we can see that e_3 is a NE, and that there are no other equilibria in the interior of Δ (since these lines intersect only once).

Analysing the boundary is slightly more delicate. Computing the best response at the corners of the simplex would immediately tell us if such corners are NE or not. The shaded regions in Figure 2 tell us that the best response near e_1 is given by e_2, and that the best response near e_2 is e_1, therefore implying that neither e_1, or e_2 are NE.

Recall that Nash Equilibria along sides are given by intersection with indifference lines. If we consider a side $\langle e_i, e_j\rangle$ then we only need to consider the correspondent indifference line $Z_{i,j}$, and analyse the best response at the intersection point.

In our case all the indifference lines intersect the side $\langle e_1, e_2\rangle$, so we automatically know we will not find Nash Equilibria on $\langle e_2, e_3\rangle$ and $\langle e_1, e_3\rangle$, and that we need to focus on the point $(\frac{1}{2}, \frac{1}{2}, 0) = Z_{1,2} \cap \langle e_1, e_2\rangle$. Such a point is a Nash Equilibrium given that the best response along $Z_{1,2}$ is given by $\langle e_1, e_2\rangle$ (except at e_3 where it is given by Δ).

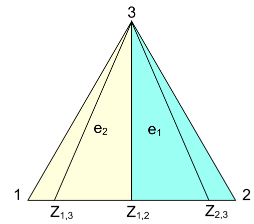

Figure 2: Indifference lines and best response for A.

The only two NE are given by e_3 and $(\frac{1}{2}, \frac{1}{2}, 0)$. Is e_3 an ESS? In order for e_3 to be an ESS we need that for all $x \in \Delta \setminus \{e_3\}$ and for $\varepsilon > 0$ small enough

$$x \cdot A(\varepsilon x + (1-\varepsilon)e_3) < e_3 \cdot A(\varepsilon x + (1-\varepsilon)e_3).$$

Notice that $x \cdot Ae_3 = 1$ for all $x \in \Delta$, hence the above claim reduces to showing

$$x \cdot Ax < e_3 \cdot Ax$$

for all $x \in \Delta \setminus \{e_3\}$. Now

$$\begin{aligned} x \cdot Ax &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 10x_2 + x_3 \\ 10x_1 + x_3 \\ 1 \end{pmatrix} = 20x_1x_2 + (1 + x_1 + x_2)x_3 \\ e_3 \cdot Ax &= 1 \end{aligned}$$

Consider the vector $\tilde{x} = \left(\begin{smallmatrix} 1/2 \\ 1/2 \\ 0 \end{smallmatrix}\right) \in \Delta \setminus \{e_3\}$, but then this gives us that $\tilde{x} \cdot A\tilde{x} = 5 > 1 = e_3 \cdot A\tilde{x}$. Therefore, e_3 is NOT an ESS.

Is $\tilde{x} = (\frac{1}{2}, \frac{1}{2}, 0)$ an ESS? For $\tilde{x}$ to be an ESS point we need that

$$y \cdot Ay < \tilde{x} \cdot Ay \tag{1}$$

holds for $y \neq \tilde{x}$ sufficiently close to $\tilde{x}$. Let $y = (\frac{1}{2} + \delta_1, \frac{1}{2} + \delta_2, -\delta_1 - \delta_2)$ where we need to remember that $\delta_1 + \delta_2 \leq 0$, and where δ_1 and δ_2 are assumed not to be zero simultaneously. Then

$$\begin{aligned} y \cdot Ay &= (\frac{1}{2} + \delta_1)(5 + 10\delta_2 - \delta_1 - \delta_2) + (\frac{1}{2} + \delta_2)(5 + 10\delta_1 - \delta_1 - \delta_2) - \delta_1 - \delta_2. \\ &= (\frac{1}{2} + \delta_1)(5 + 9\delta_2 - \delta_1) + (\frac{1}{2} + \delta_2)(5 + 9\delta_1 - \delta_2) - \delta_1 - \delta_2 \\ &= 5 + 8\delta_1 + 8\delta_2 + 18\delta_1\delta_2 - \delta_1^2 - \delta_2^2 \end{aligned}$$

Similarly

$$\tilde{x} \cdot Ay = 5 + 5\delta_1 + 5\delta_2 - \delta_1 - \delta_2 = 5 + 4(\delta_1 + \delta_2).$$

So (1) is equivalent to

$$4(\delta_1 + \delta_2) + 18\delta_1\delta_2 - \delta_1^2 - \delta_2^2 < 0. \tag{2}$$

Without loss of generality, we may assume that $|\delta_2| \leq |\delta_1|$ and $\delta_2 = \lambda\delta_1$ with $|\lambda| \leq 1$. Now of course we need to remember that $\delta_1 + \delta_2 \leq 0$ (and $(\delta_1, \delta_2) \neq (0, 0)$) so $(\lambda + 1)\delta_1 \leq 0$ and therefore either $\delta_1 < 0$ and $\lambda \in (-1, 1]$ or $\delta_1 > 0$ and $\lambda = -1$. So (2) becomes

$$4(\lambda + 1)\delta_1 + 18\lambda\delta_1^2 - (\lambda^2 + 1)\delta_1^2 < 0 \tag{3}$$

with $\delta_1 < 0$ and $\lambda \in (-1, 1]$, or $\delta_1 > 0$ and $\lambda = -1$. If $\delta_1 < 0$ and $\lambda \in (-1, 1]$ then (3) is equivalent to

$$4(\lambda + 1) + 18\lambda\delta_1 - (\lambda^2 + 1)\delta_1 > 0$$

which obviously holds for $|\delta_1|$ small. If $\delta_1 > 0$ and $\lambda = -1$ then (3) is equivalent to

$$-18\delta_1 - 2\delta_1 < 0$$

which again holds. It follows that $\tilde{x}$ is an ESS.

The last thing left to check is the presence of flow singularities. Recall that the replicator dynamics equation is

$$\dot{x_i} = x_i((Ax)_i - x \cdot Ax)$$

for $i \in \{1, 2, 3\}$, which implies

$$\left(\frac{x_i}{x_j}\right)' = \frac{x_i}{x_j}((Ax)_i - (Ax)_j)$$

for $i, j \in \{1, 2, 3\}$. Using the second formulation and for $x \in \Delta$ we have

$$\begin{aligned}
\left(\frac{x_1}{x_2}\right)' &= \frac{x_1}{x_2}(10x_2 + x_3 - 10x_1 - x_3) = 10\frac{x_1}{x_2}(x_2 - x_1) \\
\left(\frac{x_3}{x_1}\right)' &= \frac{x_3}{x_1}(x_1 - 9x_2) \\
\left(\frac{x_3}{x_2}\right)' &= \frac{x_3}{x_2}(x_2 - 9x_1).
\end{aligned} \tag{4}$$

If we were to have singularities in the interior of Δ then there would exist $x \in \mathrm{int}\Delta$ (notice all its components are in $(0, 1)$) for which

$$\begin{cases} \left(\frac{x_3}{x_1}\right)' = 0 \\ \left(\frac{x_3}{x_2}\right)' = 0 \end{cases} \implies \begin{cases} x_1 - 9x_2 = 0 \\ x_2 - 9x_1 = 0 \end{cases}.$$

Clearly this will never happen for $x_1, x_2 > 0$. Hence the singularities, if they exist, are on $\partial\Delta$. The corners of Δ are all singularity points given the structure of the replicator dynamics ODE: at a corner e_i we have that $\dot{x}_j = 0$ if $j \neq i$ since $x_j = 0$, and $\dot{x}_i = 0$ since $(Ae_i)_i = e_i \cdot Ae_i$. If we look at Equation 4, the only other point at which all equations are zero simultaneously is $\tilde{x} = (\frac{1}{2}, \frac{1}{2}, 0)$, which is the only other singularity on $\partial\Delta$.

2) We now want to investigate how the replicator dynamics for a matrix A changes if we add to its first column the vector $\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$. Let

$$B = A + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and consider the replicator dynamics associated with B

$$\begin{aligned}
\dot{x}_i &= x_i((Bx)_i) - x \cdot Bx) \\
&= x_i\left(\left(Ax + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x\right)_i - x \cdot Ax - x \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x\right) \\
&= x_i\left(\left(Ax + \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix}\right)_i - x \cdot Ax - x_1(x_1 + x_2 + x_3)\right) \\
&= x_i((Ax_i) + x_1 - x \cdot Ax - x_1) \\
&= x_i((Ax)_i - x \cdot Ax).
\end{aligned}$$

The replicator dynamics of B is the same as the replicator dynamics of A.

3) The RHS of the replicator dynamics equation is $\mathcal{C}^\infty$-regular in every entry, therefore we can apply the local Picard–Lindelöf Theorem and obtain that for any starting point $x(0) \in \Delta$ we have a unique local solution. Notice that the solution can never leave the simplex, hence its norm is always bounded. This implies that the solution cannot blow up in finite time, therefore the unique local solution we have precedently established exists for all times.

1.4 ESS points are asymptotically stable for the replicator system

Exercise 1.4:
Consider the matrix $A = \mathbf{Id}_3$, the 3×3 identity matrix. For $\hat{x} = e_j$ and $j \in \{1, 2, 3\}$, consider the function

$$P(x) = \prod_{i=1}^{3} x_i^{\hat{x}_i} = \prod_{i=1}^{3} x_i^{(e_j)_i} = x_j$$

where $x \in \Delta$. We will show that the flow tends to e_j for each $j \in \{1, 2, 3\}$. In order to simplify the calculations let $\hat{x} = e_1$, so that $P(x) = x_1$. Then

$$\begin{aligned}\frac{\dot{P}}{P}(x) &= \hat{x} \cdot Ax - x \cdot x \\ &= e_1 \cdot \mathbf{Id}_3 x - x \cdot \mathbf{Id}_3 x \\ &= e_1 \cdot x - |x|^2 = x_1 - x_1^2 - x_2^2 - x_3^2\end{aligned}$$

If now let x be near e_1, then we can write it as $x = \left(\begin{smallmatrix} 1-\varepsilon \\ \delta \\ \tau \end{smallmatrix}\right)$ for $\varepsilon, \delta, \tau \in (0, 1)$, and $\delta + \tau = \varepsilon$ so that

$$\begin{aligned}\frac{\dot{P}}{P}(x) &= 1 - \varepsilon - (1-\varepsilon)^2 - \delta^2 - \tau^2 \\ &= \varepsilon - \varepsilon^2 - \delta^2 - \tau^2 \\ &> \varepsilon - \varepsilon^2 - \varepsilon^2 \qquad \text{since } \delta^2 + \tau^2 < \epsilon^2 \\ &= \varepsilon(1 - 2\varepsilon) > 0 \quad \text{since } \varepsilon > 0 \text{ (and small).}\end{aligned}$$

Therefore, $\dot{P}(x) > 0$ for $x \in \Delta \setminus \{e_1\}$ close to e_1, so e_1 attracts nearby points. Similar computations, where we take $\hat{x} = e_2$ or e_3 in the definition of P, show that the vertices of Δ attract nearby points.

Next, we turn our attention to the boundary of Δ. So thanks to the replicator dynamics equation

$$\left(\frac{x_i}{x_j}\right)' = \frac{x_i}{x_j}(x_i - x_j)$$

where $i, j \in \{1, 2, 3\}$. For example along $\langle e_1, e_2 \rangle$ we have that the sign of $\left(\frac{x_1}{x_2}\right)'$ changes at $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Similarly, we have a sign change at $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $\left(0, \frac{1}{2}, \frac{1}{2}\right)$.

These same equations tell us that along the indifference line $Z_{i,j}$ the derivative $\left(\frac{x_i}{x_j}\right)'$ is equal to 0, and the direction of the flow is completely described by the derivative in the third component. Let us make an example. Suppose we want to consider the flow along $Z_{1,2}$, then we know that $\left(\frac{x_1}{x_2}\right)' = 0$, so we consider $\dot{x}_3$. Any $x \in Z_{1,2}$ can be written as $x = \begin{pmatrix} \frac{1-x_3}{2} \\ \frac{1-x_3}{2} \\ x_3 \end{pmatrix}$, and the replicator equation gives us

$$\begin{aligned}\dot{x}_3 &= x_3((\mathbf{Id}_3 x)_3 - x \cdot \mathbf{Id}_3 x) \\ &= x_3(x_3 - |x|^2) \\ &= x_3(x_3 - 2\left(\frac{1-x_3}{2}\right)^2 - x_3^2) \\ &= -\frac{1}{2}x_3(3x_3^2 - 4x_3 + 1) = -\frac{1}{2}x_3(x_3 - 1)(x_3 - \frac{1}{3}).\end{aligned}$$

We can conclude that $\dot{x}_3 < 0$ whenever $x_3 \in \left(0, \frac{1}{3}\right)$, and that $\dot{x}_3 > 0$ for $x_3 \in \left(\frac{1}{3}, 1\right)$. The same computations show identical behaviour along $Z_{1,3}$ and $Z_{2,3}$.

Consider now the Nash Equilibrium $\hat{x} = (1/3, 1/3, 1/3)$. How does the flow behave around it? From the previous analysis we have carried out for the flow along $Z_{i,j}$ we expect the flow to be repelled by $\hat{x}$ (star node). In order to see if our hunch is correct we will linearise the RHS of the replicator equation at $\hat{x}$. There are different equivalent ways to do so, for example see Example 1.8 in the notes. We will take a more direct approach.

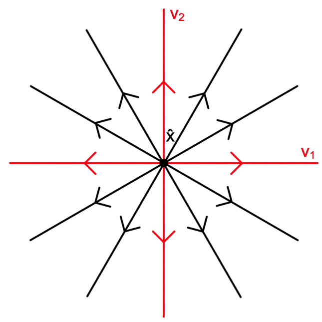

Figure 3: Linearisation around $\hat{x}$.

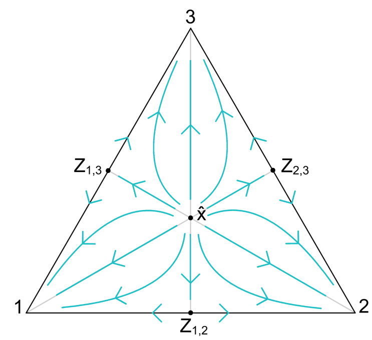

Figure 4: Flow in Δ for $A = \mathbf{Id}_3$.

Let $h \in \mathbb{R}^3$ be a vector whose entries sum up to 0, i.e. $\sum_{i=1}^3 h_i = 0$, and consider the perturbed vector $p = \hat{x} + h$. Now let $i, j, k \in 1, 2, 3$ be distinct

$$
\begin{aligned}
\dot{h_i} = \dot{p}_i &= p_i((\mathbf{Id}_3 p)_i - p \cdot \mathbf{Id}_3 p) \\
&= (\frac{1}{3} + h_i)(\frac{1}{3} + h_i - \sum_{l=1}^{3} (\frac{1}{3} + h_l)^2) \\
&= (1/3 + h_i)(\frac{1}{3}h_i - \frac{2}{3}h_j - \frac{2}{3}h_k + \mathcal{O}(h^2)) \\
&= (\frac{1}{3} + h_i)(h_i + \mathcal{O}(h^2)) \qquad \text{since } h_i = -h_j - h_k \\
&= \frac{1}{3}h_i + \mathcal{O}(h^2).
\end{aligned}
$$

Therefore, the linearisation yields the matrix

$$
L = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}
$$

with repeated eigenvalues $\lambda_1 = \lambda_2 = \frac{1}{3}$ and associated eigenvectors $v_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and $v_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. As we suspected, the point $\hat{x}$ is a star node. See Figure 3 for a representation of the flow near $\hat{x}$ and Figure 4 for the flow in Δ.

1.5 Further examples

Exercise 1.5:
We want to establish the phase portrait for the replicator equation where

$$A = \begin{pmatrix} 0 & 10 & 1 \\ 10 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We already know, thanks to Exercise 1.3 that this system has two Nash Equilibria, namely e_3, and $\tilde{x} = (\frac{1}{2}, \frac{1}{2}, 0)$, and flow singularities at each vertex of Δ and at $\tilde{x}$.

Let us study what happens along the boundary of Δ. Let $x \in \langle e_1, e_3 \rangle$, so that $x = \left(\begin{smallmatrix} x_1 \\ 0 \\ x_3 \end{smallmatrix}\right)$ where $x_1 = 1 - x_3$ for $x_3 \in (0,1)$. Along such a side we have

$$\begin{aligned} \left(\frac{x_3}{x_1}\right)' &= \left(\frac{x_3}{x_1}\right)(x_1 - 9x_2) \\ &= \frac{x_3}{x_1} x_1 = x_3 > 0 \end{aligned}$$

which means that the flows goes from e_1 towards e_3. Along $\langle e_2, e_3 \rangle$ we see that $\left(\frac{x_3}{x_2}\right)' > 0$, therefore the solution flows from e_2 towards e_3.

The last side contains a singularity, hence we expect a more interesting behaviour. For $x = \left(\begin{smallmatrix} x_1 \\ x_2 \\ 0 \end{smallmatrix}\right) \in \langle e_1, e_2 \rangle$, where $x_1 = 1 - x_2$ for $x_2 \in (0,1)$, we obtain

$$\left(\frac{x_1}{x_2}\right)' = 10 \frac{x_1}{x_2}(x_2 - x_1) = 10 \frac{(1-x_2)(2x_2-1)}{x_2}.$$

Such a function is negative between 0 and $\frac{1}{2}$, zero at $\frac{1}{2}$ (as we expected by flow singularities), and positive between $\frac{1}{2}$ and 1, so the flows is attracted by $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ from e_1 and e_2. See Figure 5 for more details.

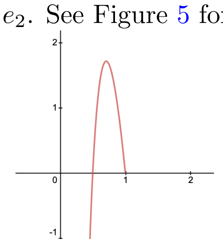

Figure 5: Graph of $(\frac{x_1}{x_2})'$ along $\langle e_1, e_2 \rangle$.

Along $Z_{2,3}$	Along $Z_{1,2}$	Along $Z_{1,3}$
$\left(\frac{x_3}{x_1}\right)' < 0$	$\left(\frac{x_3}{x_1}\right)' < 0$	$\left(\frac{x_3}{x_1}\right)' = 0$
$\left(\frac{x_3}{x_2}\right)' = 0$	$\left(\frac{x_3}{x_2}\right)' < 0$	$\left(\frac{x_3}{x_2}\right)' < 0$
$\left(\frac{x_1}{x_2}\right)' > 0$	$\left(\frac{x_1}{x_2}\right)' = 0$	$\left(\frac{x_1}{x_2}\right)' < 0$

Table 1: Flow along indifference lines.

A similar analysis can be carried out along the indifference lines $Z_{1,2}, Z_{2,3}, Z_{1,3}$ as summarised in Table 1. Therefore, the flow along the indifference lines leaves e_3 and goes towards $\langle e_1, e_2 \rangle$.

As we have proved in Exercise 1.3 part 1, the point $\tilde{x}$ is an ESS, therefore it is asymptotically stable (as the flow analysis we have just carried would suggest).

1.6 Rock–Paper–Scissors replicator game

Exercise 1.6:
1) We want to model the game of Rock–Paper–Scissors. In such game we have three strategies R, P, S. The rules are quite easy R beats S, which beats P, which beats R, and any strategy played against itself resolves to a draw. See Figure 6.

	Rock	Paper	Scissors
Rock	0	+1	-b
Paper	-b	0	1
Scissors	1	-b	0

Table 2: Payoff table.

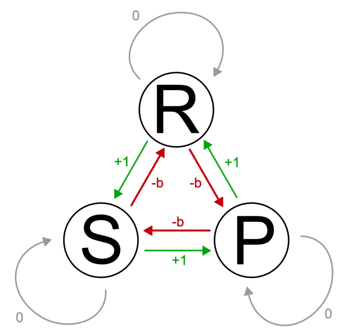

Figure 6: A schematics of the interactions between the various strategies.

The payoff for winning is $+1$, for losing $-b$, where $b > 0$, and for drawing 0. If we write all such data into a table we obtain Table 2. From this, we can read off the payoff matrix

$$A = \begin{pmatrix} 0 & +1 & -b \\ -b & 0 & 1 \\ 1 & -b & 0 \end{pmatrix}.$$

2) Consider the three vectors

$$\begin{aligned} A_1 &= \frac{1}{1+b+b^2}(1, b^2, b)^\mathsf{T} \\ A_2 &= \frac{1}{1+b+b^2}(b, 1, b^2)^\mathsf{T} \\ A_3 &= \frac{1}{1+b+b^2}(b^2, b, 1)^\mathsf{T} \end{aligned}$$

from Lemma 1.5. We claim that A_i, A_{i+1}, and e_{i+1} are collinear (where the indexes are to be taken in $\mathbb{Z}/\mathbb{Z}_3$). We will only show the computations for A_3, A_1, e_1, but every other case is identical. Recall that three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear if and only if $\mathbf{a}-\mathbf{c}$, and $\mathbf{b}-\mathbf{c}$ are parallel. Therefore

$$A_3 - e_1 = \frac{1}{1+b+b^2}\begin{pmatrix} -1-b \\ b \\ 1 \end{pmatrix}$$
$$A_1 - e_1 = \frac{b}{1+b+b^2}\begin{pmatrix} -1-b \\ b \\ 1 \end{pmatrix}$$

which means that $A_1 - e_1 = b(A_3 - e_1)$, so A_3, A_1, e_1 are collinear.

3) At the beginning of the proof of Lemma 1.5 we say that

$$\frac{1}{T}\int_0^T x(t)\cdot Ax(t)dt \to 0 \qquad \text{as } T \to +\infty, \tag{5}$$

let us show why this is true. Recall that

$$A = \begin{pmatrix} 0 & 1 & -b \\ -b & 0 & 1 \\ 1 & -b & 0 \end{pmatrix}$$

where $b > 1$. The reasons why this happens are sketched in the proof, and they are basically two

1. The payoff $x \cdot Ax$ tends to 0 as the flows $x(t)$ gets closer and closer to any vertex of Δ;
2. $x(t)$ spends most of the time close to the vertices of Δ.

To deal with 1) consider a point

$$x = \begin{pmatrix} 1-\varepsilon \\ \delta \\ \tau \end{pmatrix}$$

in Δ close to e_1, where $0 < \delta, \tau \leq \epsilon < 1$. Now

$$\begin{aligned}
|x \cdot Ax| &= \left| \begin{pmatrix} 1-\varepsilon \\ \delta \\ \tau \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -b \\ -b & 0 & 1 \\ 1 & -b & 0 \end{pmatrix} \cdot \begin{pmatrix} 1-\varepsilon \\ \delta \\ \tau \end{pmatrix} \right| = \\
&= |-b\varepsilon(1-\varepsilon) + (1-b)\delta(1-\delta)| \\
&\leq b\varepsilon(1-\varepsilon) + (1-b)\delta(1-\delta) \\
&< b\varepsilon + (b-1)\varepsilon \\
&= \varepsilon(2b-1)
\end{aligned}$$

which means that $x \cdot Ax \to 0$ as $\varepsilon \to 0$. So the closer we are to the vertices of Δ, the closer to zero the payoff is.

Next, we want to show that the speed of the flow is almost zero near the vertices of Δ, and maximal away from them. Consider the side of Δ between e_1 and e_2, and let $x = x_1 e_1 + (1 - x_1)e_2$ be a point on it, where $x_1 \in [0, 1]$. Now

$$Ax = \begin{pmatrix} 1 - x_1 \\ -bx_1 \\ x_1 - b(1 - x_1) \end{pmatrix}$$

$$x \cdot Ax = x_1(1 - x_1)(1 - b)$$

so we have, thanks to the replicator equation

$$\begin{aligned} |\dot{x}|^2 &= x_1^2((1 - x_1) - x_1(1 - x_1)(1 - b))^2 + (1 - x_1)^2(-bx_1 - x_1(1 - x_1)(1 - b))^2 \\ &= x_1^2(1 - x_1)^2[(1 - x_1(1 - b))^2 + (b + (1 - x_1)(1 - b))^2]. \end{aligned}$$

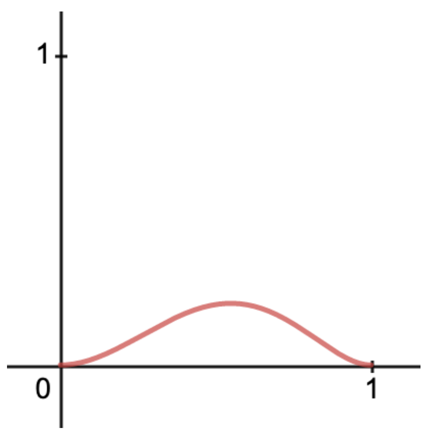

Figure 7: Graph of $|\dot{x}|^2$ for $b = 1.5$.

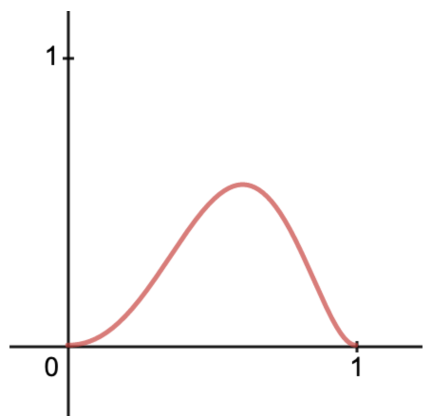

Figure 8: Graph of $|\dot{x}|^2$ for $b = 3$.

Thanks to continuity of the solution of ODEs, we can expect the velocity of the flow in int Δ close to the boundary to behave like $|\dot{x}|$ (as computed above). This function has a (local) maximum in (0,1) and tends to zero as x_1 approaches 0 or 1, i.e. the flow has maximal velocity away from the corners of Δ and gets smaller as it gets closer to them (see Figure 7 and 8 to get an idea of what this function looks like along on of the sides of Δ).

We have just established that the payoff of this game gets small near the corners of Δ, and that in these areas the flow has low speed, meaning that it spends most of the time there. Intuitively this is why the limit 5 converges to 0, but in order to give a rigorous proof of this we have to estimate how much time we spend in the corners (at least for T big enough).

We will now assume that Δ is our ambient space, and we equip it with the subset topology coming from the Euclidean topology of $\mathbb{R}^3$, i.e. all the neighbourhoods we will consider from now on are neighbourhoods in Δ. Consider the neighbourhood

$$\Omega = \bigcup_{i=1}^{3} \Omega_i = \bigcup_{i=1}^{3} B_\varepsilon(e_i)$$

of the vertices of Δ, where $B_\varepsilon(e_i)$ is the ball of radius ε around e_i. Similarly, consider the ε^2-tubular neighbourhood of $\partial\Delta$ given by

$$\Xi = \overline{B_{\varepsilon^2}(\partial\Delta)} \setminus \Omega = \overline{\bigcup_{x\in\partial\Delta} B_{\varepsilon^2}(x)} \setminus \Omega.$$

Notice that $\varepsilon > 0$ can be taken small enough so that we can apply the Hartman–Grobman theorem to $B_\varepsilon(e_i)$ for $i = 1, 2, 3$, and such that the payoff $|x \cdot Ax|$ is bounded above by ε over Ω.

As we wrote before Ω is made up by 3 components $\Omega_i = B_\varepsilon(e_i)$, and similarly Ξ is made up by 3 components. We will call Ξ_i the components in which the flows travels from Ω_{i-1} to Ω_i, where all the indexes have to be take $\mod 3$. Notice that Ξ is compact, and the flow over this set has always non-zero derivative since it is away from e_1, e_2, e_3. We can define $C \coloneqq \min_\Xi |\dot{x}|$ which tells us that $\max T_{\Xi_i} = \frac{1-2\varepsilon}{C} = K$, for T_{Ξ_i} being the time it takes the flow to get through Ξ_i. In order to estimate $\max T_{\Xi_i}$ we have used small angle approximations, linearised estimates of the flow, and we maximised the equation Time=Displacement/Speed. As time $T \to \infty$ we can see that $\max T_{\Xi_i}$ remains constant $(= K)$, meaning that the maximal time for the flow to get through Ξ_i is constant, and does not depend on how close the flow gets to $\partial\Delta$.

If we denote by T^N the amount of time that the flow takes to complete a full loop then we can break this down as

$$T^N = T^N_{\Omega_1} + T^N_{\Xi_2} + T^N_{\Omega_2} + T^N_{\Xi_3} + T^N_{\Omega_3} + T^N_{\Xi_1} \le T^N_{\Omega_1} + T^N_{\Omega_2} + T^N_{\Omega_3} + 3K = T^N_{\Omega} + 3K$$

where $T^N_{\Omega_i}$, and $T^N_{\Xi_i}$, represent the time needed to get through Ω_i, Ξ_i respectively during the loop N. We now will proceed to show that

$$\lim_{N\to\infty} \frac{T^{N+1}}{T^N} = \lim_{N\to\infty} \frac{T^{N+1}_\Omega}{T^N_\Omega} < \infty.$$

In order to compute such a limit we want estimate the time it takes the flow to traverse Ω_i. Let us start by considering Ω_1. By applying the Hartman-Grobman theorem to Ω_1 we have that the flow generated by the replicator equation is $\mathcal{C}^1$ conjugated to the linearised flow given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 1 & 1+b \\ 0 & -b \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \tag{6}$$

Notice that the eigenvalues of the matrix are $\lambda_1 = 1$ and $\lambda_2 = -b$ with associated eigenvectors $v_1 = \binom{1}{0}$, and $v_2 = \binom{1}{-1}$, respectively.

Remark. *The Hartman-Grobman theorem states that an equilibrium of a system of ODEs is locally linearisable as long as the linearisation matrix L is hyperbolic (all its eigenvalues have non-zero real part). If the flow is 2 dimensional then the conjugacy between the flow given by the original system and the one given by $\dot{\mathbf{x}} = L\mathbf{x}$ is $\mathcal{C}^1$ regular. For more general flows then the conjugacy is only α-Hölder continuous, where α depends on the eigenvalues of L.*

Let X, Y be two subsets of Euclidean spaces, and $r \in \mathbb{N}$. We say two flows $\varphi : X \to X$, and $\psi : Y \to Y$ are $\mathcal{C}^r$ conjugated if there exists a $\mathcal{C}^r$ diffeomorphism $h : X \to Y$ such that $h \circ \varphi = \psi \circ h$. If the last equality only holds over a subset of X then we say that φ and ψ are locally $\mathcal{C}^r$ conjugated.

By making a simple change of basis transformation we can (smoothly) conjugate the flow given by Equation 6 to

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \tag{7}$$

It is immediate to see that the eigenvalues of the matrix are given by $\lambda_1 = -b$, and $\lambda_2 = 1$, but now the associated eigenvectors are $v_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and $v_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ respectively. The solution to this system of ODEs for $\left(\begin{smallmatrix} x(0) \\ y(0) \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)$ is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{-bt} \\ y_0 e^{t} \end{pmatrix}.$$

As we have anticipated we want to understand how long it takes our original flow to cross Ω_1, and while doing so we will see how quickly the flow tends to $\partial\Delta$ in terms of distance from it. Let $\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \rho \\ \tau \end{smallmatrix}\right)$ for $\rho, \tau > 0$ arbitrarily small. We want to establish how long it takes to get to $\left(\begin{smallmatrix} \tau' \\ \rho \end{smallmatrix}\right)$, and what is the size of τ' compared to τ. This is done by solving

$$\begin{pmatrix} \tau' \\ \rho \end{pmatrix} = \begin{pmatrix} \rho e^{-bt} \\ \tau e^{t} \end{pmatrix}$$

which gives us $t = \ln \frac{\rho}{\tau}$ and $\tau' = \rho^{1-b}\tau^b = C_1\tau^b$. If we fix ρ and we let $t \to \infty$ we get $\tau \to 0$ which tells us that the flow tends to $\partial\Delta$, as we know. This estimates have been computed for System 7, which is $\mathcal{C}^1$ conjugated to the replicator dynamics, so thanks to the regularity of the conjugacy we know that the same asymptotics hold for the original flow. This means that for N big enough, and if $d \leq \varepsilon^2$ denotes the distance between the flow entering Ω_1 to start the N^{th} loop then the time to exit Ω_1 is $\sim \ln \frac{\varepsilon}{d}$, and that time the flow will be $\sim d^b$ away from $\partial\Delta$. These asymptotic estimates only depend on the eigenvalues of the linearisation of the flow at e_1, by proceeding similarly one can show that the situation in Ω_2, and Ω_2 is identical. We will assume that the distance flow-$\partial\Delta$ when entering Ω_{i+1} is approximately the same as when leaving Ω_i. The time to complete the loop is approximately given by

$$\begin{aligned}
T^N \sim T_\Omega^N &= T_{\Omega_1}^N + T_{\Omega_2}^N + T_{\Omega_3}^N \\
&\sim \ln\frac{\varepsilon}{d} + \ln\frac{\varepsilon}{d^b} + \ln\frac{\varepsilon}{d^{b^2}} \\
&= \ln\frac{\varepsilon^3}{d^{1+b+b^2}}.
\end{aligned}$$

Note that when the flow comes back to Ω_1 its distance to $\partial\Delta$ is $\sim d^{b^3}$, hence $T^{N+1} \sim \ln\frac{\varepsilon}{d^{b^3+b^4+b^5}}$. Since we have expressed T^N in terms of d then taking a limit as $N \to \infty$ is the same as $d \to 0$. Therefore,

$$\begin{aligned}
\lim_{N\to\infty}\frac{T^{N+1}}{T^N} &= \lim_{d\to 0}\frac{\ln\frac{\varepsilon^3}{d^{b^3+b^4+b^5}}}{\ln\frac{\varepsilon^3}{d^{1+b+b^2}}} \\
&= \lim_{d\to 0}\frac{\ln\frac{1}{d^{b^3+b^4+b^5}}}{\ln\frac{1}{d^{1+b+b^2}}} = b^3
\end{aligned}$$

So for N large enough $T^{N+1} \sim b^3 T^N$, where $b > 1$.

If we assume that T is big enough, we have that T can be approximately written as the sum of the times it takes to do N loops, or equivalently

$$T = \sum_{i=0}^{N-1} T^i \sim T^0 \sum_{i=0}^{N-1} b^{3i} = T^0\frac{1-b^{3N}}{1-b^3}.$$

This approximation allows us to compute how many loops we expect to have completed in a fixed (large) time T

$$N = \frac{\ln\left(\left(\frac{b^3-1}{T_0}\right)T + 1\right)}{3\ln b}$$

which immediately gives us that

$$\lim_{T\to\infty}\frac{N}{T} = 0.$$

We can finally prove the limit in Equation 5. Therefore

$$\begin{aligned}
\lim_{T\to\infty}\left|\frac{1}{T}\int_0^T x\cdot Ax dt\right| &\leq \lim_{T\to\infty}\frac{1}{T}\int_0^T |x\cdot Ax| dt \\
&= \lim_{T\to\infty}\frac{1}{T}\left(\int_{\sum_{j=0}^N T_\Omega^j}|x\cdot Ax|dt + \int_{\sum_{j=0}^N T_\Xi}|x\cdot Ax|dt\right) \\
&\leq \lim_{T\to\infty}\frac{\sum_{j=0}^N T_\Omega^j}{T}\varepsilon + \frac{N}{T}\max_{\Xi}|x\cdot Ax| \leq \varepsilon
\end{aligned}$$

since $\frac{\sum_{j=0}^N T_\Omega^j}{T} \sim \frac{\sum_{j=0}^N T^j}{T} \to 1$ and $\frac{N}{T} \to 0$. Since ε can be taken arbitrarily small we are done.

1.7 Hypercycle equation and permanence

Exercise 1.7:
We want to show that $\gamma_k = \sum_{j=0}^{n-1} c_j \lambda^{jk}$, where $k = 0, 1, \ldots, n$ and $\lambda = e^{\frac{2\pi i}{n}}$, are the eigenvalues of

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}.$$

We know, by the proof of Lemma 1.6, that the corresponding eigenvector to γ_k is

$$v_k = \begin{pmatrix} 1 \\ \lambda^k \\ \vdots \\ \lambda^{(n-1)k} \end{pmatrix}.$$

It is just a matter of multiplying C and v_k, and show that the product equals $\gamma_k v_k$. For $k = 0, 1, \ldots, n-1$ we have

$$\begin{aligned} Cv_k &= \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda^k \\ \vdots \\ \lambda^{(n-1)k} \end{pmatrix} \\ &= \begin{pmatrix} c_0 + c_1\lambda^k + \ldots + c_{n-1}\lambda^{(n-1)k} \\ \lambda_k(c_{n-1}\lambda^{(n-1)k} + c_0 + \ldots + c_{n-2}\lambda^{(n-2)k}) \\ \vdots \\ \lambda^{(n-1)k}(c_1\lambda^k + c_2\lambda^{2k} + \ldots + c_0) \end{pmatrix} \\ &= \sum_{j=0}^{n-1} c_j\lambda^{jk} \begin{pmatrix} 1 \\ \lambda^k \\ \vdots \\ \lambda^{(n-1)k} \end{pmatrix} = \gamma_k v_k. \end{aligned}$$

1.8 Existence and the number of Nash Equilibria

Exercise 1.8
1) We want to check the Poincaré-Hopf formula holds for simple flows X on some surface M. Recall that the formula states

$$\sum_{\substack{x \in M \\ X(x)=0}} i_X(x) = \chi(M)$$

where $i_X(x)$ is the index of X at x and $\chi(M)$ is the Euler characteristic of M.

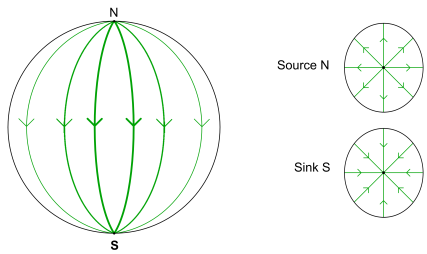

Figure 9: North-South flow on $\mathbb{S}^2$.

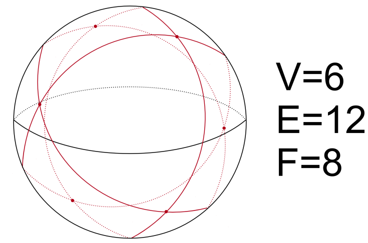

Figure 10: A triangulation of $\mathbb{S}^2$.

Let $M = \mathbb{S}^2$ be the two dimensional sphere, and consider the north-south flow X on it. As we can see from Figure 9 this flow has to singularities at N (the north pole) and S (the south pole). N is a source, whereas S is a sink, which means that $i_X(N) = i_X(S) = +1$.

Recall that the Euler characteristic of a surface M can be computed as

$$\chi(M) = V - E + F$$

where V is the number of vertices, E the number of edges, and F the number of faces of a triangulation of M. The Euler characteristic is independent from the choice of triangulation (as long as you are not collapsing triangles). For more information about Euler characteristic and triangulations look up CW-complexes or simplicial complexes.

The triangulation of $\mathbb{S}^2$ in Figure 10 tells us that $\chi(\mathbb{S}^2) = 6 - 12 + 8 = +2$. Now if we put everything together

$$\sum_{\substack{x \in \mathbb{S}^2 \\ X(x)=0}} i_X(x) = i_X(N) + i_X(S) = 1 + 1 = 2 = \chi(\mathbb{S}^2).$$

Let us consider a different surface. Let M be the two dimensional torus $\mathbb{T}^2$, and let X be the north-south flow as in Figure 11.

The flow X has now 4 singularities, namely a source N, a sink S and two saddle points N', S'. Their indexes are

$$\begin{aligned} i_X(N) &= +1; \\ i_X(S') &= -1; \\ i_X(N') &= -1; \\ i_X(S) &= +1. \end{aligned}$$

The Euler characteristic of $\mathbb{T}^2$ is readily computed thanks to the triangulation shown in Figure 12

$$\chi(\mathbb{T}^2) = 9 - 27 + 18 = 0.$$

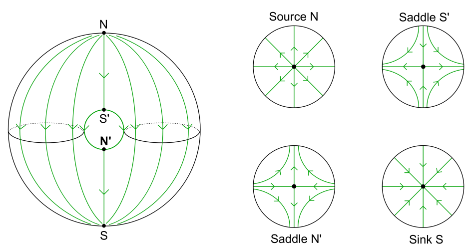

Figure 11: North-South flow on $\mathbb{T}^2$.

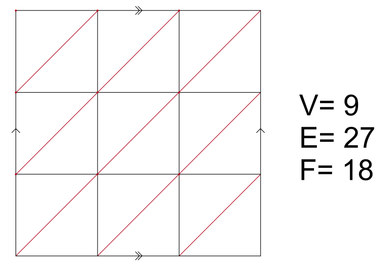

Figure 12: A triangulation of $\mathbb{T}^2$.

Now let us put everything together as before we have

$$\sum_{\substack{x\in\mathbb{T}^2 \\ X(x)=0}} i_X(x) = i_X(N) + i_X(S') + i_X(N') + i_X(S) = 1 - 1 - 1 + 1 = 0 = \chi(\mathbb{T}^2).$$

2) See Example 1.19 in the notes.

3) Consider, for $\varepsilon > 0$, the perturbed flow

$$\dot{x}_i = x_i((Ax)_i - x \cdot Ax) + \varepsilon$$

and assume that the original flow only presents regular singularities (the linearisation matrix at the singularity is invertible). We claim that under this assumption all the Nash Equilibria of the original flow on the boundary move towards the interior of the simplex under the perturbed flow, and the other singularities of the system move outwards.

Let us assume that Δ is a 3 dimensional simplex in order to simplify the discussion. It is important to notice that the singularity of the original replicator equation remain the same under the perturbed flow and they move smoothly as ε varies. The assumption that every singularity is regular implies that there are only finitely many singularities, and that they are all isolated.

Assume that $\hat{x}$ is a Nash Equilibrium on $\partial\Delta$ for the original game. Hence there exists $i \in \{1, 2, 3\}$ for which $\dot{\hat{x}}_i = 0$. As was shown in the proof of Theorem 1.4, the i-th component of the vector field $X_\varepsilon(x)$ takes the form $x_i z_i + \epsilon +$h.o.t. (higher order terms) where $z_i = (A\hat{x})_i - \hat{x} \cdot A\hat{x}$. Since $z_i < 0$ when $\hat{x}$ is a Nash equilibrium, the singularity $\hat{x}_\epsilon$ for $X_\varepsilon(x)$ near $\hat{x}$ has a positive i-th component (and so moves to the interior of Δ). Similarly, if $\hat{x}$ is not a Nash Equilibrium then $z_i \geq 0$, but since we have assumed that the vector field is regular, we have $z_i > 0$. It follows that in this case $\hat{x}$ moves to outside Δ.

2 Two Players Games

2.1 Two conventions for the payoff matrices

Exercise 2.1:
1) Consider the two players game given by the matrix

$$G = \begin{pmatrix} (1,-1) & (0,0) \\ (0,0) & (-1,1) \end{pmatrix}$$

where we have adopted the 2^{nd} convention (see the notes). From G we can read off the two matrices determining this game

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and we can use them to compute the best response for both players. Recall that $(\hat{x}, \hat{y})$ is a Nash Equilibrium if and only if $\hat{x} \in \mathcal{BR}_A(\hat{y})$ and $\hat{y} \in \mathcal{BR}_B(\hat{x})$. For $x, y \in \Delta = \langle e_1, e_2 \rangle$

$$\begin{aligned}
\mathcal{BR}_A(y) &= \operatorname*{arg\,max}_{x \in \langle e_1, e_2 \rangle} x \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y \\
&= \operatorname*{arg\,max}_{x \in \langle e_1, e_2 \rangle} x_1 y_1 - x_2 y_2 = \{e_1\} \\
\mathcal{BR}_B(x) &= \operatorname*{arg\,max}_{y \in \langle e_1, e_2 \rangle} x \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} y \\
&= \operatorname*{arg\,max}_{x \in \langle e_1, e_2 \rangle} -x_1 y_1 + x_2 y_2 = \{e_2\}
\end{aligned}$$

Hence we can immediately see that $e_1 \in \mathcal{BR}_A(e_2)$ and $e_2 \in \mathcal{BR}_B(e_1)$, so (e_1, e_2) is the only Nash Equilibrium for G.

2) Consider a two-person game (A, B), and denote by $\Delta_A \times \Delta_B$ its phase space. Let $(\hat{x}, \hat{y}) \in \text{int } \Delta_A \times \Delta_B$ be a Nash Equilibrium for the game (A, B). If we work with the second notation then we know that $\hat{x}$ maximises the product $x \cdot A\hat{y}$ for $x \in \Delta_A$, and that $\hat{y}$ maximises the product $\hat{x} \cdot By$ for $y \in \Delta_B$. Therefore, for any i, j we have

$$e_i \cdot A\hat{y} \leq \hat{x} \cdot A\hat{y} \qquad \hat{x} \cdot Be_j \leq \hat{x} \cdot B\hat{y}.$$

The two vectors $\hat{x}, \hat{y}$ can be written as the linear combinations $\hat{x} = \sum_i \lambda_i e_i$, and $\hat{y} = \sum_j \rho_j e_j$, where $\lambda_i, \rho_j > 0$ for all i, j since we have assumed that the Nash Equilibrium is contained in the interior of our phase space, and $\sum_i \lambda_i = \sum_j \rho_j = 1$ since we are working with probability vectors. Then if we sum over the two previous inequalities we obtain

$$\begin{aligned}
\hat{x} \cdot A\hat{y} &= \sum_i \lambda_i e_i \cdot A\hat{y} \leq \sum_i \lambda_i \hat{x} \cdot A\hat{y} = \hat{x} \cdot A\hat{y} \\
\hat{x} \cdot B\hat{y} &= \sum_j \rho_j \hat{x} \cdot Be_j \leq \sum_j \rho_j \hat{x} \cdot B\hat{y} = \hat{x} \cdot B\hat{y}.
\end{aligned}$$

In order to get a strict inequality in the previous derivation we would need at least one i and/or j such that $e_i \cdot A\hat{y} < \hat{x} \cdot A\hat{y}$, and/or $\hat{x} \cdot Be_j < \hat{x} \cdot B\hat{y}$, but both these conditions are clearly impossible. Therefore we can conclude that for all i and all j

$$(A\hat{y})_i = e_i \cdot A\hat{y} = \hat{x} \cdot A\hat{y} = c \qquad (\hat{x}^{\mathsf{T}}B)_j = \hat{x} \cdot Be_j = \hat{x} \cdot B\hat{y} = \tilde{c}$$

where $c, \tilde{c} \in \mathbb{R}$ are constants.

Remark. *We have just showed an equivalent statement to Lemma* 1.2 *in the lecture notes, but for the specific case of a Nash Equilibrium point contained in the interior of the state space. We can do better. If a Nash Equilibrium* $(\hat{x}, \hat{y})$ *is NOT in the interior of* $\Delta_A \times \Delta_B$ *then from the above proof we can conclude that*

$$\begin{cases} (A\hat{y})_i = c & \textit{whenever } \hat{x}_i \neq 0 \\ (\hat{x}^{\mathsf{T}}B)_j = \tilde{c} & \textit{whenever } \hat{y}_j \neq 0 \end{cases}$$

where $c, \tilde{c} \in \mathbb{R}$ *are constants. This gives us a complete reformulation of Lemma 1.2 for 2 player games.*

3) Consider the two player game encoded by the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix},$$

and let us choose to follow the second convention for 2 players games. We will firstly compute the Nash Equilibria of these matrices and then we will check if they are ESS. The Best Response maps are

$$\mathcal{BR}_A(y) = \arg\max_{x \in \Delta} x \cdot Ay = \arg\max_{x \in \Delta} 2 - (1 - y_1)x_1 = \begin{cases} \{e_2\} & \text{if } y \neq e_1 \\ \Delta & \text{if } y = e_1 \end{cases}$$

$$\mathcal{BR}_B(x) = \arg\max_{y \in \Delta} x \cdot By = \arg\max_{y \in \Delta} 2 - x_2 y_1 = \begin{cases} \{e_2\} & \text{if } x \neq e_1 \\ \Delta & \text{if } x = e_1 \end{cases}$$

from which we can read that (e_1, e_1) and (e_2, e_2) are Nash Equilibria: $e_1 \in \Delta = \mathcal{BR}_A(e_1)$ and $e_1 \in \Delta = \mathcal{BR}_B(e_1)$ and also $e_2 \in \mathcal{BR}_A(e_2)$ and $e_2 \in \mathcal{BR}_B(e_2)$. Note that (e_2, e_1) is an NE for this game as well. Since e_1 corresponds to strategy i, and e_2 to strategy ii, we can conclude that the strategies (i, i) and (ii, ii) are NE.

Next we want to see if such strategies are Evolutionary Stable. Recall the definition: $(\hat{x}, \hat{y})$ is an ESS if for all $\varepsilon > 0$ and all $(x, y) \in (\Delta_A \setminus \{\hat{x}\}) \times (\Delta_B \setminus \{\hat{y}\})$ then

$$\begin{aligned} x \cdot A(\varepsilon y + (1-\varepsilon)\hat{y}) &< \hat{x} \cdot A(\varepsilon y + (1-\varepsilon)\hat{y}) \\ (\varepsilon x + (1-\varepsilon)\hat{x}) \cdot By &< (\varepsilon x + (1-\varepsilon)\hat{x}) \cdot B\hat{y}. \end{aligned}$$

Firstly we will show that $(\hat{x}, \hat{y}) = (e_1, e_1)$ is NOT an ESS. Fix $\varepsilon > 0$, and take the point $\left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right)$ then this choice yields

$$\begin{aligned}
x \cdot A(\varepsilon y + (1-\varepsilon)\hat{y}) &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 - \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 - \frac{\varepsilon}{2} \\ 2 \end{pmatrix} \\
&= 2 - \frac{\varepsilon}{4} > 2 - \frac{\varepsilon}{2} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 - \frac{\varepsilon}{2} \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 - \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2} \end{pmatrix} = \hat{x} \cdot A(\varepsilon y + (1-\varepsilon)\hat{y})
\end{aligned}$$

which means that (e_1, e_1) is not an ESS.

On the other hand, the Nash Equilibrium $(\hat{x}, \hat{y}) = (e_2, e_2)$ is an ESS. Let $\varepsilon \in (0, 1)$, and take any $(x, y) \in (\Delta_A \setminus \{e_2\}) \times (\Delta_B \setminus \{e_2\})$ then

$$\begin{aligned}
x \cdot A(\varepsilon y + (1-\varepsilon)\hat{y}) &= \begin{pmatrix} x_1 \\ 1 - x_1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \varepsilon y_1 \\ 1 - \varepsilon y_1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 1 + \varepsilon y_1 \\ 2 \end{pmatrix} \\
&= x_1(1 + \varepsilon y_1) + 2(1 - x_1) \\
&< 2x_1 + 2(1 - x_1) = 2 \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 + \varepsilon y_1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \varepsilon y_1 \\ 1 - \varepsilon y_1 \end{pmatrix} = \hat{x} \cdot A(\varepsilon y + (1-\varepsilon)\hat{y})
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon x + (1-\varepsilon)\hat{x}) \cdot By &= \begin{pmatrix} \varepsilon x_1 \\ 1 - \varepsilon x_1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ 1 - y_1 \end{pmatrix} \\
&= 2\varepsilon x_1 + (2 - y_1)(1 - \varepsilon x_1) \\
&< 2\varepsilon x_1 + 2(1 - \varepsilon x_1) = 2 \\
&= \begin{pmatrix} \varepsilon x_1 \\ 1 - \varepsilon x_1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{y} \cdot B(\varepsilon x + (1-\varepsilon)\hat{x})
\end{aligned}$$

which confirms that (e_2, e_2), or (ii, ii) is an ESS.

2.2 Two players replicator dynamics

Exercise 2.2:
Consider the two players game given by the matrix

$$G = \begin{pmatrix} (1,-1) & (0,0) \\ (0,0) & (-1,1) \end{pmatrix}$$

then, as we have done in the previous question, we can retrieve the two matrices defining this game

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using Equations (17) from Section 2.2 of the notes we can write down the replicator equations of this game as

$$\begin{aligned} \dot{x_i} &= x_i((Ay)_i - x \cdot Ay) \\ &= x_i((-1)^{i+1} y_i - x_1 y_1 + x_2 y_2) \\ \dot{y_j} &= y_j((x^{\mathsf{T}} B)_j - x \cdot By) \\ &= y_j((-1)^j x_j + x_1 y_1 - x_2 y_2) \end{aligned}$$

where $x, y \in \Delta = \langle e_1, e_2 \rangle$, and $i, j \in \{1, 2\}$. Because of the specific shape of our phase space Δ we can rewrite these two equations using the fact that $x_2 = 1 - x_1$ and $y_2 = 1 - y_1$ for $x_1, y_1 \in [0,1]$

$$\begin{aligned} \dot{x_1} &= x_1(1 - x_1) \\ \dot{y_1} &= -y_1(1 - y_1). \end{aligned}$$

We can then proceed with the usual phase diagram analysis, as for any system of ODEs. The phase space of this system is the unit square $I^2 = [0,1] \times [0,1] \subset \mathbb{R}^2$. From now on we will drop the indexes in the equations. The derivative of x is zero along $\{x = 0\}$ and $\{x = 1\}$, whereas $\dot{y}$ is zero along $\{y = 0\}$ and $\{y = 1\}$. The four vertices of I^2 are singularities for the flow ($\dot{x} = \dot{y} = 0$). Along $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$ we have $\dot{y} < 0$, and along $(0,1) \times \{0\}$ and $(0,1) \times \{1\}$, we have $\dot{x} > 0$. We can therefore conclude that $(0,1)$ is a sink (in black in the figure), $(0,0)$ and $(1,1)$ are saddles (in red in the figure), and $(1,0)$ is a sink (in white in the figure). The flow flows travels from close to $(0,1)$ towards $(1,0)$ without ever touching the boundary of the unit square. See Figure 13 for a sketch of the flow.

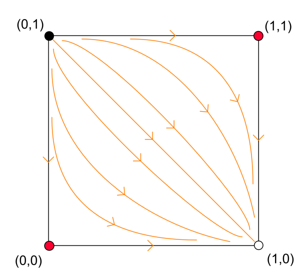

Figure 13: Flow for the game determined by G

2.3 Symmetric games

Exercise 2.3:
Consider a symmetric 2×2 game with payoff (square) matrices A and B, where $A = B^{\mathsf{T}}$. Suppose that $x(0) = y(0)$. We are going to show that if Player 1 plays strategy x against strategy y then that is equal to Player 2 playing x against y. We are going to work with the 2^{nd} convention for the replicator equation. Recall that if A, B are square matrices then $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$. Therefore

$$\begin{aligned}\dot{x_i} &= x_i((Ay)_i - x \cdot Ay) = x_i((B^{\mathsf{T}} y)_i - x \cdot B^{\mathsf{T}} y) \\ &= x_i((y^{\mathsf{T}} B)_i - x^{\mathsf{T}} B^{\mathsf{T}} y) = x_i((y^{\mathsf{T}} B)_i - y^{\mathsf{T}} B x) \\ &= x_i((y^{\mathsf{T}} B)_i - y \cdot Bx) = \dot{y_i}.\end{aligned}$$

Under the assumption $x(0) = y(0)$ we can conclude that, by uniqueness of solutions of ODEs, that $x(t) = y(t)$ for all times. Recall that we have showed in part 3 of Exercise 1.3 that a solution always exist to every initial value problem (starting in Δ) and that such a solution exists for all times.

This question can be approached from a more geometric perspective. Consider the product space $\Delta \times \Delta$ and more specifically its diagonal

$$D := \{(x, y) \in \Delta \times \Delta \mid x = y\}.$$

We want to show that the space D is invariant under the action of $\dot{x} - \dot{y}$. Using Equations (18) from the lecture notes we can write

$$\begin{aligned}\dot{x} - \dot{y} &= x_i((Ay)_i - x \cdot Ay) - y_i((Ax)_i - x \cdot Ay) \\ &= (x_i - y_i)((A(x + y))_i - x \cdot Ay) - (x_i(Ax)_i - y_i(Ay)_i)\end{aligned}$$

which tells us that $\dot{x} - \dot{y}|_D = 0$. This means that the vector field $\dot{x} - \dot{y}$ is tangent to D at every point, hence there is no normal component pointing outwards from D. Therefore, D is invariant under $\dot{x} - \dot{y}$, and so if the flow $(x(t), y(t))$ starts on D, i.e. $x(0) = y(0)$, then $(x(t), y(t))$ is in D for all times t.

2.4 The 2×2 case

Exercise 2.4:
1) Consider the system of ODEs given by Equations (19) in Section 2.4 of the notes

$$\begin{aligned}\dot{x} &= x(1 - x)(\alpha_1 - y(\alpha_1 + \alpha_2)) \\ \dot{y} &= y(1 - y)(\beta_1 - x(\beta_1 + \beta_2)).\end{aligned} \tag{8}$$

We want to understand all the different (non-degenerate) phase portraits that can arise from this system. Our state space is the usual unit square $I^2 = [0, 1] \times [0, 1]$ in $\mathbb{R}^2$, endowed with the subspace topology. Firstly, notice that along the boundary ∂I^2 at least one of the two derivatives is zero.

{x=0}	$\beta_1 > 0 \Rightarrow \dot{y} > 0$
{y=0}	$\alpha_1 > 0 \Rightarrow \dot{x} > 0$
{x=1}	$\beta_2 > 0 \Rightarrow \dot{y} < 0$
{y=1}	$\alpha_2 > 0 \Rightarrow \dot{x} < 0$

Table 3: Behaviour of Equation 8 along ∂I^2.

Table 3 summarises the behaviour of the flow restricted on the boundary. If we now look at the interior of the unit square we have two other lines along which one of the equations in 8 equals zero, namely $\{x = \frac{\beta_1}{\beta_1+\beta_2}\}$ along which $\dot{y} = 0$, and $\{y = \frac{\alpha_1}{\alpha_1+\alpha_2}\}$ along which $\dot{x} = 0$. Notice that the point $\theta = (\frac{\beta_1}{\beta_1+\beta_2}, \frac{\alpha_1}{\alpha_1+\alpha_2})$ is an equilibrium point for the system. For now we will assume that θ is in int I^2: this is equivalent to $\alpha_1\alpha_2 > 0$, and $\beta_1\beta_2 > 0$. In Figure 14 we have reported the direction of the flow when intersecting the nullclines $\{x = \frac{\beta_1}{\beta_1+\beta_2}\}$ (in red on the left) and $\{y = \frac{\alpha_1}{\alpha_1+\alpha_2}\}$ (in blue on the right).

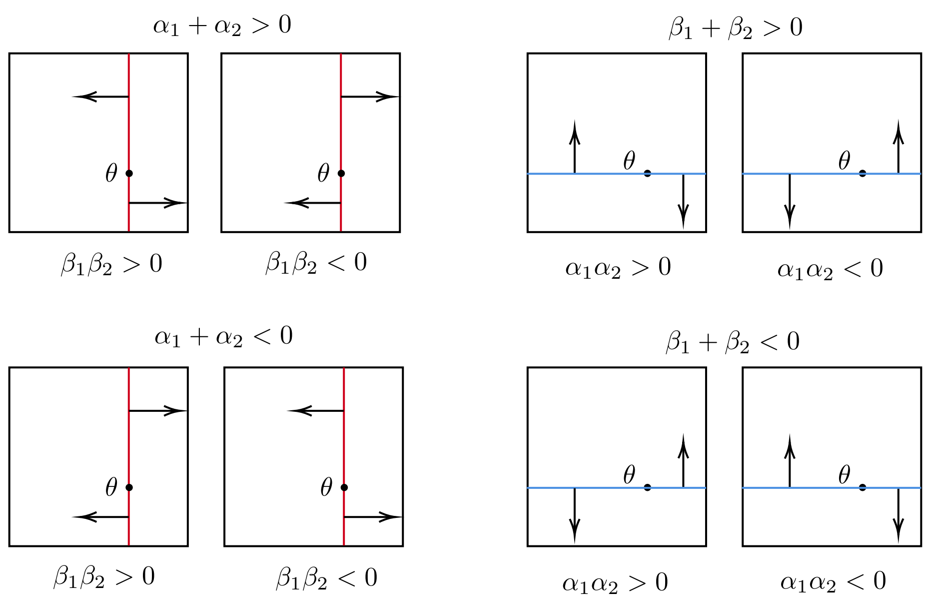

Figure 14: Possible flow directions along the nullclines in int I^2.

We can clearly see that the direction of the flow when crossing the nullclines only depends on $\alpha_1\alpha_2$ and $\beta_1\beta_2$. All the possible phase portraits that Equation 8 are described in Proposition 2.1 in the lecture notes. Given the assumption $\theta \in$ int I^2 we have made before, we are interested for now in understanding the portraits associated with case (i) and (iii). Set $\beta_1 > 0$ therefore fixing the direction of our flow (one gets the same portraits but with the flow direction reversed for $\beta_1 < 0$, as suggested in Figure 14). From Table 3 we know that if $\alpha_1\beta_1 > 0$ then the points $(0, 0)$, and $(1, 1)$ are sources, whilst $(0, 1)$, and $(1, 0)$ are sinks. On the other hand, $\alpha_1\beta_1 < 0$, translates to the flow travelling clockwise around ∂I^2. We can conclude that the left phase portrait in Figure 13 in the lecture notes corresponds to case (i), whereas the right one corresponds to (iii).

In Figure 16 you can see the direction of the derivatives in the various quadrants for the cases we have just discussed (under the underlying assumption $\beta_1 > 0$.

What happens if $\theta \notin \text{int } I^2$? If either $\alpha_1\alpha_2 < 0$ or $\beta_1\beta_2 < 0$ we obtain a *dominated strategy* type of system, case (ii) in Proposition 2.1. Assume, for the sake of discussion, that $\alpha_1 < 0$, and $\alpha_2 > -\alpha_2, \beta_1 > 0$, and $\beta_2 > 0$ (every other case is either similar or simpler). Then we have $\alpha_1\alpha_2 < 0$, $\beta_1\beta_2 > 0$, the nullcline $\{x = \frac{\beta_1}{\beta_1+\beta_2}\}$ is still in int I^2, so we obtain a phase diagram as in Figure 15. The yellow dot in the top right corner represents the dominating strategy, whereas the red dotted line is $\{x = \frac{\beta_1}{\beta_1+\beta_2}\}$. Picking different values for the α's and β's will surely change the dominating strategy, the direction of the flow, and the presence of nullclines, but the overall shape of the phase diagram will always be the same.

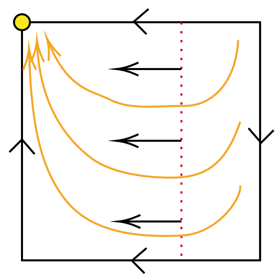

Figure 15: Dominated Strategy

A full explanation of the terminology can be found in *Hofbauer, Sigmund – Evolutionary games and population dynamics*. The term *dominated strategy* is illustrated in Section 8.3, whereas *battle of the sexes* or *coordination game* is explained in Section 10.2. The term *zero-sum case* includes not only zero-sum games, but all games in which the total payoff between players is zero, i.e. the net change of global wealth is zero.

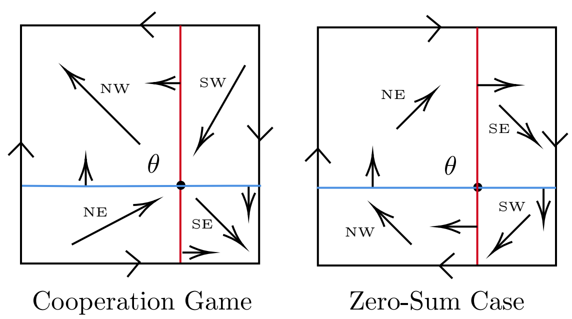

Figure 16: Direction of the flow for case (i) and (iii) assuming $\beta_1 > 0$.

We will now analyse some specific examples of games for every type of phase portrait described in Proposition 2.1. Please note we will adopt the second convention from now, and we will denote by $\Delta_A \times \Delta_B$ the total phase space (I^2 is a reparameterisation of such space). Let

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

be the matrices describing the *stag hunt* game. Then

$$\alpha_1 = -1 \quad \alpha_2 = -1, \quad \beta_1 = -1, \quad \beta_2 = -1$$

hence $\alpha_1\alpha_2 > 0$, $\beta_1\beta_2 > 0$, $\alpha_1\beta_1 > 0$: this is a coordination game. We have two pure dominating strategies, and a Mixed Nash Equilibrium in the interior of I^2. The Mixed Nash Equilibrium in I^2 is given by $\theta = \left(\frac{\beta_1}{\beta_1+\beta_2}, \frac{\alpha_1}{\alpha_1+\alpha_2}\right)$ which corresponds to $\left(\left(\frac{\beta_1}{\beta_1+\beta_2}, \frac{\beta_2}{\beta_1+\beta_2}\right), \left(\frac{\alpha_1}{\alpha_1+\alpha_2}, \frac{\alpha_2}{\alpha_1+\alpha_2}\right)\right) = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. The other Pure Nash Equilibria can be computed by looking at the best responses

$$\begin{aligned}
\mathcal{BR}_A(e_1) &= \arg\max_{x\in\Delta} x \cdot A\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \arg\max_{x\in\Delta} 4x_1 + 3x_2 = \{e_1\} \\
\mathcal{BR}_A(e_2) &= \arg\max_{x\in\Delta} x \cdot A\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \arg\max_{x\in\Delta} x_1 + 2x_2 = \{e_2\} \\
\mathcal{BR}_B(e_1) &= \arg\max_{y\in\Delta} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \cdot By = \arg\max_{y\in\Delta} 4y_1 + 3y_2 = \{e_1\} \\
\mathcal{BR}_B(e_2) &= \arg\max_{y\in\Delta} \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \cdot By = \arg\max_{y\in\Delta} y_1 + 2y_2 = \{e_2\}
\end{aligned}$$

so (e_1, e_1) and (e_2, e_2) are the equilibria we were looking for, which corresponds to the strategies (C, C), and (D, D).

Another coordination game example is given by the *battle of sexes* game described by

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

The coefficients for this game are given by

$$\alpha_1 = -2, \qquad \alpha_2 = -3, \qquad \beta_1 = -3, \qquad \beta_2 = -2$$

which confirms that this is a coordination game since $\alpha_1\alpha_2 > 0$, $\beta_1\beta_2 > 0$, and $\alpha_1\beta_1 > 0$. As before we have a Mixed Nash Strategy given by $\left(\left(\frac{3}{5}, \frac{2}{5}\right), \left(\frac{2}{5}, \frac{3}{5}\right)\right)$. The two Pure Nash Strategies are given again by (e_1, e_1) and (e_2, e_2).

Consider the classic *Prisoner's Dilemma* game described by the matrices

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

given in Example 0.3 in the lecture notes. For this game we have that

$$\alpha_1 = -1, \quad \alpha_2 = 1, \quad \beta_1 = -1, \quad \beta_2 = 1$$

which implies $\alpha_1\alpha_2 = -1 < 0$ and $\beta_1\beta_2 = -1 < 0$. The Prisoner's Dilemma falls under the dominated strategy category, therefore there is one dominating pure strategy given by the Nash Equilibrium (e_2, e_2). As a sanity check

$$\begin{aligned}
\mathcal{BR}_A(e_2) &= \arg\max_{y\in\Delta} y \cdot Ae_2 = \arg\max_{y\in\Delta} y_2 = \{e_2\} \\
\mathcal{BR}_B(e_2) &= \arg\max_{x\in\Delta} e_2 \cdot Bx = \arg\max_{x\in\Delta} x_2 = \{e_2\}
\end{aligned}$$

as we claimed.

Finally, let us look at zero sum type of games. The first we want to look at is described by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which give us coefficients

$$\alpha_1 = 1, \qquad \alpha_2 = -1, \qquad \beta_1 = -1, \qquad \beta_2 = 1.$$

Since $\alpha_1\alpha_2 < 0$, and $\beta_1\beta_2 < 0$ then we have a dominated strategy type of game. As before, we only have one Pure Nash Equilibrium

$$\mathcal{BR}_A(e_2) = \operatorname*{arg\,max}_{y\in\Delta} y \cdot Ae_2 = \operatorname*{arg\,max}_{y\in\Delta} -y_2 = \{e_1\}$$
$$\mathcal{BR}_B(e_1) = \operatorname*{arg\,max}_{x\in\Delta} e_1 \cdot Bx = \operatorname*{arg\,max}_{x\in\Delta} -x_1 = \{e_2\}$$

given by (e_1, e_2).

The last game we want to analyse is described by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which give us coefficients

$$\alpha_1 = -1, \qquad \alpha_2 = -1, \qquad \beta_1 = 1, \qquad \beta_2 = 1.$$

Since $\alpha_1\alpha_2 > 0$, $\beta_1\beta_2 > 0$, and $\alpha_1\beta_1 < 0$ then we have a zero sum game with interior Nash Equilibrium, given once again by $\left(\left(\frac{\beta_1}{\beta_1+\beta_2}, \frac{\beta_2}{\beta_1+\beta_2}\right), \left(\frac{\alpha_1}{\alpha_1+\alpha_2}, \frac{\alpha_2}{\alpha_1+\alpha_2}\right)\right) = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. We will see in the next section that for this game the solution to this specific type of game is described by simple close periodic orbits (topological circles). This means that θ is not asymptotically stable (only Lyapunov stable).

2) Recall that we proved in part 2 of Exercise 1.3 that adding constant column vectors to a matrix does not change its replicator dynamics. Therefore, it is possible that two different matrices induce a phase portrait belonging to the category of zero sum games.

3) As hinted in the question, we will firstly linearise our system around the equilibrium $\theta = \left(\frac{\beta_1}{\beta_1+\beta_2}, \frac{\alpha_1}{\alpha_1+\alpha_2}\right)$. Consider the perturbed point $\tilde{\theta} = \left(\frac{\beta_1}{\beta_1+\beta_2} + \varepsilon_1, \frac{\alpha_1}{\alpha_1+\alpha_2} + \varepsilon_2\right)$ for $\varepsilon_1, \varepsilon_2 > 0$ small, then the equations for the replicator dynamics will give us

$$\dot{\varepsilon}_1 = -\frac{\beta_1\beta_2(\alpha_1+\alpha_2)}{(\beta_1+\beta_2)^2}\varepsilon_2 + \mathcal{O}(\varepsilon^2)$$
$$\dot{\varepsilon}_2 = -\frac{\alpha_1\alpha_2(\beta_1+\beta_2)}{(\alpha_1+\alpha_2)^2}\varepsilon_1 + \mathcal{O}(\varepsilon^2)$$

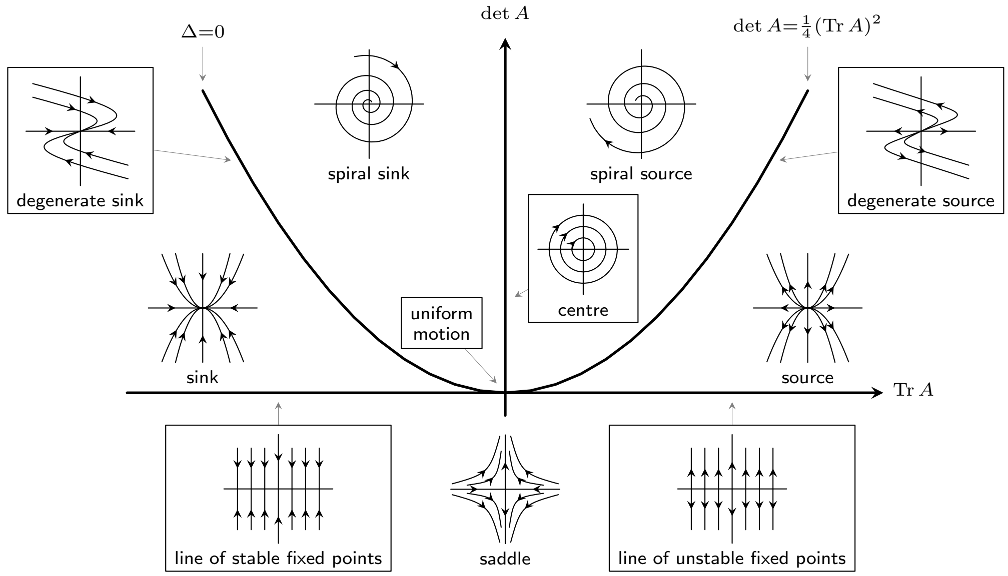

Figure 17: Classification of Phase Portraits in the $(\det A, \operatorname{Tr} A)$-plane [1]

which leads us to the linearisation matrix at θ

$$L = \begin{pmatrix} 0 & -\frac{\beta_1\beta_2(\alpha_1+\alpha_2)}{(\beta_1+\beta_2)^2} \\ -\frac{\alpha_1\alpha_2(\beta_1+\beta_2)}{(\alpha_1+\alpha_2)^2} & 0 \end{pmatrix}.$$

This matrix has a very special shape since its trace, $\operatorname{Tr} L$, is always zero, which severely restricts the possible flow behaviour near θ.

By looking at the Poincaré Diagram in Figure 17 we see that there are only two possibilities for the flow close to θ (excluding the degenerate case): either $\det L$ is positive and we have that the flow generates concentric ellipses (purely imaginary eigenvalues), or $\det L$ is negative and θ is a saddle point (the real parts of the two eigenvalues have opposite sign).

To start with, the determinant of L is given by

$$\det L = -\frac{\alpha_1\alpha_2\beta_1\beta_2}{(\alpha_1+\alpha_2)(\beta_1+\beta_2)}$$

therefore the conditions of Proposition 2.1 will uniquely determine the positivity of it.

For zero sum case with interior Nash Equilibrium $\alpha_1\alpha_2 > 0$, $\beta_1\beta_2 > 0$, and $\alpha_1\beta_1 > 0$, which means that either α_1, α_2 are both positive, and β_1, β_2 are negative, or vice-versa. In both cases $\det L$ is positive (beware of the minus sign

[1] Adapted from: https://tex.stackexchange.com/questions/347201/

in front of it), and so θ is the centre of concentric ellipses (the ratio between the sizes of minor and major axes of the ellipses depends on the ration of the modulus of the imaginary eigenvalues).

For coordination games we know that $\alpha_1\alpha_2 > 0$, $\beta_1\beta_2 > 0$, and $\alpha_1\beta_1 > 0$, which means that $\alpha_1, \alpha_2, \beta_1, \beta_2$ have all the same sign. This is sufficient to show that $\det L$ is negative and hence θ is a saddle point.

We will now directly show that the orbits of the replicator dynamics circle around θ. Consider the Lyapunov function

$$P(x, y) = x^{-\beta_1}(1-x)^{-\beta_2} y^{\alpha_1}(1-y)^{\alpha_2},$$

and we claim that this function is constant along the solutions of the replicator dynamics whenever $\alpha_1\alpha_2 > 0$, $\beta_1\beta_2 > 0$, and $\alpha_1\beta_1 > 0$. Notice that P is always positive in I^2, and vanishes on its boundary. If we compute the time logarithmic derivative of P

$$\begin{aligned}
\frac{\dot{P}}{P}(x, y) = \dot{\log P} = \frac{d}{dt}(-\beta_1 \log x - \beta_2 \log(1-x) + \alpha_1 \log y + \alpha_2 \log(1-y)) \\
&= -\beta_1 \frac{\dot{x}}{x} + \beta_2 \frac{\dot{x}}{1-x} + \alpha_1 \frac{\dot{y}}{y} - \alpha_2 \frac{\dot{y}}{1-y} \\
&= -(\alpha_1 - y(\alpha_1 + \alpha_2))(\beta_1 - \beta_1 x - \beta_2 x) + (\beta_1 - x(\beta_1 + \beta_2))(\alpha_1 - \alpha_1 y - \alpha_2 y) \\
&= -(\alpha_1 - y(\alpha_1 + \alpha_2))(\beta_1 - x(\beta_1 + \beta_2)) + (\alpha_1 - y(\alpha_1 + \alpha_2))(\beta_1 - y(\beta_1 + \beta_2)) \\
&= 0
\end{aligned}$$

we see that $\dot{P} = 0$ along the orbits of the solution to Equation 8.

This means that P is constant along orbits, therefore the orbits of Equation 8 are level sets of P.

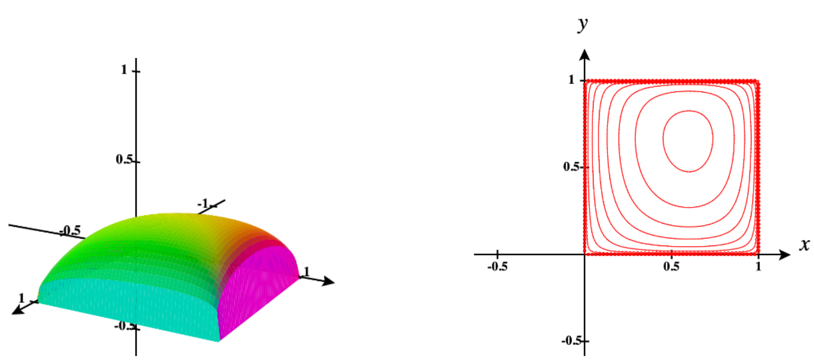

Figure 18: 3D and contour plot of the function P for $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{6}, \beta_1 = -\frac{1}{2}, \beta_2 = -\frac{1}{3}$.

In Figure 18 we can see a contour plot for the function P for some arbitrarily chosen values of $\alpha_1, \alpha_2, \beta_1$, and β_2. We can see that the level sets of P are closed simple lines, or topological circles. Those represent the shape of the orbits of the flow induced by Equation 8 in I^2.

4) All the hard work is now done. We have showed, by computing the linearization matrix at θ that the equilibrium point can only either be a saddle (unstable) or orbits cycle around it as in Figure 18 (Lyapunov stable). As we proved in 1.1 in the lecture notes, Evolutionary Stable Strategy are asymptotically stable equilibria for the replicator dynamics, but since θ in our case is either unstable or Lyapunov stable (this is weaker than asymptotically stable), then we can conclude that these games admit no ESS in the interior of I^2.

2.5 A 3×3 replicator dynamics systems with chaos

Exercise 2.5:
1) Fix $\varepsilon \in (0, 1)$ (in order to simplify the calculations), and the two matrices

$$A = \begin{pmatrix} \varepsilon & -1 & 1 \\ 1 & \varepsilon & -1 \\ -1 & 1 & \varepsilon \end{pmatrix} \qquad B = \begin{pmatrix} -\varepsilon & -1 & 1 \\ 1 & -\varepsilon & -1 \\ -1 & 1 & -\varepsilon \end{pmatrix}$$

which describe a 3×3 game with two players. We wish to compute the Nash Equilibria of such a game. In order to maintain consistency with the lecture notes we will adopt the first convention. The state space will be denoted by $\Delta_A \times \Delta_B$. As usual, we will firstly show that we have only one Nash Equilibrium in the interior of $\Delta_A \times \Delta_B$, and then we will move to the boundary. Please note that we will freely use the letters i, j, k to denote indices, these have to be understood as all different elements of $\mathbb{Z}/3\mathbb{Z}$.

Let us consider the indifference lines in Δ_A

$$\begin{aligned} Z^A_{1,2} &= \{(Ay)_1 = (Ay)_2\} = \{(3-\varepsilon)y_1 + (3+\varepsilon)y_2 = 2\} \\ Z^A_{2,3} &= \{(Ay)_2 = (Ay)_3\} = \{(3-\varepsilon)y_2 + (3+\varepsilon)y_3 = 2\} \\ Z^A_{1,3} &= \{(Ay)_1 = (Ay)_3\} = \{(3+\varepsilon)y_1 + (3-\varepsilon)y_3 = 2\} \end{aligned}$$

and in the simplex Δ_B

$$\begin{aligned} Z^B_{1,2} &= \{(Bx)_1 = (Bx)_2\} = \{(3+\varepsilon)x_1 + (3-\varepsilon)x_2 = 2\} \\ Z^B_{2,3} &= \{(Bx)_2 = (Bx)_3\} = \{(3+\varepsilon)x_2 + (3-\varepsilon)x_3 = 2\} \\ Z^B_{1,3} &= \{(Ax)_1 = (Ax)_3\} = \{(3-\varepsilon)x_1 + (3+\varepsilon)x_3 = 2\}. \end{aligned}$$

In Figure 19 we reported all the indifference lines we have just computed, together with the best response for every convex region. To estimate the best response in each of these region it is enough to compute the best response at each corner which accounts to

$$\mathcal{BR}_A(e_i) = \{e_{i+1}\}$$

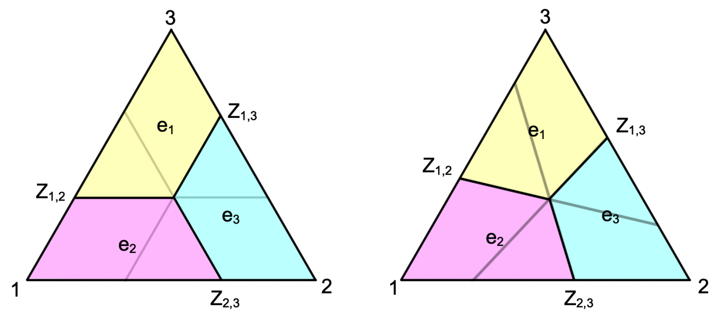

Figure 19: Indifference lines and best response in Δ_A (left), and in Δ_B (right).

and

$$\mathcal{BR}_B(e_i) = \{e_{i+1}\}.$$

Along every indifference line $Z^A_{i,j}$ there is a portion along which $(Ay)_k \leq (Ay)_i = (Ay)_j$ (denoted by a black segment), and a portion along which the opposite inequality holds (denoted by a light grey segment). The same notation has been adopted in Δ_B. Henceforth, when we will say "indifference line" will refer to the black segment of that indifferent line, i.e. we abuse notation and redefine $Z^A_{i,j} \coloneqq Z^A_{i,j} \cap \{(Ay)_k \leq (Ay)_i = (Ay)_j\}$, and $Z^B_{i,j} \coloneqq Z^B_{i,j} \cap \{(Bx)_k \leq (Bx)_i = (Ay)_j\}$. As you will see in Chapter 4, the indifference lines are discontinuity lines for the best response map, hence why we only worried about computing the best response at each corner.

The indifference lines, in both simplices, meet at the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This means that the point $((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ is an interior Nash Equilibrium for this system. This clearly is the only internal Nash Equilibrium, since the indifference lines do not intersect again.

The last thing we are left with is to check for Nash Equilibria along the boundary of our space. These points can only appear as intersections of indifference lines and a side. Notice that $Z^A_{i,j}$ intersects the side $\langle e_{i-1}, e_{j-1}\rangle$, and the best response at the intersection point is given by $\mathcal{BR}_A(Z^A_{i,j} \cap \langle e_{i-1}, e_{j-1}\rangle) = \langle e_i, e_j\rangle$. We find an identical picture in Δ_B, meaning that $\mathcal{BR}_B(Z^B_{i,j} \cap \langle e_{i-1}, e_{j-1}\rangle) = \langle e_i, e_j\rangle$. Therefore

$$\begin{aligned}
&\mathcal{BR}_A(Z^A_{i,j} \cap \langle e_{i-1}, e_{j-1}\rangle) = \langle e_i, e_j\rangle \\
&\mathcal{BR}_B(Z^B_{i+1,j+1} \cap \langle e_i, e_j\rangle) = \langle e_{i+1}, e_{j+1}\rangle
\end{aligned}$$

but since the intersection between indifference lines and sides does not happen at the corners of the simplices (remember $\varepsilon \in (0, 1)$), we can conclude we have no Nash Equilibria on the boundary of the state space. The only Nash Equilibrium for this game is given by

$$\left(\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix},\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}\right).$$

2) We now want to understand the flow described by the replicator dynamics induced by the matrices A and B. More specifically, we want to study the direction of the flow along the edges connecting the vertices of the space $\Delta_A \times \Delta_B$, and show that it corresponds to the one represented in Figure 14 in the Lecture Notes. Firstly, recall that we are considering the system of differential equations

$$\begin{aligned}\dot{x}_i &= x_i((Ay)_i - x \cdot Ay)\\ \dot{y}_j &= y_j((Bx)_j - y \cdot Bx)\end{aligned}$$

and in order to understand the direction of the flow we want to look at the following ratios

$$\begin{aligned}\left(\frac{x_i}{x_j}\right)' &= \frac{\dot{x}_i x_j - x_i \dot{x}_j}{x_j^2} = \frac{x_i}{x_j}\left[(Ay)_i - (Ay)_j\right]\\ \left(\frac{y_i}{y_j}\right)' &= \frac{\dot{y}_i y_j - y_i \dot{y}_j}{y_j^2} = \frac{y_i}{y_j}\left[(Bx)_i - (Bx)_j\right]\end{aligned}$$

where x will always denote an element from Δ_A, and y an element from Δ_B, and $i \neq j$. We will now proceed to calculate a few of these ratios. Recall that we have that R is associated to e_1, P to e_2, and S to e_3. Let us say that we want to understand the direction of the flow between the points $(P,P) = (e_2, e_2)$ and $(P,S) = (e_2, e_3)$, then this means that we are interested in the sign of $(y_2/y_3)'$ along the constraint $x = e_2$ (notice sign $(y_2/y_3)' = -$sign $(y_3/y_2)'$). Hence for $y \in \langle e_2, e_3 \rangle \setminus \{e_2, e_3\}$

$$\left.\left(\frac{y_2}{y_3}\right)'\right|_{x=e_2} = \frac{y_2}{y_3}((Be_2)_2 - (Be_2)_3) = -(1+\varepsilon)\frac{y_2}{y_3} < 0$$

which translates to the flow moving from (e_2, e_2) towards (e_2, e_3) in $\Delta_A \times \Delta_B$. This means that the flow goes from (P,P) to (P,S).

Similarly we can show that the flow goes from (R,P) to (P,P). In order to see this let us compute for $x \in \langle e_1, e_2 \rangle \setminus \{e_1, e_2\}$, and $y = e_2$

$$\left.\left(\frac{x_1}{x_2}\right)'\right|_{y=e_2} = \frac{x_1}{x_2}\left[(Ae_2)_1 - (Ae_2)_2\right] = -(1+\varepsilon)\frac{x_1}{x_2} < 0$$

which confirms that the flow goes from (R,P) to (P,P). Similar calculations give us the direction of the flow along all the edges of the graph in Figure 14.

3) The space $\Delta_A \times \Delta_B$ is 6 dimensional, and it can be reduced to 4 dimensions using the definition of simplex, for example by discarding the 3^{rd} component of

the vectors $x \in \Delta_A$, and $y \in \Delta_B$ since $x_3 = 1 - x_1 - x_2$, and $y_3 = 1 - y_1 - y_2$. Even if we reduce $\Delta_A \times \Delta_B$ to a 4 dimensional object, in order to represent it on a sheet of paper we need to project it into $\mathbb{R}^2$ or $\mathbb{R}^3$. One of the ways to represent $\Delta_A \times \Delta_B$ in $\mathbb{R}^2$ can be seen in Figure 14 in the Lecture Notes. In order to replicate such a graph we need to find the right projection matrix. Recall that Rock corresponds to e_1, Paper to e_2, and Scissors to e_3. Henceforth, we will work under the identification $\left(\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix} \right), \left(\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \end{smallmatrix} \right) \right) = (x_1, x_2, x_3, y_1, y_2, y_3)^\mathsf{T}$.

The matrix

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

induces a linear map from $\mathbb{R}^6$ to $\mathbb{R}^2$. The projection induced by X is too restrictive since we loose too much information: consider $(R, P) = (1, 0, 0, 0, 1, 0)$ and $(R, S) = (1, 0, 0, 0, 0, 1)$, then

$$X(R, P) = X(R, S) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

More specifically, $X(R, \cdot) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, $X(P, \cdot) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right)$, and $X(S, \cdot) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)$, independently from the last three entries of the vectors, i.e. independently from the component coming from Δ_B.

In order to obtain a more useful and meaningful projection of $\Delta_A \times \Delta_B$ we aim at a matrix similar to X, but under which the nine points $(R, R), (R, P), (R, S), (P, R), (P, P), (P, S), (S, R), (S, P), (S, S)$ have all distinct images. For example, consider

$$X = \begin{pmatrix} 3.65 & -1.35 & 1.35 & 5.35 & 1.35 & 1.45 \\ 0.40 & 0.40 & 4.60 & 1.90 & -0.40 & 4.40 \end{pmatrix}$$

then this give us exactly what we want

$$\begin{array}{lll} X(R, R) = (9.00, 2.30) & X(R, P) = (5.00, 0.00) & X(R, S) = (5.10, 4.80) \\ X(P, R) = (4.00, 2.30) & X(P, P) = (0.00, 0.00) & X(P, S) = (0.10, 4.80) \\ X(S, R) = (6.70, 6.50) & X(S, P) = (2.70, 4.20) & X(S, S) = (2.80, 9.00). \end{array}$$

Notice this matrix was computed to generate the diagram in Figure 14 in the Lecture notes.

4) See the appendix of the lecture notes.

3 Iterated Prisoner Dilemma (IRP) and the Role of Reciprocity

3.1 Repeated games with unknown time length

Exercise 3.1:
Recall that the bimatrix for the Prisoner's Dilemma is given by

$$\begin{pmatrix} (-1,-1) & (-3,0) \\ (0,-3) & (-2,-2) \end{pmatrix}$$

where $(-2,-2)$ is the payoff if both prisoners defect. As we have seen in the notes the Nash Equilibrium of this game, given by both parties always defecting, is rather sub-optimal since it leads to a fairly poor payoff. We will now investigate how playing different strategies (namely how cooperating) can usually increase the total payoff if a game is played for a long time (as $t \to \infty$).

Now suppose that Player 1 defects with probability $p \in [0,1]$ and Player 2 with probability $q \in [0,1]$. All the computations will be carried out for Player 1, but given the symmetry of the game the whole discussion immediately extends to Player 2. The payoff at round n is given by

$$\begin{aligned} A_n =&(-2)pq + (-1)(1-p)(1-q) + (-3)(1-p)q+ \\ &+ (0)p(1-q) = p - 2q - 1. \end{aligned}$$

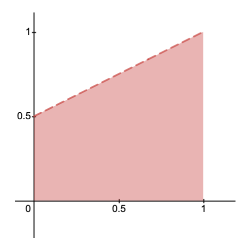

Figure 20: Probability square for (p, q)

Therefore the total payoff, as explained in the notes, is given by

$$A(\omega) = \sum_{i=1}^{\infty} A_i \omega^{i-1} = \frac{p - 2q - 1}{1-\omega}$$

where $\omega \in (0,1)$ is the chance of play the following turn. The expected payoff is therefore given by

$$\mathbb{E}(\text{Payoff}_A) = \frac{A(\omega)}{\frac{1}{1-\omega}} = p - 2q - 1.$$

For $p = 1$, and $q = 1$ we have that $\mathbb{E}(\text{Payoff}_A) = -2$, which precisely corresponds to the expected payoff if both Players always defect. In Figure 20 we have the probability unit square $I^2 = [0,1] \times [0,1]$ with p along the x–axis, and q along the y–axis. The red shaded region represents all the tuples (p, q) which give an expected payoff to Player 1 greater than -2. The shaded region accounts for $3/4$ of the total area, meaning that if they choose to sometime cooperate ($p < 1$) then it is likely that they will get a higher payoff (in the long run) that playing always defect.

3.2 The three strategies AllC, AllD, TFT

Exercise 3.2:
Consider the standard donation game. In Section 3.1 we have analysed the various payoffs corresponding to the strategies always defect (AllD), always cooperate (allC), tit for tat (TFT). Let us consider a new strategy TFTT: a player defects only when the other player defects twice. We want to establish a payoff matrix as Matrix (21) in Section 3.2. As before, $\omega \in (0, 1)$ represents the probability of playing a new round. We will assume that for TFT, and TFTT strategy the player will start by cooperating.

If Player 1 plays $TFTT$ then whenever Player 2 plays AllC, TFT, or TFTT Player 1's payoff is given by $\frac{b-c}{1-\omega}$, since both players are constantly cooperating. The interesting case is whenever Player 2 plays AllD. In this case we have that Player 1's payoff is given by

$$A_1 = -c, \quad A_2 = -c \quad A_n = 0, \quad \text{for } n \geq 3$$

so that $A(\omega) = -c(1 + \omega)$.

Symmetrically if Player 1 plays AllD against TFTT, then we have that their payoff is given by

$$A_1 = b, \quad A_2 = b, \quad A_n = 0 \quad \text{for } n \geq 3$$

so that $A(\omega) = b(1 + \omega)$.

The payoff matrix is given by

$$\frac{1}{1-\omega}\begin{pmatrix} b-c & -c & b-c & b-c \\ b & 0 & b(1-\omega) & b\frac{1+\omega}{1-\omega} \\ b-c & -c(1-\omega) & b-c & b-c \\ b-c & -c\frac{1+\omega}{1-\omega} & b-c & b-c \end{pmatrix}$$

where the fourth row represents Player 1 playing TFTT, and the fourth column represents Player 2 playing TFTT.

3.3 The replicator dynamics associated to a repeated game with the AllC, AllD, TFT strategies

Exercise 3.3:
We want to calculate the Evolutionary Stable Strategies and Nash Equilibria of the matrix

$$A = \begin{pmatrix} -c & -c & b\omega - c \\ 0 & 0 & 0 \\ -c & -c(1-\omega) & b\omega - c \end{pmatrix}.$$

We will assume that $\omega \in (0, 1)$. Notice that the matrix we just wrote has the same Evolutionary Stable Strategies and Nash Equilibria as Matrix (21) in Section 3.1.

There are two cases we have to consider. Firstly assume $b\omega < c$. When calculating the best response to a strategy $x \in \Delta$ we have

$$\mathcal{BR}(x) = \arg\max_{y \in \Delta} y \cdot Ax = \arg\max_{y \in \Delta} (b\omega x_3 - c)y_1 + (b\omega x_3 - c + \omega c x_2)y_3.$$

Thanks to the assumptions $b\omega < c$ and $\omega < 1$ it follows that

- $b\omega x_3 - c < cx_3 - c = -c(1 - x_3) \leq 0$;
- $b\omega x_3 - c + \omega c x_2 < cx_3 - c + cx_2 = -c(1 - x_1 - x_2) \leq 0$

which means that both the coefficients we are trying to minimise are negative, therefore

$$\mathcal{BR}(x) = \{e_2\} \qquad \text{for all } x \in \Delta.$$

The only Nash Equilibrium in this case is e_2, but it actually is a strict Nash Equilibrium

$$x \cdot Ae_2 = -cx_1 + (\omega c - c)x_3 < -cx_1 \leq 0 = e_2 \cdot Ae_2$$

where $x \in \Delta \setminus \{e_2\}$. Therefore, if $b\omega < c$ and $\omega < 1$ then e_2 is a strict Nash Equilibrium, therefore an ESS and Nash Equilibrium.

Now we will assume $b\omega > c$. As we have seen before we have for $x \in \Delta$

$$Ax = \begin{pmatrix} b\omega x_3 - c \\ 0 \\ b\omega x_3 - c + c\omega x_2 \end{pmatrix} = \begin{pmatrix} (Ax)_1 \\ 0 \\ (Ax)_1 + c\omega x_2 \end{pmatrix}.$$

We can compute the indifference lines

$$\begin{aligned} Z_{1,2} &= \{x_3 = \frac{c}{b\omega}\} \\ Z_{2,3} &= \{cx_2 + bx_3 = \frac{c}{\omega}\} \\ Z_{1,3} &= \{x_2 = 0\} = \langle e_1, e_3 \rangle, \end{aligned}$$

which intersect at the point $\tilde{q} = \left(\frac{b\omega - c}{b\omega}, 0, \frac{c}{b\omega}\right)$, which is automatically a Nash Equilibrium. Clearly, there are no NE in the interior of Δ.

Let us compute the best response at the various corners of the simplex. We obtain

$$\begin{aligned} \mathcal{BR}(e_1) &= \arg\max_{y \in \Delta} y \cdot Ae_1 \\ &= \arg\max_{y \in \Delta} -c(y_1 + y_3) = \{e_2\} \\ \mathcal{BR}(e_2) &= \arg\max_{y \in \Delta} y \cdot Ae_2 \\ &= \arg\max_{y \in \Delta} -cy_1 - c(1-\omega)y_3 = \{e_2\} \qquad \text{since } \omega < 1 \\ \mathcal{BR}(e_3) &= \arg\max_{y \in \Delta} y \cdot Ae_3 \\ &= \arg\max_{y \in \Delta} (b\omega - c)(y_1 + y_3) = \langle e_1, e_3 \rangle \quad \text{since } b\omega - c > 0 \end{aligned}$$

which tell us that e_2, e_3 are Nash Equilibria.

Similar calculations tell us the best response along the indifference lines, which we have reported in Figure 21. From it we can read off the remaining Nash Equilibria. If $x \in Z_{1,3} = \langle e_1, e_3 \rangle$, and $x_3 > \frac{c}{b\omega}$, then the best response is given by $\langle e_1, e_3 \rangle$, which implies we have a line of NE, between $\tilde{q}$, and e_3 (included). The last candidate as a Nash Equilibrium is given by $q = \left(0, \frac{b\omega - c}{\omega(b-c)}, \frac{c(1-\omega)}{\omega(b-c)}\right)$, which corresponds to the intersection between $Z_{2,3}$ and the side $\langle e_2, e_3 \rangle$. Along the indifference line $Z_{2,3}$ the Best Response is $\langle e_2, e_3 \rangle$ (except at $\tilde{q}$, hence q is a NE.

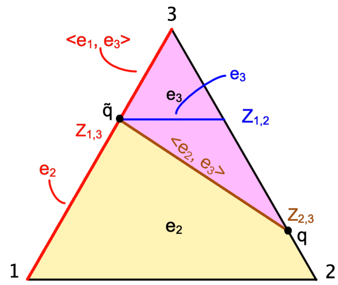

Figure 21: Indifference lines and $\mathcal{BR}$ when $b\omega > c > 0$.

The Nash Equilibria of this system, denoted NE, are

$$\text{NE} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{b\omega - c}{(b-c)\omega} \\ \frac{c(1-\omega)}{(b-c)\omega} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 - p_3 \\ 0 \\ p_3 \end{pmatrix} \right\}_{p_3 \in [\frac{c}{b\omega}, 1]},$$

and in order to simplify notation we will refer to the last family of vectors as Γ, so that $\text{NE} = \{e_1, q\} \cup \Gamma$.

We are left with showing which of these points are ESS. As before, e_2 is an ESS since by Lemma 1.3 in the notes, we need to show that for y close to e_2 we have that $y \cdot Ay < e_2 \cdot Ay = 0$. So if we let $0 \leq \delta, \tau$ and $0 < \varepsilon$, where $\delta + \tau = \varepsilon$, and $y = \left(\begin{smallmatrix} \delta \\ 1-\varepsilon \\ \tau \end{smallmatrix} \right)$ then

$$\begin{aligned} y \cdot Ay &= \begin{pmatrix} \delta \\ 1 - \varepsilon \\ \tau \end{pmatrix} \cdot \begin{pmatrix} b\omega\tau - c \\ 0 \\ b\omega\tau - c + c\omega - c\omega\varepsilon \end{pmatrix} = \\ &= -c\delta - c\tau + c\omega\tau + \mathcal{O}(\varepsilon^2) \\ &\leq -c\varepsilon(1-\omega) + \mathcal{O}(\varepsilon^2) < 0 = e_2 \cdot Ay \end{aligned}$$

where the last inequality follows by taking ε small enough. Next, we will show that any point in $\Gamma \setminus \{e_3\}$ is not an ESS. Recall that a point $\hat{x}$ is an Evolutionary Stable Strategy if for all $x \in \Delta \setminus \{\hat{x}\}$ one has for $\varepsilon > 0$ small enough that

$$x \cdot A(\varepsilon x + (1-\varepsilon)\hat{x}) < \hat{x} \cdot A(\varepsilon x + (1-\varepsilon)\hat{x}).$$

Fix $\varepsilon > 0$, let $\hat{x} = \left(\begin{smallmatrix} 1-p_3 \\ 0 \\ p_3 \end{smallmatrix}\right) = p$ where $p_3 \in [\frac{c}{b\omega}, 1)$, and let $x = e_3$ then

$$e_3 \cdot A(\varepsilon e_3 + (1-\varepsilon)p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot A \begin{pmatrix} (1-\varepsilon)(1-p_3) \\ 0 \\ \varepsilon + (1-\varepsilon)p_3 \end{pmatrix} = -c + b\omega(\varepsilon + (1-\varepsilon)p_3)$$
$$p \cdot A(\varepsilon e_3 + (1-\varepsilon)p) = \begin{pmatrix} 1-p_3 \\ 0 \\ p_3 \end{pmatrix} \cdot A \begin{pmatrix} (1-\varepsilon)(1-p_3) \\ 0 \\ \varepsilon + (1-\varepsilon)p_3 \end{pmatrix} = -c + b\omega(\varepsilon + (1-\varepsilon)p_3).$$

So it follows that no point in $\Gamma \setminus \{e_3\}$ is an ESS. Similarly, we can show that e_3 is not an ESS either. Let $\left(\begin{smallmatrix} y_1 \\ 0 \\ 1-y_1 \end{smallmatrix}\right) = y \in \langle e_1, e_3 \rangle \setminus \{e_3\}$, then we have

$$y \cdot Ay = y_1(b\omega y_1 - c) + (1-y_1)(b\omega y_1 - c) = b\omega y_1 - c = e_3 \cdot Ay$$

which immediately tells us that e_3 is not an ESS either.

Finally, we will show that q is not an Evolutionary Stable Strategy. Recall that by Theorem 1.1 in the lecture notes any ESS is an asymptotically stable equilibrium for the replicator dynamics. This is not the case for q. Consider a point $\left(\begin{smallmatrix} 0 \\ x_2 \\ 1-x_2 \end{smallmatrix}\right) = x \in \langle e_2, e_3 \rangle$, then we have by the replicator equation

$$\begin{aligned} \dot{x}_2 &= x_2(0 - (1-x_2)(b\omega(1-x_2) - c + c\omega x_2)) \\ &= x_2(1-x_2)(\omega(b-c)x_2 - (c - b\omega)). \end{aligned}$$

Therefore, $\dot{x}_2$ is negative over $(0, \frac{b\omega - c}{(b-c)\omega})$, and positive over $(\frac{b\omega-c}{(b-c)\omega}, 1)$ which is equivalent to saying that q repels points on $\langle e_1, e_3 \rangle$. Since q is not asymptotically stable along $\langle e_1, e_3 \rangle$, then it cannot be an ESS.

We can conclude that the only Evolutionary Stable Strategy of this game is e_2.

Since we are still assuming $b\omega > c$, we can see in Figure 14 of the Lecture Notes that the simplex Δ is partitioned into two invariant subsets. This figure represents the replicator dynamics for $\frac{1}{1-\omega}\left(\begin{smallmatrix} b-c & -c & b-c \\ b & 0 & b(1-\omega) \\ b-c & -c(1-\omega) & b-c \end{smallmatrix}\right)$, which is obtained through adding or removing multiples of the vector $\mathbb{1}$ from A. As we have seen in Exercise 1.3 part 2., the replicator dynamics of these two matrices are identical. We will denote the two sets in the partition as

$$\begin{aligned} \Xi_d &:= \left\{ x \in \Delta \,|\, x_3 < \tfrac{c(1-\omega)}{(b-c)\omega} \right\} \\ \Xi_u &:= \left\{ x \in \Delta \,|\, x_3 > \tfrac{c(1-\omega)}{(b-c)\omega} \right\}. \end{aligned}$$

In order to simplify notation we will denote $\frac{c(1-\omega)}{(b-c)\omega}$ by $\hat{x}_3$. We will now quickly show that the line $\{x_3 = \hat{x}_3\}$ is invariant under the replicator dynamics

$$\begin{aligned}(Ax)_3 - x \cdot Ax_3 &= (Ax)_1 + c\omega x_2 - (Ax)_1(x_1 + x_3) - c\omega x_2 x_3\\ &= (Ax)_1(1 - x_1 - x_3) + c\omega x_2(1 - x_3)\\ &= x_2((Ax)_1 + c\omega(1 - x_3))\\ &= x_2(b\omega x_3 - c + c\omega - c\omega x_3)\\ &= x_2(x_3(\omega(b - c)) - c(1 - \omega))\\ &= \frac{x_2}{\omega(b - c)}(x_3 - \hat{x}_3).\end{aligned}$$

Since $\dot{x}_3 = x_3((Ax)_3 - x \cdot Ax) = \frac{x_2 x_3}{\omega(b-c)}(x_3 - \hat{x}_3)$ then we see that on $\{x_3 = \hat{x_3}\}$ the derivative $\dot{x}_3$ is zero, which means that the flow is constrained along the line. Both Ξ_d and Ξ_u are invariant.

In Ξ_d we have only one Nash Equilibrium e_2, which is also an Evolutionary Stable Strategy. Therefore e_2 is an asymptotically stable equilibrium for the flow starting in Ξ_d. The point e_2 in the simplex corresponds to the strategy allD, hence the most optimal (and stable) strategy to play in Ξ_d is to always defect.

The situation is slightly more delicate in Ξ_u. Here we have a line of Nash Equilibria along $\langle e_1, e_3\rangle \cap \Xi_u$, namely Γ. Every point in Γ is an equilibrium for the replicator dynamics, and attracts point in the interior of Δ, as showed by the vector field in the left diagram in Figure 15 in the lecture notes. Therefore, depending on the Initial Value Problem the flow in Ξ_u can end up reaching a different point in Γ, meaning that the recommended strategy depends on the initial conditions one chooses, and it is a mixed strategy (between allC and TFT). The only time we can get a pure strategy is if one starts in $\langle e_2, e_3\rangle \cap \Xi_u$, then the flow tends to e_3, or TFT.

Exercise 3.4:
1) Consider the replicator dynamics defined by the matrix

$$A = \begin{pmatrix} 0 & -1 & \delta\sigma \\ 1 & 0 & -\kappa\sigma \\ \delta & -\kappa & 0 \end{pmatrix}$$

where $\delta = \omega\varepsilon, \kappa = 1 - \omega + \omega k\varepsilon, \sigma = \frac{b\theta - c}{c - c\theta}$, and $\theta = \omega(1 - (k+1)\varepsilon)$ are positive constants. The replicator equations associated with A are

$$\begin{aligned}\dot{x}_1 &= x_1(-x_2 + \delta\sigma x_3 - x_3(1+\sigma)(\delta x_1 - \kappa x_2))\\ \dot{x}_2 &= x_2(x_1 - \kappa\sigma x_3 - x_3(1+\sigma)(\delta x_1 - \kappa x_2))\\ \dot{x}_3 &= x_3(1 - x_3(1+\sigma))(\delta x_1 - \kappa x_2).\end{aligned}$$

Consider the Lyapunov function $P(x_1, x_2, x_3) = x_1^A x_2^B x_3^C(1 - (1+\sigma)x_3)$, where $A = \frac{\kappa}{\theta}, B = \frac{\delta}{\theta}$, and $C = -\frac{1}{\theta}$. We claim that P is constant along the orbits of the solution of the previous system of ODEs. In order to prove this we will show that the (logarithmic) derivative of P is zero. Hence,

$$\begin{aligned}
\frac{\dot{P}}{P}(x_1, x_2, x_3) &= \dot{\log P} = \frac{d}{dt}(A \log x_1 + B \log x_2 + C \log x_3 + \log(1-(1+\sigma)x_3)) \\
&= A\frac{\dot{x}_1}{x_1} + B\frac{\dot{x}_2}{x_2} + C\frac{\dot{x}_3}{x_3} - \frac{(1+\sigma)\dot{x}_3}{1-(1+\sigma)x_3} \\
&= \ -[1+A+B+C](x_3(1+\sigma)(\delta x_1 - \kappa x_2)) + \\
&\quad + [-Ax_2 + \delta\sigma A x_3 + Bx_1 - \kappa\sigma B x_3 + \delta C x_1 - \kappa C x_2] \\
&= 0
\end{aligned}$$

where we get zero in the last equality since

$$\begin{aligned}
1 + A + B + C =& 1 + \frac{\kappa}{\theta} + \frac{\delta}{\theta} - \frac{1}{\theta} \\
=& \frac{1 - \omega + \omega k\varepsilon + \omega\varepsilon - 1 + \omega - \omega k\varepsilon - \omega\varepsilon}{\theta} = 0
\end{aligned}$$

and

$$\begin{aligned}
-Ax_2 + \delta\sigma A x_3 + Bx_1 - &\kappa\sigma B x_3 + \delta C x_1 - \kappa C x_2 \\
&= -\frac{\kappa}{\theta}x_2 + \frac{\delta\sigma\kappa}{\theta}x_3 + \frac{\delta}{\theta}x_1 - \frac{\kappa\sigma\delta}{\theta}x_3 - \frac{\delta}{\theta}x_1 + \frac{\kappa}{\theta}x_2 = 0.
\end{aligned}$$

We can conclude that P is constant along the orbits of the system we wrote down at the beginning of this solution. Therefore, the level sets of the function P describe the shape of the flow in Δ.

4 The Best Response Dynamics

4.1 Rock-Scissor-Paper game and some other examples

Exercise 4.1:
a) Consider the matrix

$$A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$

and the Lyapunov function $V(x) = \max_i (Ax)_i$. As illustrated in Example 4.2 in the Lecture Notes, the simplex Δ can be divided into three regions over which the Best Response is single-valued

$$\begin{aligned} \Xi_1 &= \{x \in \Delta \mid \mathcal{BR}(x) = \{e_1\}\} \setminus (Z_{1,2} \cup Z_{1,3}) \\ \Xi_2 &= \{x \in \Delta \mid \mathcal{BR}(x) = \{e_2\}\} \setminus (Z_{1,2} \cup Z_{2,3}) \\ \Xi_3 &= \{x \in \Delta \mid \mathcal{BR}(x) = \{e_3\}\} \setminus (Z_{1,3} \cup Z_{2,3}). \end{aligned}$$

Figure 22 in the Lecture Notes reports the level set $\{x \mid V(x) = 0\}$ which is given by the union of three segments (in light blue, in the right simplex of Figure 22). We will now show that those light blue segment are segments of lines in Δ. The blue segments are given by imposing $V(x) = 0$ in the three regions Ξ_1, Ξ_2, and Ξ_3. Indeed, in Ξ_1 we have that $V|_{\Xi_1}(x) = (Ax)_1 = 6x_2 - 4x_3$, hence the blue segment is given by

$$L_1 = \{6x_2 - 4x_3 = 0\} \cap \Xi_1,$$

where $\{6x_2 - 4x_3 = 0\}$ can be see as a plane in $\mathbb{R}^3$ intersecting the simplex Δ. Note that $e_1 \in \{6x_2 - 4x_3 = 0\}$. Following the same reasoning we get that in Ξ_2 the blue segment is given by

$$L_2 = \{-3x_1 + 5x_3 = 0\} \cap \Xi_2;$$

whereas in Ξ_3 it is given by

$$L_3 = \{-x_1 + 3x_2 = 0\} \cap \Xi_3.$$

Once again notice that $e_2 \in \{-3x_1 + 5x_3 = 0\}$, and $e_3 \in \{-x_1 + 3x_2 = 0\}$. We can see L_1, L_2, and L_3 as three lines in Δ going through e_1, e_2, and e_3 respectively, restricted to the appropriate regions where $\mathcal{BR}$ is single-valued.

b) The next step is to show that L_1, L_2, and L_3 are invariant under the flow in Ξ_1, Ξ_2, and Ξ_3, respectively. In order to see this we will show that the flow $\dot{x}$ restricted to L_i has the same direction as the line L_i. For example in Ξ_1 we have

$$\dot{x}|_{L_1} = \mathcal{BR}(x) - x|_{L_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 - \frac{5}{2}x_2 \\ x_2 \\ \frac{3}{2}x_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2}x_2 \\ -x_2 \\ -\frac{3}{2}x_2 \end{pmatrix}$$

and the normal vector to the plane $\{6x_2 - 4x_3 = 0\}$ is given by $\hat{n}_1 = \left(\begin{smallmatrix} 0 \\ 6 \\ -4 \end{smallmatrix}\right)$. By taking the dot product we see that

$$\dot{x}|_{L_1} \cdot \hat{n}_1 = -6x_2 + 4\frac{3}{2}x_2 = 0$$

which means that the flow $\dot{x}$ along L_1 has no normal component to L_1, therefore L_1 is invariant under the flow in Ξ_1. We can carry out similar calculations in Ξ_2, and Ξ_3. Indeed,

$$\dot{x}|_{L_2} = \mathcal{BR}(x) - x|_{L_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} x_1 \\ 1 - \frac{8}{5}x_1 \\ \frac{3}{5}x_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ \frac{8}{5}x_1 \\ -\frac{3}{5}x_1 \end{pmatrix}$$

and the normal to the plane $\{-3x_1 + 5x_3 = 0\}$ is given by

$$\hat{n}_2 = \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix}.$$

As before, the dot product between these two vectors is zero

$$\dot{x}|_{L_2} \cdot \hat{n}_2 = 3x_1 - 5\frac{3}{5}x_1 = 0.$$

Finally, in Ξ_3 we have

$$\dot{x}|_{L_3} = \mathcal{BR}(x) - x|_{L_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3x_2 \\ x_2 \\ 1 - 4x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ -x_2 \\ 4x_2 \end{pmatrix}$$

and the normal to the plane $\{-x_1 + 3x_2 = 0\}$ is given by

$$\hat{n}_3 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}.$$

As before, the dot product between these two vectors is zero

$$\dot{x}|_{L_3} \cdot \hat{n}_3 = 3x_2 - 3x_2 = 0.$$

We can conclude that the segment L_i in Ξ_i is invariant under the Best Response dynamics.

c) We will now turn our attention to the derivative of the function V. Let $x \in \Xi_i$ then we have that $\mathcal{BR}(x) = \{e_i\}$ hence

$$\dot{V}(x) = e_i \cdot A\dot{x} = e_i \cdot (\mathcal{BR}(x) - x) = e_i \cdot Ae_i - e_i \cdot Ax = A_{ii} - V(x) = -V(x)$$

since $A_{ii} = 0$. Hence $\dot{V} = -V$ in the regions where the Best Response is single-valued and constant.

4)The ODE we have just computed $\dot{V}(x) = -V(x)$, only holds for $x \in \bigcup_i \Xi$. This equation, together with IVP $V(x(0)) = V(x_0)$ for $x_0 \in \Xi_i$ tells that $V(x) = V(x_0)e^{-t}$, as long as $x \in \bigcup_i \Xi_i$, which seems to suggest that the solution to this ODE tends to zero exponentially fast. Unfortunately the solutions will have to cross the indifference, and along such lines the equality $\dot{V} = -V$ will not hold. For example, along $Z_{2,3}$ we have that $\dot{V}(x) = \frac{d}{dt} \max_i Ax_i = e_2 \cdot A\dot{x} = e_2 \cdot A(\mathcal{BR}(x) - x)$ since $(Ax)_2 = (Ax_3)$ and they are maximal. We immediately see that now $\dot{V}$ is multi-valued, and therefore we cannot have $\dot{V} = -V$.

In conclusion V decays to 0 exponentially fast, as long as it does not hit an indifference line. Unfortunately, V tends to 0 as t tends to infinity, and a crossing of an indifference line is unavoidable.

4.2 Two player best response dynamics

Exercise 4.2:
We are now interested in the Best Response dynamics for two players. We consider the matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the system of (possibly multivalued) ODEs

$$\begin{aligned} \dot{x} &= \mathcal{BR}_A(y) - x \\ \dot{y} &= \mathcal{BR}_B(x) - y. \end{aligned}$$

In Example 4.3 we proved that this system has a unique NE in the interior of Δ, namely $E = (E^A, E^B) = \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right)$, and that the Lyapunov function $V(x, y) = \mathcal{BR}_A(y) \cdot Ay + x \cdot B\mathcal{BR}_B(x)$ is such that $V(x, y) \geq V(E^A, E^B) = 0$. Let us adopt the second convention. By a simple computation we can see that

$$\mathcal{BR}_A(y) = \begin{cases} \{e_2\} & \text{if } y_1 > y_2 \\ \Delta & \text{if } y_1 = y_2 = \frac{1}{2} \\ \{e_1\} & \text{if } y_1 < y_2 \end{cases} \qquad \mathcal{BR}_B(x) = \begin{cases} \{e_1\} & \text{if } x_1 > x_2 \\ \Delta & \text{if } x_1 = x_2 = \frac{1}{2} \\ \{e_2\} & \text{if } x_1 < x_2 \end{cases}$$

which immediately tells us that E is the only NE in general. Notice that $V(x, y) = 0$ if and only if (x, y) is a NE for (A, B). We know that if we supplement the ODE $\dot{V} = -V$ with the initial condition $(x(0), y(0)) = (x_0, y_0)$, we get the solution $V(x(t), y(t)) = e^{-t}V(x_0, y_0)$. Assuming that our initial condition is not E, then $V(x_0, y_0) > 0$, which means that $V(x(t), y(t))$ reaches 0 as $t \to \infty$.

In Example 4.1, we find ourselves in a similar situation as we just described. Again, we have that $V(x) > V(E)$ for $x \in \Delta \setminus \{E\}$, and that $\dot{V} = -V$, hence $V(x(t)) = e^{-t}V(x_0)$. The main difference is that now $V(E) = \frac{a-b}{3}$, so if we assume that $a > b$ then $V(E) > 0$. This means that if we assume $x_0 \neq E$, our solution starting at x_0 reaches E in finite time $t = \ln\left(\frac{V(x_0)}{V(E)}\right) < \infty$. If $a = b$

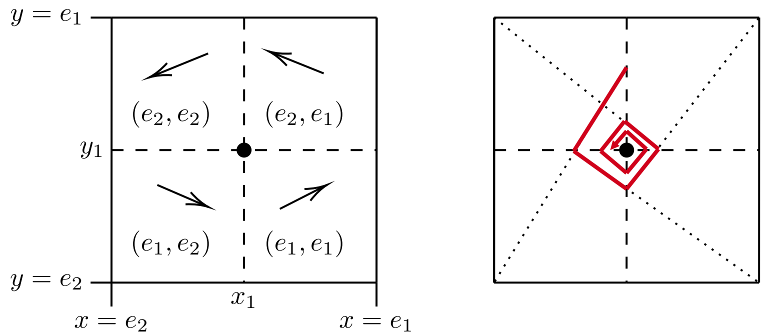

Figure 22: In the left square we reported the direction of the flow, and the values of the Best Response map in the regions where it is constant. Notice we plotted x_1 on the horizontal axis, against y_1 on the vertical one. In the right square we plotted the flow converging to the equilibrium $(\frac{1}{2}, \frac{1}{2})$.

then the solution converges to E as time tends to infinity. If $a < b$ we have the appearance of the Shapley triangle to which our solution converges to, as explained in Example 4.1.

As we have just seen, the flow associated to the Best Response dynamic for the game described by A and B tends towards to $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$ as showed in Figure 22. The velocity of the flow does not go to zero! We will denote the four regions in the squares in Figure 22 using cardinal directions: starting from the top right region, and moving anticlockwise we have North-West (NW), South-West (SW), South-East (SE), and North-East (NE). Consider the NW region $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, here the Best Response function is single valued and equals (e_2, e_2) which leads to the system

$$\dot{x} = \begin{pmatrix} -x_1 \\ 1 - x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_1 \end{pmatrix}$$
$$\dot{y} = \begin{pmatrix} -y_1 \\ 1 - y_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_1 \end{pmatrix}.$$

Hence the velocity v_{NW} in this quadrant is given by

$$|v_{\mathrm{NW}}|^2 = |\dot{x}|^2 + |\dot{y}|^2 = 2x_1^2 + 2y_1^2$$

which tends to 1 as the flow tends to its equilibrium. This velocity is always strictly positive in this quadrant.

Similar we obtain the velocities

$$v_{\mathrm{SW}} = 2x_2^2 + 2y_1^2$$
$$v_{\mathrm{SE}} = 2x_2^2 + 2y_2^2$$
$$v_{\mathrm{NE}} = 2x_1^2 + 2y_2^2$$

which are always strictly greater than zero if our flow starts in the interior of the phases space, and all tend to 1 as the solution tends to its equilibrium.

4.3 Convergence and non-convergence to Nash Equilibrium for Best Response Dynamics

Exercise 4.3:
1) Let A be 3×3 matrix, let $\alpha, \beta, \gamma \in \mathbb{R}$, and $c > 0$. We will show that for any $y \in \Delta$, then $\mathcal{BR}_{A'}(y) = \mathcal{BR}_A(y)$ where

$$A' := cA + \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix}.$$

This statement can be proved through a direct calculation

$$\begin{aligned}
\mathcal{BR}_{A'}(y) &= \arg\max_{x \in \Delta} x \cdot A'y = \arg\max_{x \in \Delta} x \cdot \left(cA + \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix} \right) y \\
&= \arg\max_{x \in \Delta} \left[x \cdot (cA)y + x \cdot \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix} y \right] \\
&= \arg\max[c(x \cdot Ay) + (\alpha y_1 + \beta y_2 + \gamma y_3)(x_1 + x_2 + x_3)] \\
&= c \arg\max_{x \in \Delta}(x \cdot Ay) + \alpha y_1 + \beta y_2 + \gamma y_3 \\
&= \mathcal{BR}_A(y),
\end{aligned}$$

as we claimed.

2) Consider the two matrices

$$A = \begin{pmatrix} 1 & 0 & \beta \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -\beta & 1 & 0 \\ 0 & -\beta & 1 \\ 1 & 0 & -\beta \end{pmatrix}$$

we will show that we for $\beta = \phi = \frac{\sqrt{5}-1}{2}$, the reciprocal of the golden ratio, the matrix B can be rescaled to give a zero-sum game. A quick remark on the chosen value for β

$$\phi^2 = \frac{6 - 2\sqrt{5}}{4} = 1 + \frac{1 - \sqrt{5}}{2} = 1 - \phi. \tag{9}$$

Consider the matrix

$$\tilde{B} = \phi \left(B - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) = \begin{pmatrix} -\phi(\beta+1) & 0 & -\phi \\ -\phi & -\phi(\beta+1) & 0 \\ 0 & -\phi & -\phi(\beta+1) \end{pmatrix}$$

which by the first part of this question we know yields the same Best Response (and therefore Best Response dynamics) as B. Clearly we have

$$A + \tilde{B} = \begin{pmatrix} 1 - \phi(\beta+1) & 0 & \beta - \phi \\ \beta - \phi & 1 - \phi(\beta+1) & 0 \\ 0 & \beta - \phi & 1 - \phi(\beta+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

since $\beta = \phi$, and $1 - \phi(1+\beta) = 1 - \phi - \phi^2 = 0$ by Equation 9.

3) This solution is adapted from the lecture notes. Take the Shapley periodic orbit $\gamma : \mathbb{R} \to \Delta \times \Delta \subset \mathbb{R}^6$ of the Best Response dynamics associated to the two player Rock–Paper–Scissors game corresponding to $\beta = 0$. Note that $\mathcal{BR}_A(e_i) = e_i$ and $\mathcal{BR}_B(e_i) = e_{i+1}$ (note that we using the 2^{nd} notation for the matrices). Let π_A, π_B be the projections of $\Delta_A \times \Delta_B \subset \mathbb{R}^6$ onto the two triangles shown in Figure 23. The blue triangles drawn in this figure correspond

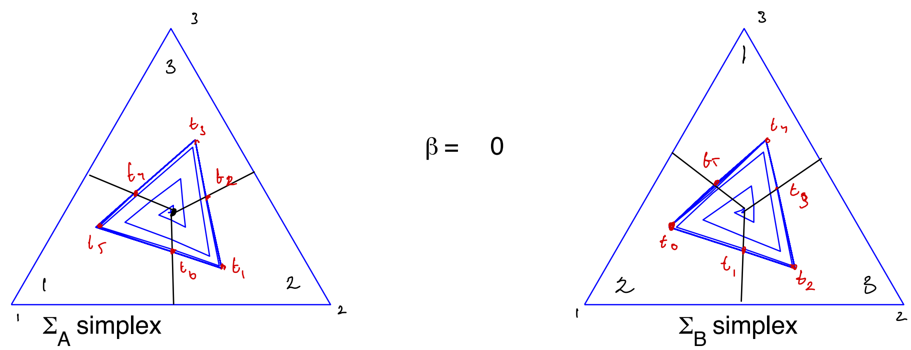

Figure 23: The projection of the Shapley's periodic orbit γ onto the two simplices Δ_A and Δ_B. The numbers inside the triangle Δ_A denote to corner to which player B will be heading, and the number inside the triangle Δ_B where player A is heading.

to the projections $\pi_A(\gamma)$ and $\pi_B(\gamma)$ of γ. Let T be the period of γ and let $0 = t_0 < t_1 < \cdots < t_5 < t_6 = T$ be the times when $\pi_A(\gamma(t))$ or $\pi_B(\gamma(t))$ are contained in one of the indifference lines. When $\pi_A(\gamma(t))$ lies in an indifference line at $t = t'$, then $t \mapsto \pi_B(\gamma(t))$ changes from moving towards one corner for $t < t'$ close to t' to moving towards another corner for $t > t'$ close to t'. In fact, for each t we have that $\gamma(t)$ intersects at most one of the indifference lines. The points $\pi_A(\gamma(t_i))$ and $\pi_B(\gamma(t_i))$ are indicated in the figure, and note that the points move anti-clockwise in the triangles and head towards e_i in Δ_A when $\gamma(t)$ is in the region in Δ_B marked with i (and vice versa). The curve γ is a solution of the piece-wise smooth ODE:

$$\dot{\gamma}(t) = \begin{pmatrix} e_i \\ e_j \end{pmatrix} - \gamma(t) \text{ for } t \in (t_i, t_{i+1}).$$

Here $\left(\begin{smallmatrix} e_i \\ e_j \end{smallmatrix}\right)$ are best response choices. i.e., $e_i = \mathcal{BR}_A(\pi_B(\gamma(t))$ and $e_j = \mathcal{BR}_B(\pi_A(\gamma(t))$. The solution of this ODE is

$$\gamma(t) = (1 - e^{-t}) \begin{pmatrix} e_i \\ e_j \end{pmatrix} + e^{-t}\gamma(0) \text{ for } t \in (t_i, t_{i+1}). \tag{10}$$

So $(t_i, t_{i+1}) \ni t \mapsto \gamma(t)$ is a straight line in $\mathbb{R}^6$ (which is contained in $\Delta_A \times \Delta_B$). Let us take a closer look at what these lines: the best response choices depend on time in the following way

$$\begin{pmatrix} e_i \\ e_j \end{pmatrix} = \begin{cases} (e_2, e_2)^\mathsf{T} & \text{if } t \in (t_0, t_1) \\ (e_3, e_2)^\mathsf{T} & \text{if } t \in (t_1, t_2) \\ (e_3, e_3)^\mathsf{T} & \text{if } t \in (t_2, t_3) \\ (e_1, e_3)^\mathsf{T} & \text{if } t \in (t_3, t_4) \\ (e_1, e_1)^\mathsf{T} & \text{if } t \in (t_4, t_5) \\ (e_2, e_1)^\mathsf{T} & \text{if } t \in (t_5, t_0). \end{cases}$$

We will now see that all these vectors describe a regular hexagon. First of all we know that γ is composed of six sides and that the length of the first two sides is given by

$$\gamma(t_1) - \gamma(t_0) = \lambda((e_2, e_2) - \gamma(t_0))$$

and

$$\gamma(t_2) - \gamma(t_1) = \lambda((e_3, e_2) - \gamma(t_1)),$$

where λ is as in the notes. Hence, using the formulas for $\gamma(t_i)$

$$||\gamma(t_1) - \gamma(t_0)|| = \lambda||(e_2, e_2) - \gamma(t_0)|| = \lambda C||(\theta^3, \theta^3 - C, \theta, \theta^4, 1 - C, \theta^2)||$$

and

$$||\gamma(t_2) - \gamma(t_1)|| = \lambda||(e_3, e_2) - \gamma(t_1)|| = \lambda C||(\theta^2, \theta^4, 1 - C, \theta^3, \theta^3 - C, \theta)||$$

and this implies that

$$||\gamma(t_1) - \gamma(t_0)|| = ||\gamma(t_2) - \gamma(t_1)||$$

and therefore by symmetry

$$||\gamma(t_{i+1}) - \gamma(t_i)|| = ||\gamma(t_1) - \gamma(t_0)||$$

for all i.

5 Fictitious Play: a Learning Model

5.1 Best response and fictitious play

Exercise 5.1:
1) Consider the fictitious play dynamics

$$\dot{p}(s) = \frac{1}{s}(\mathcal{BR}_A(q(s)) - p(s))$$
$$\dot{q}(s) = \frac{1}{s}(\mathcal{BR}_B(p(s)) - q(s))$$

where the quantities p, and q depend on the previously played strategies, as explained in the lecture notes. This is a non-autonomous systems of ODEs (notice the factor $\frac{1}{s}$ in the RHS!), but this can be reduced to an autonomous system if we substitute $s = e^t$. Indeed, if we define $\tilde{p}(t) = p(e^t)$, and $\tilde{q}(t) = q(e^t)$ then by imposing $s = e^t$ we get the following system of ODEs

$$\dot{\tilde{p}}(t) = e^t \dot{p}(e^t) = e^t \frac{1}{e^t}(\mathcal{BR}_A(q(e^t)) - p(e^t)) = \mathcal{BR}_A(\tilde{q}(t)) - \tilde{p}(t)$$
$$\dot{\tilde{q}}(t) = e^t \dot{q}(e^t) = e^t \frac{1}{e^t}(\mathcal{BR}_B(p(e^t)) - q(e^t)) = \mathcal{BR}_B(\tilde{p}(t)) - \tilde{q}(t)$$

which describes the Best Response dynamics. If we consider the Shapley dynamics of Example 4.5 then we have that the orbits of the fictitious play $(p(t), q(t))$ are the same as the orbits of the best response $(\tilde{p}(t), \tilde{q}(t))$, since we have only smoothly reparameterised time.

2) For $\beta = 0$ we know that the Shapley Best Response dynamics shows the presence of closed periodic orbits, i.e.

$$(\tilde{p}(t), \tilde{q}(t)) = (\tilde{p}(t + T), \tilde{q}(t + T)) \tag{11}$$

where T represents the period, or the amount of time needed to complete a full lap of the periodic orbit. Since the fictitious play can be seen as a reparameterisation of the Best Response dynamics then the system present the same closed periodic orbit, but its characterisation is slightly different. The speed along such orbit decreases, which means that it will take us longer, and longer to complete a full lap of the orbit. Indeed, instead of having a characterisation like in Equation 11, we have

$$(\tilde{p}(t), \tilde{q}(t)) = (\tilde{p}(t + T), \tilde{q}(t + T))$$
$$\iff (p(e^t), q(e^t)) = (p(e^{t+T}), q(e^{t+T}))$$
$$\iff (p(s), q(s)) = (p(e^T s), q(e^T s))$$

which exactly tells us that it will take us exponentially longer to complete a full lap compared to the precedent lap.

5.2 The no–regret set

Exercise 5.2:
Try to understand how to the CCE set from Chapter 5 and the CE set from Chapter 7 are related. Why do we say that CCE $\subseteq$ CE? What probability distribution would you be more inclined to follow and why?

5.3 Fictitious play converges to the no–regret set CCE

Exercise 5.3:
Open ended question.

5.4 FP orbits often give better payoff than Nash

Exercise 5.4:
1) Consider the two matrices

$$A = A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = B_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

giving rise to the Shapley system for the Best Response dynamics. As we have seen in Exercise 5.1, the periodic orbit (or Shapley triangles) under the $\mathcal{BR}$ dynamics that was analysed in Example 4.5 in the lecture notes corresponds to a closed orbit γ for the FP dynamics (note that γ is periodic under the BR dynamics). By the very own definition of closed orbit, if the initial condition for our FP dynamics is along such an orbit, then the flow is constrained there. Therefore the limits of the points $\gamma(t)$ lie in Shapley's periodic orbit. By Theorem 5.1, the probability distribution

$$p_{ij}(t) = \frac{1}{t}\int_0^t x_i(s)y_j(s)ds$$

converges to the CCE set.

From previous analyses, we know that the point $(\hat{p}, \hat{q}) = \left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$ is a Nash Equilibrium for the system. By Lemma 5.1 we know that the every element in the set of Nash Equilibria corresponds an element in the CCE set. Therefore, the CCE set for the Shapley system with $\beta = 0$ is composed of at least two elements.

The last thing we are left to check is that, indeed, these two probability matrices are different. The element in the CCE set given by the NE equilibrium $(\hat{p}, \hat{q})$ is simply

$$\hat{P} = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9}. \end{pmatrix}$$

We will now show that some of the entries of the probability matrix $P(t)$ associated to the Shapley orbit of the FP dynamics have value different from $\frac{1}{9}$.

We will use the same notation and some of the results in part 3 of Exercise 4.3. Let $\gamma(t) = (p(t), q(t))$ be the Shapley periodic orbit, and suppose that to complete a lap of this periodic orbit takes e^T time, i.e. $\gamma(t_0) = (p(t_0), q(t_0)) = (p(e^T t_0), q(e^T t_0)) = \gamma(e^T t_0)$.

One approach to this is to analyse the strategies played in terms of the FP dynamics. Recall that we the strategies played under the FP dynamics are given by

$$\begin{aligned} x(t) &\in \mathcal{BR}_A(q(t)) \\ y(t) &\in \mathcal{BR}_B(p(t) \end{aligned}$$

and notice that since we are restricting our attention to a flow over the closed orbit γ then we have that the Best Response function is piece-wise constant outside of 6 points on γ (the corners of the hexagon) and it is equal to

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{cases} (e_2, e_2)^\mathsf{T} & \text{if } t \in (t_0, t_1) \\ (e_3, e_2)^\mathsf{T} & \text{if } t \in (t_1, t_2) \\ (e_3, e_3)^\mathsf{T} & \text{if } t \in (t_2, t_3) \\ (e_1, e_3)^\mathsf{T} & \text{if } t \in (t_3, t_4) \\ (e_1, e_1)^\mathsf{T} & \text{if } t \in (t_4, t_5) \\ (e_2, e_1)^\mathsf{T} & \text{if } t \in (t_5, t_0) \end{cases}.$$

We are now seeing the orbit γ as a closed periodic orbit of period T. If we now want to compute the entry $p_{1,3}$ of the CCE distribution P given by γ we have

$$p_{1,2}(T) = \frac{1}{T} \int_0^T x_1(t) y_2(t)\, dt = 0$$

since there exists no time where both the 1st entry of $x(t)$ and the 2nd entry of $y(t)$ are simultaneously non-zero. This comes from the very specific form of the Best Response function along the periodic orbit: if $x(t) = e_i$ then $y(t)$ can only be equal to e_i or e_{i-1}, and never e_{i+1} (the indexes are to be take $\mod 3$).

Another approach is to note that from

$$p_{ij}(t) = \frac{1}{t} \int_0^t x_i(s) y_j(s) ds$$

it follows that $\sum_j p_{ij}(t) = \frac{1}{t}\int_0^t x_i(s) = p(t)$ and $\sum_i p_{ij}(t) = \frac{1}{t}\int_0^t y_j(s) = q(t)$ where $p(t)$ and $q(t)$ are so that $\gamma(t) = (p(t), q(t))$. It follows that the marginals of the probability matrix $P(t)$ do not converge as $t \to \infty$.

2) Suppose that our solution for the FP dynamics associated to the Shapley system is bound to the periodic orbit we precedently discussed. By Proposition 5.1 we know that

$$\begin{aligned} \lim_{t\to\infty} \hat{u}^A(T) - \max_{\overline{p}\in\Delta} \overline{p} \cdot Aq(T) = 0 \\ \lim_{t\to\infty} \hat{u}^B(T) - \max_{\overline{q}\in\Delta} p(T) \cdot B\overline{q} = 0. \end{aligned}$$

The orbit $(p(s), q(s))$ is constrained along the Shapley triangle, and from Exercise 5.1 we know that $(p(s), q(s)) = (p(e^K s), q(e^K s))$, for $K > 0$. This means that the quantities $\max_{\overline{p}\in\Delta} \overline{p} \cdot Aq(T)$ and $\max_{\overline{q}\in\Delta} p(T) \cdot B\overline{q}$ just need to be computed over one lap of the flow along the Shapley periodic orbit instead for all times. Therefore

$$\lim_{T\to\infty} \max_{\overline{p}\in\Delta} \overline{p} \cdot Aq(T) = \max_{T\in[t_0, e^K t_0]} \max_{\overline{p}\in\Delta} \overline{p} \cdot Aq(T) = c_A$$
$$\lim_{T\to\infty} \max_{\overline{q}\in\Delta} p(T) \cdot B\overline{q} = \max_{T\in[t_0, e^K t_0]} \max_{\overline{q}\in\Delta} p(T) \cdot B\overline{q} = c_B$$

by compactness. We can conclude that the average payoffs for FP dynamics flows along the Shapley periodic orbit converge

$$\lim_{t\to\infty} \hat{u}^A = c_A \qquad \lim_{t\to\infty} \hat{u}^B = c_B.$$

5.5 Discrete fictitious dynamics

Exercise 5.5:
Try to adapt the algorithms in Appendix B for the discrete FP dynamics of the Shapley system.

6 Reinforcement Learning

6.1 Set-up of reinforcement learning

Exercise 6.1:
1) As explained in the exercise let a doctor be prescribing either a Placebo or a Medicine to patients of type I or II. Suppose that the matrix describing this scenario is given by

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{I} & \mathrm{II} \end{array} \\ \begin{array}{c} \mathrm{M} \\ \mathrm{P} \end{array} & \begin{pmatrix} 10 & 0 \\ 5 & 5 \end{pmatrix} \end{array}.$$

Let q be the probability of a patient being of type I and $1-q$ of type II. Then assuming that the doctor prescribes M with probability p and P with probability $1-p$ the payoff is

$$\begin{pmatrix} p \\ 1-p \end{pmatrix} \cdot \begin{pmatrix} 10 & 0 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix} \cdot \begin{pmatrix} 10q \\ 5 \end{pmatrix}$$

So if $q < 1/2$ then taking $p = 0$ (i.e. prescribing a placebo) gives the best payoff whereas if $q > 1/2$ then taking $p = 1$ (i.e. prescribing the medicine) gives the best payoff.

2) The Python code and graphs indicating what outcomes to expect when the doctor uses the Erev-Roth model to determine which medication to prescribe can be found in the appendix of the lecture notes.

6.2 The Arthur model in the 2×2 setting

Exercise 6.2:
1) Let us start with the coordination game first. We wish to show that the set of singularities $S = \{(0,0),(0,1),(1,0),(1,1),(\theta_1,\theta_2)\} \subset [0,1] \times [0,1]$ is internally chain recurrent. The dynamics in this case is given by the system of replicator equations

$$\begin{aligned} \dot{x} &= x(1-x)[\alpha_1 - y(\alpha_1 + \alpha_2)] \\ \dot{y} &= y(1-y)[\beta_1 - x(\beta_1 + \beta_2)] \end{aligned}$$

and the point $\theta = (\theta_1, \theta_2) = \left(\frac{\beta_1}{\beta_1+\beta_2}, \frac{\alpha_1}{\alpha_1+\alpha_2}\right)$ is an internal Nash Equilibrium. Let us denote by $\phi_t = (x(t), y(t))$ the flow for this system. Notice that for any point $s \in S$ we have that $\dot{x}|_s = \dot{y}|_s = 0$. This means that the set S is composed of fixed points, i.e. $\phi_t(s) = s$ for all $t \geq 0$. Notice this makes S automatically invariant.

We will now show that if s is a fixed point, then s is chain recurrent. This means that for all $\delta, T > 0$ there exists a (δ, T)-pseudo orbit of ϕ_t connecting s to itself. Fix $\delta, T > 0$ then let $x_0 = s$ and $t_0 = T + 1$ then we simply have that $\phi_{t_0}(x_0) = \phi_{T+1}(s) = s$, hence

$$t_0 > T \qquad \text{and} \qquad d(\phi_{t_0}(x_0), x_0) = 0 < \delta,$$

as we wanted. Thus, every point in S is chain recurrent, making S internally chain recurrent.

Let us now move to the second case. We will now generalise what we have just showed. In the classification of 2×2 games this second system we are considering is a zero sum case with an internal Nash Equilibrium. In this case we are asked to show that the entire phase space $M = [0,1] \times [0,1]$ is internally chain recurrent. The main observation that has to be made here is that the phase space is foliated by periodic orbits of ϕ_t: for any $x \in M$ there exists a time $t(x)$, called period, such that $\phi_{t(x)}(x) = x$. We will denote the periodic orbit passing through x as $\gamma_x = \phi_{[0,t(x)]}(x)$. If $x \in \partial([0,1] \times [0,1])$ then $\gamma_x = \partial([0,1] \times [0,1])$. Notice that the existence of these periodic orbits is given by the last part of section 3 of Exercise 2.4.

We will now show that any point belonging to a periodic orbit is chain recurrent. Let $x \in M$, and fix $\delta, T > 0$. We will now define $x_0 = x$ and if $t(x) < T$ then we will let $t_0 = T + (t(x) - k)$ where $k = T \mod t(x)$ (clearly $k, T, t(x) \in \mathbb{R}_{\geq 0}$), otherwise $t_0 = 2t(x)$: in both cases $t_0 > T$, and $t(x)$ divides t_0. Therefore we have that

$$t_0 \geq T \qquad \text{and } d(\phi_{t_0}(x_0, x_0) = d(\phi_{lt(x)}(x), x) = d(x, x) = 0 < \delta$$

where $l = \frac{T}{t(x)} \in \mathbb{Z}$. We can conclude that every point in M is chain recurrent, and therefore M is internally chain recurrent.

2) Recall the assumption of Proposition 6.1: $a_{ij}, b_{i1j>C>0}$ for all i, j. First of all we want to show that there exists $\alpha, \alpha' > 1$ such that

$$1 - \frac{\alpha}{t} < 1 - \frac{a_{1j}}{Ct + a_{1j}} < 1 - \frac{\alpha'}{t}$$

for t big enough. We can rewrite the two inequalities as

$$\frac{\alpha'}{t} < \frac{a_{1j}}{Ct + a_{1j}} < \frac{\alpha}{t}.$$

We will prove them in order, left to right.

By assumption we know that $a_{ij} > C$ for all i, j, hence define $\varepsilon := \min_{i,j} a_{ij} - C > 0$. For any $\delta \in (0, \varepsilon)$ notice the following

$$\frac{a_{1j}}{C + \delta} > \frac{a_{1j}}{C + \varepsilon} = \frac{a_{1j}}{\min_{i,j} a_{i,j}} \geq 1,$$

for any j. Now fix a positive δ smaller than ε, then there exists a time t_0 such that $\frac{\max_{i,j} a_{ij}}{t} \leq \delta$ for all $t \geq t_0$. Therefore

$$\frac{a_{1j}}{Ct + a_{1j}} = \left(\frac{a_{1j}}{C + a_{1j}/t}\right)\frac{1}{t} \geq \left(\frac{a_{1j}}{C + \max_{i,j} a_{i,j}/t}\right)\frac{1}{t} \geq \left(\frac{a_{1j}}{C + \delta}\right)\frac{1}{t} \geq \left(\frac{\min_{i,j} a_{ij}}{C + \delta}\right)\frac{1}{t}$$

for $t \geq t_0$, and if we define $\alpha' := \frac{\min_{i,j} a_{i,j}}{C+\delta} > 1$ we can conclude that

$$\frac{a_{1j}}{Ct + a_{1j}} \geq \frac{\alpha'}{t}.$$

Notice that we could tweak the denominator in the definition of α' in order to get a sharp inequality above.

For the second inequality it is enough to remember that $a_{1j} > 0$, and that $\frac{a_{ij}}{C} > 1$ for all i, j. Indeed if we let $\alpha = \frac{\max_{i,j} a_{ij}}{C} > 1$ we obtain

$$\frac{a_{1j}}{Ct + a_{1j}} < \frac{a_{1j}}{Ct} \leq \left(\frac{\max_{i,j} a_{ij}}{C}\right)\frac{1}{t} = \frac{\alpha}{t}.$$

We can therefore conclude that for $t \geq t_0$

$$1 - \frac{\alpha}{t} < 1 - \frac{a_{1j}}{Ct + a_{1j}} < \frac{\alpha'}{t},$$

as we wanted. Recall that we define the sequence $(d^t)_t$ as $d^{t+1} = \left(\frac{a_{1j}}{Ct+a_{1j}}\right) d^t$, we will now mimic this construction. Define the sequence $(\tilde{d}^t)_t$ as

$$\begin{aligned} \tilde{d}^1 &:= d^1 \\ \tilde{d}^{t+1} &:= \left(1 - \frac{\alpha'}{t}\right)\tilde{d}^t \qquad \text{for all } t. \end{aligned}$$

Clearly

$$\frac{d^{t+1}}{d^t} = 1 - \frac{a_{1j}}{Ct + a_{1j}} < 1 - \frac{\alpha'}{t} = \frac{\tilde{d}^{t+1}}{\tilde{d}^t} \tag{12}$$

for all $t \geq t_0$. Similarly we can compare the two sequences: if $d^{t_0} \leq \tilde{d}^{t_0}$ then we automatically have that $d^t \leq \tilde{d}^t$ for all $t \geq t_0$, if, on the other hand, $d^{t_0} > \tilde{d}^{t_0}$ then there exists a time $\tau \geq t_0$ for which $d^t \leq \tilde{d}^t$ for all $t \geq \tau$ thanks to Inequality 12. Therefore, if we can show that the series $\sum_{t=1}^{\infty} \tilde{d}^t$ converges, then this will give us control over the two series tails $\sum_{t=t_0}^{\infty} d^t$ and $\sum_{t=\tau}^{\infty} d^t$ which will tell us that $\sum_{t=1}^{\infty} d^t$ converges.

In order to show that $\sum_{t=1}^{\infty} d^t$ converges we will use the Raabe test. In order for such test to work we need to check its two conditions.

1. $\lim_{t\to\infty} \left|\frac{\tilde{d}^t}{\tilde{d}^{t+1}}\right| = \lim_{t\to\infty} \frac{1}{1-\frac{\alpha'}{t}} = 1$, as we need;
2. $\lim_{t\to\infty} t\left(\left|\frac{\tilde{d}^t}{\tilde{d}^{t+1}}\right|\right) = \lim_{t\to\infty} \frac{\alpha' t}{t-\alpha'} = \alpha' > 1$;

since the second limit tends to a finite value bigger than 1 we can conclude that by the Raabe test the series $\sum_{t=1}^{\infty} \tilde{d}^t$ converges. Therefore, the series $\sum_{t=1}^{\infty} d^t$ converges, and this concludes our proof.

6.3 The Erev-Roth model

Exercise 6.3:
1) Consider the system of differential equations

$$\begin{aligned}
\dot{p}_i &= \frac{p_i}{a(t)}[(Aq)_i - p \cdot Aq] \\
\dot{q}_j &= \frac{q_j}{b(t)}[(Bq)_j - q \cdot Bp] \\
\dot{a} &= -a + p \cdot Aq \\
\dot{b} &= -b + q \cdot Bp
\end{aligned}$$

and we wish to study the singularities of it. Assume that p and q are n dimensional probability vectors. Firstly notice that in order to have a singularity we need

$$\begin{aligned}
a &= p \cdot Aq \\
b &= q \cdot Bp
\end{aligned}$$

where a, b are constant (since $\dot{a} = \dot{b} = 0$ for all times). In order to avoid complicated behaviours which will requite a much more in depth analysis we will assume that a and b are nonzero. The other $2n$ ODEs are always zero whenever

$$\begin{aligned}
\dot{p}_i = 0 &\Longleftrightarrow \begin{cases} p_i = 0 \\ (Ap)_i = p \cdot Aq \quad \text{if } p_i \neq 0 \end{cases} \\
\dot{q}_j = 0 &\Longleftrightarrow \begin{cases} q_j = 0 \\ (Aq)_j = q \cdot Bp \quad \text{if } q_j \neq 0 \end{cases}
\end{aligned}$$

for all $i, j = 1, 2, \ldots, n$. Since p and q are probability vectors, there exists at least one entry p_i and one entry q_j which are nonzero, meaning that at least for those indexes $(Ap)_i = p \cdot Aq = a$ and $(Bq)_j = q \cdot Bp = b$.

To conclude the singularities of the aforementioned system of ODEs are given by the simultaneous system of equations

$$\begin{aligned}
(Ap)_i &= p \cdot Aq \\
(Bq)_j &= q \cdot Bp \\
a &= p \cdot Aq \\
b &= q \cdot Bp.
\end{aligned}$$

for all i such that $p_i \neq 0$ and for all j such that $q_j \neq 0$. Unfortunately we are not able to produce anything more insightful about the location of the singularities for such a general system.

2) As we have seen before, approaching this problem from a purely theoretical

point of view will not take us far. A computational approach to this type of problems is usually preferred. Try to play around with different matrices and see how different singularities can arise in different places. If you use 3×3 matrices you can use the visualisation code you developed in Exercise 2.5 to understand where the singularities lie in the phase space Δ.

6.6 Q-Learning with softmax

Exercise 6.4:
For some more background on how these picture have been obtained we wish to redirect you to the paper *Frequency Adjusted Multi-agent Q-learning* by Michael Kaisers and Karl Tuyls (In Proc. of 9th Intl. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2010), pp.309–315).

7 No Regret Learning

7.1 The correlated equilibrium (CE) set

Exercise 7.1:
1) Let (p, q) be a Nash Equilibrium for a game determined by the matrices (A, B). If we define our probability distribution matrix $S = (s_{ij}) = p_i q_j$, then in order to check if this is a Correlated Equilibrium it is just a matter of working through the definition of CE, that we will now rewrite in vector form. In order to check if S is a CE we would need to show

$$\sum_k a_{i'k} s_{ik} = \sum_k a_{i'k} p_i q_k = p_i (Aq)_{i'} \leq p_i (Aq)_i = \sum_k a_{ik} p_i q_k = \sum_k a_{ik} s_{ik}$$
$$\sum_k b_{kj'} s_{kj} = \sum_k b_{kj'} p_k q_j = q_j (p^\mathsf{T} B)_{j'} \leq q_j (p^\mathsf{T} B)_j = \sum_k b_{kj} p_k q_j = \sum_k b_{kj} s_{kj}$$

From the remark in the solution of Exercise 2.1.2 we know that for a Nash Equilibrium (p, q) and constants $c, c' \in \mathbb{R}$ we have $(Aq)_i = c$ for all i such that $p_i \neq 0$, and symmetrically $(p^\mathsf{T} B)_j = c'$ for all j for which $q_j \neq 0$. Notice that $p \cdot (Aq) = c$ and that $q \cdot (p^\mathsf{T} B) = c'$ since p, q are probability vectors, and by the aforementioned property.

Now if $p_i = 0$ then we trivially have $p_i (Aq)_{i'} = 0 = p_i (Aq)_i$, and similarly for $q_j = 0$. Hence assume that $p_i, q_j \neq 0$. Then

$$p_i (Aq)_{i'} = p_i (e_{i'} \cdot Aq) \leq p_i (p \cdot Aq) = p_i c = p_i (Aq)_i$$

by the definition of (p, q) being a Nash Equilibrium. Similarly

$$q_j (p^\mathsf{T} B)_{j'} = q_j (B^\mathsf{T} p \cdot e_{j'}) = q_j (p \cdot B e_{j'}) \leq q_j (p \cdot Bq) = q_j c' = q_j (p^\mathsf{T} B)_j$$

as we claimed, where the inequality follows again by (p, q) is a Nash Equilibrium.

2) Consider the battle of the sexes game (coordination game) with bimatrix

$$\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}.$$

Recall that in Exercise 2.4 we analysed all the possible phase diagrams for 2×2 replicator games. Using the same notation as in Exercise 2.4 we know that

$$\alpha_1 = -1 \qquad \alpha_2 = -2 \qquad \beta_1 = -2 \qquad \beta_2 = -1$$

which does confirm that we are considering a coordination game since $\alpha_1 \alpha_2 > 0$, $\beta_1 \beta_2 > 0$, and $\alpha_1 \beta_1 > 0$. The Mixed Nash Equilibrium is represented on $I^2 = [0,1] \times [0,1]$ by $\theta = \left(\frac{\beta_1}{\beta_1 + \beta_2}, \frac{\alpha_1}{\alpha_1 + \alpha_2} \right) = \left(\frac{2}{3}, \frac{1}{3} \right)$, hence in the phase space $\Delta_A \times \Delta_B$ it corresponds to $(p^A, p^B) = \left(\left(\frac{2}{3}, \frac{1}{3} \right), \left(\frac{1}{3}, \frac{2}{3} \right) \right)$. The same question immediately tells us that the other two (Pure) Nash Equilibria are given by $((0,1),(0,1)) = (F, F)$, which corresponds to payoff $(1,2)$, and by $((1,0),(1,0)) = (T, T)$ which corresponds to the payoff $(2,1)$.

Let us break down the precedent bimatrix into two matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

and let P be a joint distribution. Therefore, we have

$$\begin{aligned}
2p_{21} = a_{11}p_{21} + a_{12}p_{22} = \sum_k a_{1k}p_{2k} \le \sum_k a_{2k}p_{2k} = a_{21}p_{21} + a_{22}p_{22} = p_{22} \\
p_{12} = a_{21}p_{11} + a_{22}p_{12} = \sum_k a_{2k}p_{1k} \le \sum_k a_{1k}p_{1k} = a_{11}p_{11} + a_{12}p_{12} = 2p_{11} \\
2p_{21} = b_{12}p_{11} + b_{22}p_{21} = \sum_k b_{k2}p_{k1} \le \sum_k b_{k1}p_{k1} = b_{11}p_{11} + b_{21}p_{21} = p_{11} \\
p_{12} = b_{11}p_{12} + b_{21}p_{22} = \sum_k b_{k1}p_{k2} \le \sum_k b_{k2}p_{k2} = b_{12}p_{12} + b_{22}p_{22} = 2p_{22}.
\end{aligned}$$

Since $2p_{21} \le p_{22}$ and $2p_{21} \le p_{11}$ if follows that $p_{21} \le \frac{1}{2}\min(p_{11}, p_{22})$, and similarly since $p_{12} \le 2p_{11}$ and $p_{12} \le 2p_{22}$, it follows that $p_{12} \le 2\min(p_{11}, p_{22})$.

Suppose that the joint distribution P is induced by one of the Nash Equilibria, then we want to numerically show that P is a Correlated Equilibrium. For example if we consider (p^A, p^B) then we have $p_{11} = p_1^A p_1^B = \frac{2}{9}$, $p_{12} = p_1^A p_2^B = \frac{4}{9}$, $p_{21} = p_2^A p_1^B = \frac{1}{9}$, $p_{22} = p_2^A p_2^B = \frac{2}{9}$. Clearly all the inequalities are respected

$$\begin{aligned}
\frac{1}{9} = p_{21} \le \frac{1}{2}\min(p_{11}, p_{22}) = \frac{1}{2}\frac{2}{9} = \frac{1}{9} \\
\frac{4}{9} = p_{12} \le 2\min(p_{11}, p_{22}) = 2\frac{2}{9} = \frac{4}{9}
\end{aligned}$$

hence the distribution induced by (p^A, p^B) is a CE.

If we now consider the Nash Equilibrium $(T, T) = ((1, 0), (1, 0))$, we have

$$p_{11} = 1, \qquad p_{12} = 0, \qquad p_{21} = 0, \qquad p_{22} = 0$$

which means that since $\min(p_{11}, p_{22}) = p_{22} = 0$, and $0 = p_{12} = p_{21} \le \frac{1}{2}\min(p_{11}, p_{22}) = 0$ the probability distribution P induced by this Nash Equilibrium is a CE.

Finally, the last Nash Equilibrium we are left to check is $(F, F) = ((0, 1), (0, 1))$ which gives us the following entries for P

$$p_{11} = 0, \qquad p_{12} = 0, \qquad p_{21} = 0, \qquad p_{22} = 1.$$

Following the same argument as for (T, T) we have that the probability distribution induced by (F, F) is a CE, as we expected.

Therefore all the three probability distributions corresponding to the Nash Equilibria for this game are in the CE set. If we denote by $P_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ the distribution induced by the NE (T, T) and by $P_2\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ the one induced by the NE (F, F) then we can see that for any $\sigma \in [0, 1]$ the matrix

$$\tilde{P} = \sigma P_1 + (1 - \sigma)P_2 = \begin{pmatrix} \sigma & 0 \\ 0 & 1 - \sigma \end{pmatrix}$$

is in the CE set, by simply checking the inequalities we have derived before. We can conclude that the CE set is infinite. Clearly if $\sigma = \frac{1}{2}$ then $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ an element of the CE set.

Consider the joint probability distribution described by $\tilde{P} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. The expected payoff for the first player is given by summing up all the possible payoffs multiplied by the probability that such payoffs are achieved. Therefore

$$\mathbb{E}(\text{Payoff}_A|\tilde{P}) = 2\frac{1}{2} + 0 + 0 + 1\frac{1}{2} = \frac{3}{2}$$

since $((1,0),(1,0))$ is played with probability $\frac{1}{2}$ and it has payoff 2, whereas $((0,1),(0,1))$ is played with probability $\frac{1}{2}$ with payoff 1. A similar computation gives us that $\mathbb{E}(\text{Payoff}_B|\tilde{P}) = 1\frac{1}{2} + 2\frac{1}{2} = \frac{3}{2}$. The expected payoff for the distribution $\tilde{P}$ is $(\frac{3}{2}, \frac{3}{2})$.

The expected payoff for the Mixed Nash Equilibrium θ which induces a probability distribution $P_\theta = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{2}{9} \end{pmatrix}$ is given by

$$\mathbb{E}(\text{Payoff}_A|P_\theta) = 2\frac{2}{9} + 0\frac{4}{9} + 0\frac{1}{9} + 1\frac{2}{9} = \frac{2}{3}$$
$$\mathbb{E}(\text{Payoff}_B|P_\theta) = 1\frac{2}{9} + 0\frac{4}{9} + 0\frac{1}{9} + 2\frac{2}{9} = \frac{2}{3}$$

so this Nash Equilibrium is outperformed by $\tilde{P}$.

Since $a_{12} = a_{21} = b_{12} = b_{21} = 0$, in order to maximise the expected payoff we need to look at probability distributions with shape $Q = \begin{pmatrix} \sigma & 0 \\ 0 & 1-\sigma \end{pmatrix}$ where $\sigma \in [0,1]$. If $\sigma = 1$ then $\mathbb{E}(\text{Payoff}_A|Q) = \mathbb{E}(\text{Payoff}_A|P_1)$ is maximised (and the total expected payoff is $(2,1)$), whereas for $\sigma = 0$ then $\mathbb{E}(\text{Payoff}_B|Q) = \mathbb{E}(\text{Payoff}_B|P_2)$ is maximised (and the expected payoff is $(1,2)$). In general, the expected payoff is given by $(2\sigma + (1-\sigma), \sigma + 2(1-\sigma)) = (\sigma+1, 2-\sigma)$, which means that every σ gives a Pareto optimal expected payoff: any improvement to one of the player's payoff will negatively affect the other player payoff.

Next we want to show that playing the mixed Nash Equilibrium truly leads to the worst payoff. Consider a probability matrix of the form

$$\hat{P} = \begin{pmatrix} \frac{2}{9} + \varepsilon_1 & \frac{4}{9} + \varepsilon_2 \\ \frac{1}{9} + \varepsilon_3 & \frac{2}{9} + \varepsilon_4 \end{pmatrix}$$

where $\sum_i \varepsilon_i = 0$ and

$$-\frac{4}{9} \leq \varepsilon_2 \leq 2\min(\varepsilon_1, \varepsilon_4)$$
$$-\frac{1}{9} \leq \varepsilon_3 \leq \frac{1}{2}\min(\varepsilon_1, \varepsilon_4)$$

since we want $\hat{P}$ to be a CE. Notice that ε_1 and ε_4 cannot be both negative, or that would imply that ε_2 and ε_3 are negative as well, contradicting the assumption that $\sum_i \varepsilon_i = 0$. If we turn our attention to the expected payoffs given $\hat{P}$ we will see that these amount to

$$\mathbb{E}(\text{Payoff}_A|\hat{P}) = \frac{2}{3} + 2\varepsilon_1 + \varepsilon_4$$
$$\mathbb{E}(\text{Payoff}_A|\hat{P}) = \frac{2}{3} + \varepsilon_1 + 2\varepsilon_4.$$

If both $\varepsilon_1, \varepsilon_4$ are positive then the expected payoff given $\hat{P}$ is higher than the one expected given P_θ. If we instead assume that ε_1 is positive, and ε_4 is negative then $\varepsilon_1 = -\varepsilon_2 - \varepsilon_2 - \varepsilon_3 > 0$ and

$$\mathbb{E}(\text{Payoff}_A|\hat{P}) = \frac{2}{3} + 2\varepsilon_1 + \varepsilon_4 = \frac{2}{3} - 2\varepsilon_1 - 2\varepsilon_2 - \varepsilon_4 > \frac{2}{3}$$
$$\mathbb{E}(\text{Payoff}_A|\hat{P}) = \frac{2}{3} + \varepsilon_1 + 2\varepsilon_4 = \frac{2}{3} - \varepsilon_1 - \varepsilon_2 + \varepsilon_4 > \frac{2}{3} - 2\varepsilon_4 - \frac{1}{2}\varepsilon_4 + \varepsilon_4 > \frac{2}{3},$$

which confirms that this would give a better payoff than playing according to P_θ. Notice that assuming ε_1 negative, and ε_4 positive leads to the same result given the symmetry in the coefficients of ε_1 and ε_4 in the functions $\mathbb{E}(\text{Payoff}_A|\hat{P})$, and $\mathbb{E}(\text{Payoff}_A|\hat{P}$. We can conclude that playing according to P_θ is always leading to the worse payoff.

Given the Pareto optimality for $\tilde{P}$ and the payoffs for $\hat{P}$, we can conclude that whenever the intermediator recommends playing according to a distribution P in the CE set for which p_{12} and p_{21} are not simultaneously zero, then the payoffs are not maximised. Indeed, if the intermediator recommends strategies such as (T, F) or (F, T) then no players will see any benefit in terms of payoff to such strategy. On the other hand, if they recommend to play according to matrices like $\tilde{P}$ then is clear that the intermediator in this game will always favour one particular player over the other (unless $\sigma = \frac{1}{2}$ then both players will be treated in the same way and get the same payoff).

3) Consider the game of chicken described by a bimatrix

$$\begin{pmatrix} (6,6) & (2,7) \\ (7,2) & (0,0) \end{pmatrix}$$

which can be decomposed into

$$A = \begin{pmatrix} 6 & 2 \\ 7 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & 7 \\ 2 & 0 \end{pmatrix}.$$

As we did for the previous exercise, we can determine the phase portrait of this two players 2×2 game by looking at the coefficients α, β as defined in Exercise 2.4. If we adopt the second convention, we have

$$\alpha_1 = 2, \qquad \alpha_2 = 1, \qquad \beta_1 = 2, \qquad \beta_2 = 1$$

which translates to $\alpha_1\alpha_2 > 0$, $\beta_1\beta_2 > 0$, and $\alpha_1\beta_1 > 0$. Therefore, the game of chicken is a coordination game. We have three Nash Equilibria: two pure, and one mixed. The mixed Nash Equilibria is given by $\theta = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$, whereas the two Pure Nash Equilibria $(D, C) = ((0,1), (1,0))$, and $(C, D) = ((1,0), (0,1))$.

As before we want to show some CE inequalities. These are obtain through the following computations

$$\begin{aligned}
6p_{21} + 2p_{22} &= \sum_k a_{1k} p_{2k} \leq \sum_k a_{2k} p_{2k} = 7p_{21} \\
7p_{11} &= \sum_k a_{2k} p_{1k} \leq \sum_k a_{1k} p_{1k} = 6p_{11} + 2p_{12} \\
6p_{12} + 2p_{22} &= \sum b_{k1} p_{k2} \leq \sum_k b_{k2} p_{k2} = 7p_{12} \\
7p_{11} &= \sum_k b_{k2} p_{k1} \leq \sum_k b_{k1} p_{k1} = 6p_{11} + 2p_{21}.
\end{aligned}$$

These inequalities can be rewritten as

$$p_{22} \leq \frac{1}{2} \min(p_{12}, p_{21}) \qquad p_{11} \leq 2 \min(p_{12}, p_{21}).$$

If we consider the probability distribution $P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$ then we immediately see that

$$\begin{aligned}
0 = p_{22} &< \frac{1}{2} \min(p_{12}, p_{21}) = \frac{1}{6} \\
\frac{1}{3} = p_{11} &< 2 \min(p_{12}, p_{21}) = \frac{2}{3}
\end{aligned}$$

so P is in the Correlated Equilibrium set. The expected payoff when playing $P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$ is given by

$$\begin{aligned}
\mathbb{E}(\text{Payoff}_A | P) &= \frac{1}{3} 6 + \frac{1}{3} 2 + \frac{1}{3} 7 = 5 \\
\mathbb{E}(\text{Payoff}_B | P) &= \frac{1}{3} 7 + \frac{1}{3} 2 + \frac{1}{3} 6 = 5.
\end{aligned}$$

On the other hand, the probability distribution induced by the Nash Equilibrium $(D, C) = ((0,1), (1,0))$ is given by $P_1 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ with payoff $(7, 2)$, and similarly the probability distribution induced by the Nash Equilibrium $(C, D) = ((1,0), (0,1))$ is given by $P_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ with payoff $(2, 7)$. The Mixed Nash Equilibrium $\theta = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$ induces a probability distribution $P_\theta = \begin{pmatrix} \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} \end{pmatrix}$ with payoff $\left(\frac{14}{3}, \frac{14}{3}\right)$. Playing according to P allows for a higher payoff for both players than playing P_θ, and this Correlated Equilibrium does not advantage a player over the other.

Indeed, when playing P the trusted intermediator is recommending both players to play strategies that generate positive payoff with equal probabilities. The intermediator is not siding with any player (the expected payoff is the same for both parties) and they are only strongly discouraging the two players to play (D,D) given its $(0,0)$ payoff.

7.2 Hart and Mas-Colell's regret matching

Exercise 7.2:
Consider the battle of the sexes game with bimatrix

$$\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}$$

where the first action corresponds to watching Football, and the second one to watching Tennis. Figure 15 in the notes represents the behaviour of the four functions $\mathrm{DIFF}^t_A(F,T), \mathrm{DIFF}^t_A(T,F), \mathrm{REGRET}^t_A(F,T)$, and $\mathrm{REGRET}^t_A(T,F)$ With f^t reported along the x-axis, and the values of the functions along the y-axis . All these functions intersect at $f^t = \frac{2}{3}$ (and they are all zero at that point). Recall that $f^t = f^t_B(T)$, so we have that $\begin{pmatrix} f^t_B(F) \\ f^t_B(T) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$. This corresponds to the second component of the the interior Nash Equilibrium of this game, namely $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$. Whenever Player 2 chooses to play a mixed strategy where the proportion of times they play T or F is determined by the internal Nash Equilibrium of B, then player 1 has no preferred way of replying to the strategies of player 2, since $\mathrm{REGRET}^t_A(F,T) = 0$, and $\mathrm{REGRET}(T,F)^t_A(T,F) = 0$. This is in accordance with the idea that the Nash Equilibrium is somewhat optimal for player 2.

We now want to show that for $t \geq 1$ we have $|f^{t+1} - f^t| < \frac{1}{t}$. By definition we see that

$$\begin{aligned} f^{t+1} &= \frac{1}{t+1} \#\{1 \leq i \leq t+1 \,|\, y^i = T\} \\ &\leq \frac{1}{t+1}(\#\{1 \leq i \leq t \,|\, y^i = T\} + 1) = \frac{t}{t+1} f^t + \frac{1}{t+1} \end{aligned}$$

and similarly

$$\begin{aligned} f^{t+1} &= \frac{1}{t+1} \#\{1 \leq i \leq t+1 \,|\, y^i = T\} \\ &\geq \frac{1}{t+1} \#\{1 \leq i \leq t \,|\, y^i = T\} = \frac{t}{t+1} f^t \end{aligned}$$

therefore

$$f^{t+1} - f^t \leq \frac{t}{t+1} f^t + \frac{1}{t+1} - f^t = \frac{1}{t+1} - \frac{1}{t+1} f^t \leq \frac{1}{t+1}$$

and

$$f^{t+1} - f^t \geq \frac{t}{t+1} f^t - f^t = -\frac{f^t}{t+1} \geq -\frac{1}{t+1}$$

since $f^t \in [0,1]$. We can rewrite the last inequalities more compactly as

$$\left| f^{t+1} - f^t \right| \leq \frac{1}{t+1} < \frac{1}{t}.$$

Notice this condition is too weak: it does not allow us to say that the sequence of frequencies $(f^t)_t$ tend to any limit. Indeed, it can happen that the frequency arbitrarily oscillates between 0 and 1. This has an impact on no-regret algorithm. Such algorithm is based on looking at the probabilities p_j^{t+1} and $p_{j^*}^{t+1}$ in order to make a decide on what is the best move to play, and in our case we can see that these probabilities depend on the frequencies f^t (REGRET depends on f^t). Since $(f^t)_t$ does not need to converge, neither do the sequences $(p_j^t)_t$ and $(p_{j^*}^t)_t$. This means that we do not have to reach a point in our game where playing one specific strategy will be the answer to minimising regret. This situation might appear seem quite bleak, but there is an upside to this whole situation. By Hart and Mas-Colell Theorem (Theorem 7.1 in the Lecture Notes) we know that if a player follows the no-regret algorithm, then they will (almost surely) asymptotically get zero regret for their moves, this means that even if the probabilities p_j^{t+1} and $p_{j^*}^{t+1}$ can vary depending on the how the second player decides to play, if one sticks to what they recommend, then it is (almost) always possible to get very little regret on the long run. One can think of this algorithm as dynamically adapting itself with respect to the second player choice of strategies.

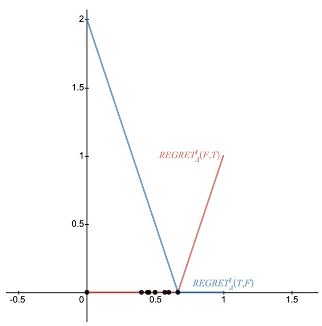

Figure 24: Player A regret when Player B chooses strategy T at prime times (11 time steps).

Let us take a look at a more numerical examples, hoping that it will make the whole discussion clearer. Suppose that player B plays strategy T at prime times then the first few rounds of this game will look something like

Time t	1	2	3	4	5	6	7	8	9	10	11	...
Strategy$_B$	F	T	T	F	T	F	T	F	F	F	T	...
Frequency f^t	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{4}{9}$	$\frac{2}{5}$	$\frac{5}{11}$	...
Recom. strategy$_A$	F	F	T/F	F	F	F	F	F	F	F	F	...

Table 4: No-Regret algorithm for when player B plays T at prime times.

What is happening in this case is that player B always plays F, and occasionally they play strategy T, at random (since it is not possible to predict when the next prime number will appear, for time large enough). The no-regret algorithm does not get thrown off from this sporadic appearance of strategy T from player B, but instead it immediately updates the probabilities $p_{j^*}^{t+1}$ and p_j^{t+1} so that player A can minimise regret after what could be considered an unexpected action.

We have just analysed what happens when player B mostly sticks to one strategy, and plays the other one at random times. We can similarly look at the case where player B sticks for a long time to playing one strategy, and at some random time switches to the other strategy, and keeps playing it for an even longer amount of time. One way to model this is to consider an increasing sequence of real numbers $(n_i)_i$ tending to infinity such that $n_{i+1} \geq n_i^2$, and then letting player B play strategy F (resp. T) when i is even (resp. odd) for times $e^{n_i}, \ldots, e^{n_{i+1}-1}$. Notice that

$$\lim_{n\to\infty} \frac{e^{n_{i+1}} - e^{n_i}}{e^{n_i}} = \lim_{n\to\infty} e^{n_{i+1}-n_i} - 1 \geq \lim_{n\to\infty} e^{n_i^2 - n_i} - 1 = +\infty$$

which roughly tells us that the amount of times B will play a different strategy from before is quite considerable (strategies are not sporadically play as before). The same argument as before holds even in this case: the no-regret algorithm adapts so that the regret is going to tend to zero almost surely.

As a final remark, please notice that the Hart and Mas-Colell algorithm does not take into account the payoff for the second player, it is only interested in the payoff of the player following the algorithm. As explained before, following this algorithm will almost surely minimise the regret connected your to choices, this algorithm does not try to minimise the payoff of your opponent, or to maximise their regret.

7.3 Min-max solutions and zero-sum games

Exercise 7.2:
1) Consider the zero-sum game described by the bimatrix

$$G = \begin{pmatrix} (4,-4) & (-2,2) \\ (-5,5) & (6,-6) \end{pmatrix}.$$

and in order to simplify notation, we will denote the first strategy by T and the second one by F. The expected payoffs for Alice (player 1) against Bob (player 2) whenever she plays a random strategy with probability $(p, 1-p)$ is

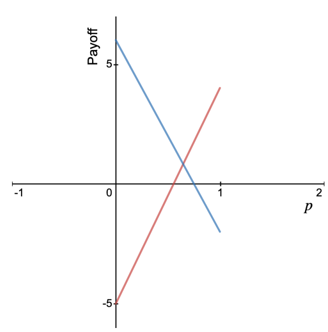

Figure 25: Expected Payoff for Alice playing against Bob 1$^{\text{st}}$ strategy (in red) and and 2$^{\text{nd}}$ strategy (in blue).

given by $\mathbb{E}(\text{Payoff}_A \mid \text{Bob plays strategy 1}) = \mathbb{E}(\text{Payoff}_A \mid (\cdot, T)) = 9p - 5$ and $\mathbb{E}(\text{Payoff}_A \mid \text{Bob plays strategy 2}) = \mathbb{E}(\text{Payoff}_A \mid (\cdot, F)) = 6 - 8p$. These two functions are represented in Figure 25, and we have that they meet at the point $p = \frac{11}{17}$, where Alice would expect a payoff of $\frac{14}{17}$ against Bob. This payoff is independent from Bob's choice of strategy. We can repeat the same analysis for Bob. Suppose that Bob plays strategy 1 with probability q, and strategy 2 with probability $1 - q$ then their expected payoffs are given by $\mathbb{E}(\text{Payoff}_B|(T, \cdot)) = -4q+2(1-q) = 2-6q$ if Alice plays T, and $\mathbb{E}(\text{Payoff}_B|(F, \cdot)) = 5q+-6(1-q) = 11q - 6$ if Alice plays F. The two expected payoffs are given by two lines meeting at $q = \frac{8}{17}$. As before, if $q = \frac{8}{17}$, then Bob can expect a payoff of $-\frac{14}{17}$, independently from Alice's strategy.

2) Consider the zero-sum game associated to the matrix

$$A = \begin{pmatrix} 4 & 1 & -4 \\ 3 & 2 & 5 \\ 0 & 1 & 7 \end{pmatrix}$$

where we assume the second convention. As explained in the text of the exercise, the entry $a_{22} = 2$ is a *saddle-point* of the game since it is the biggest entry in its column, and the smallest in its row. This property implies that the pure strategy (e_2, e_2) is a Nash Equilibrium.

Recall that in the Lecture Notes you studied that a point $(\hat{x}, \hat{y})$ is a Nash Equilibrium for a zero sum game defined by a matrix A if

$$\min_y \hat{x} \cdot Ay = v = \max_x x \cdot A\hat{y}.$$

Given a saddle point a_{ij} then the product $e_i \cdot Ay$ is equal to $(e_i^\intercal A)y$ and $e_i^\intercal A$ corresponds to the i^{th} row of A, and similarly if we look at the product $x \cdot Ae_j$ then Ae_j corresponds to the j^{th} column of A. Since we have assumed that a_{ij} is the smallest element in its row, then $\min_y e_i^\intercal Ay = a_{ij}$, and given that we assumed that it was also the biggest entry in its column then $\max_x x \cdot Ay = a_{ij}$, which by the result we recalled from the lecture notes exactly means that (e_i, e_j) is a Nash Equilibrium.

Let us look at a more concrete example. For the matrix A reported above we have

$$\min_y e_2 \cdot Ay = \min_y \begin{pmatrix} 3, & 2, & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 2$$

$$\max_x x \cdot Ae_2 = \max_x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 2$$

which confirms that (e_2, e_2) is a Nash Equilibrium for the zero-sum two player games given by the matrices $(A, -A)$.

7.4 Another way of thinking of the min-max theorem

Exercise 7.4:
1) We wish to prove the following proposition.

Proposition 7.1. *Given a function* $A : \Delta \times \Delta \to \mathbb{R}$ *then* $\min_p \max_q A(p, q) = \max_q \min_p A(p, q)$ *if and only if for any* v *in the image of* A

i) $\exists p, \forall q$ *such that* $A(p, q) \leq v$

ii) $\forall q, \exists p$ *such that* $A(p, q) \leq v$.

We will prove the two directions of the "if and only if" implication.

$\Longleftarrow$) In order to reach a contradiction, suppose that the min-max equality $\min_p \max_q A(p, q) = \max_q \min_p A(p, q)$ does not hold, whilst $i)$ and $ii)$ are equivalent (either both true or both false). Notice that

$$\min_{p(q)} A(p, q) \leq A(p, q) \leq \max_{q(p)} A(p, q)$$

where the LHS is independent of p (and the RHS is independent of q). By minimising p throughout the inequality we get

$$\min_{p(q)} A(p, q) \leq \min_p \max_{q(p)} A(p, q)$$

and if we now maximise q we have

$$\max_q \min_{p(q)} A(p, q) \leq \min_p \max_{q(p)} A(p, q).$$

Since we have assumed that the min-max equality does not hold, we have the strict inequality

$$\max_q \min_{p(q)} A(p, q) < \min_p \max_{q(p)} A(p, q).$$

and therefore there exists $v \in \mathbb{R}$ so that

$$\max_q \min_{p(q)} A(p, q) < v < \min_p \max_{q(p)} A(p, q). \tag{13}$$

Since $\max_q \min_{p(q)} A(p, q) < v$, this means that for all q, there exists p so that $A(p, q) < v$. Since p, q are taken from a compact domain, there exists $v' < v$ ($v' := \max_q \min_{p(q)} A(p, q)$) such that for all q there exists a p for which $A(p, q) \leq v'$. At the beginning of this proof we assumed that $i)$ and $ii)$ are equivalent, and we have just showed that $ii)$ holds, therefore $i)$ has to hold as well: there exists a p such that $A(p, q) \leq v'$ for all q. This last statement is equivalent to $\min_p \max_{q(p)} A(p, q) \leq v' < v$, which contradicts inequality 13.

$\Longrightarrow$) This implication is reached by noticing that minimisation corresponds to the existential quantifier and maximisation corresponds to the universal quantifier.

We now want to show that if we assume $\min_p \max_q A(p,q) = \max_q \min_p A(p,q)$ then both conditions $i)$ and $ii)$ are either both true or both false, i.e. they are equivalent. In order to reach a contradiction, suppose there exists $v \in \mathbb{R}$ so that $i)$ holds, and $ii)$ does not. The first condition being true yields $\min_p \max_{q(p)} A(p,q) \leq v$. On the other hand, we have (by negating $ii)$): $\exists q^*$ such that $\forall p$ we have $A(p,q^*) > v$. Hence $\min_p A(p,q^*) > v$, which implies

$$\max_q \min_{p(q)} A(p,q) \geq \min_p A(p,q^*) > v.$$

By applying the min-max equality we end up with

$$v \geq \min_p \max_{q(p)} A(p,q) = \max_q \min_{p(q)} A(p,q) > v$$

which is a contradiction.

Let us now assume there exists $v \in \mathbb{R}$ so that $ii)$ holds and that $i)$ does not. Now $ii)$ yields $\max_q \min_{p(q)} A(p,q) \leq v$. The negation of $i)$ translates to $\forall p$ then $\exists q^*$ such that $A(p,q^*) > v$, which means that $\max_{q(p)} A(p,q) > v$ for any p, and in particular $\min_p \max_{q(p)} A(p,q) > v$. As before, this leads to a contradiction.

We can conclude that if the min-max equality holds then either $i)$ and $ii)$ are either both true or both false.

2) Consider the function

$$\begin{aligned} A : [0,1] \times [0,1] &\to [0,1] \\ (p,q) &\mapsto pq \end{aligned}$$

and we want to show $\max_q \min_p A(p,q) = \min_p \max_q A(p,q)$. For such a simple function this is quite straightforwards: the minimum of pq for either p or q is always 0, independently from the value of the other variable (this is only true since we work over $[0,1] \times [0,1]$. Therefore $\max_q \min_p A(p,q) = \min_p \max_q A(p,q) = 0$.

We wish to show that the equivalent statement of the min-max theorem stated at the beginning of this question holds for this toy model. Since A maps onto $[0,1]$ then choosing $v < 0$ makes no sense. Fix $v \geq 0$, then fix $p \leq v$ then we have that for all $q \in [0,1]$

$$A(p,q) = pq \leq p \leq v \qquad\qquad \text{since } q \leq 1, \text{ and } p \leq v$$

therefore $i)$ holds. Similarly, for $v \geq 0$ fixed and any $q \in [0,1]$ then choose $p \geq v$ then

$$A(p,q) = pq \leq p \leq v \qquad\qquad \text{since } q \leq 1, \text{ and } p \leq v$$

hence $ii)$ holds as well, as we expected.

3) Consider now the function

$$A : [0,1] \times [0,1] \to \mathbb{R}$$
$$(p,q) \mapsto \begin{cases} p+q & \text{if } p+q \leq 1 \\ 2-(p+q) & \text{otherwise.} \end{cases}$$

Firstly we can see that

$$A(p,q) \leq \begin{cases} 1 & \text{if } p+q \leq 1 \\ 2-1 & \text{if } p+q \geq 1 \end{cases} = 1.$$

hence $\max_q A(p,q) = 1$ (equality is reached, for example, when $q = 1-p$ for any $p \in [0,1]$).

Similarly, if we try to minimise $A(p,q)$ for p we get

$$A(p,q) \geq \begin{cases} 0+q & \text{if } 0+q \leq 1 \\ 2-(1+q) & \text{if } 1+q \geq 1 \end{cases} = \begin{cases} q & \text{if } q \leq 1 \\ 1-q & \text{if } q > 0 \end{cases}$$

where the first function is minimised for $p = 0$, and the second one for $p = 1$. Therefore, we can write more compactly $\min_p A(p,q) = \min(q, 1-q)$. Now the minmax theorem does not hold anymore since

$$\min_p \max_q A(p,q) = \min_p(1) = 1$$
$$\max_q \min_p A(p,q) = \max_q \min(q, 1-q) = \frac{1}{2}.$$

Some convexity and concavity properties on the function $A(p,q)$ are required in order for the minmax theorem to hold. Let M, and N be two subsets of a topological vector spaces $\mathfrak{U}$, and $\mathfrak{V}$ (these are just vector spaces equipped with a topology so that vector addition and scalar multiplication are continuous with respect to the chosen topology). Assume the scalar field to be either $\mathbb{R}$ or $\mathbb{C}$ equipped with the Euclidean (or Standard) topology.

Definition 7.1. A function f on $M \times N$ is *quasi-concave* in N if $\{y \mid f(x,y) \geq c\}$ is a convex set for any $x \in M$ and $c \in \mathbb{R}$. Similarly, a function f on $M \times N$ is *quasi-convex* in M if $\{x \mid f(x,y) \leq c\}$ is a convex set for any $y \in N$ and $c \in \mathbb{R}$.

Theorem 7.1 (Sion's Minmax Theorem [2])**.** *Let M be a compact convex subset of $\mathfrak{U}$, and let N be a subset of $\mathfrak{V}$. If f is a real-valued function on $M \times N$ with*

- *$f(x,\cdot)$ upper semicontinuous and quasi-concave on N, $\forall x \in M$;*
- *$f(\cdot,y)$ lower semicontinuous and quasi-convex on M, $\forall y \in N$;*

then

$$\min_M \sup_N f(x,y) = \sup_N \min_M f(x,y).$$

[2] Sion, M. (1958) On general Minmax Theorems. *Pacific Journal of Mathematics.* 8(1), 171-176.

7.5 A vectored valued payoff game

Exercise 7.5: 1) Let Δ represent our canonical simplex and consider a function

$$A : \Delta \times \Delta \to \mathbb{R}^k.$$

Let $\mathcal{C}$ be a convex subset of $\mathbb{R}^k$, such that for each $q \in \Delta$ there exists a $p \in \Delta$ such that $A(p, q) \in \mathcal{C}$. In the question we are asked if we can infer that there exists a $p \in \Delta$ such that for all $q \in \Delta$ we have $A(p, q) \in \mathcal{C}$. As the hint suggests, this is not true if the range has dimension greater or equal to 2. Let us provide a counterexample.

Let $\Delta = [0, 1]$, consider the function

$$\begin{aligned} A : \Delta \times \Delta &\to [0, 1]^2 \subset \mathbb{R}^2 \\ (p, q) &\mapsto (p, q) \end{aligned}$$

and let $\mathcal{C}$ be the diagonal of $[0, 1]^2$, i.e. $\mathcal{C} := \{(x, x) \mid x \in [0, 1]\}$. Since $\mathcal{C}$ is an interval it is automatically convex. In this case, for any $q \in [0, 1]$ set $p := q \in [0, 1]$ then $A(p, q) = (p, q) = (q, q) \in \mathcal{C}$. Unfortunately, there exists no p for which $A(p, q) \in \mathcal{C}$ for all q. Suppose such a p existed, then take $q = p + \frac{1}{4} \mod 1$. In this situation $A(p, q) = (p, q) = (p, p + \frac{1}{4} \mod 1)$ which is clearly not contained in $\mathcal{C}$. You can think of our choice of q as a vertical translation of $\mathcal{C}$ by $\frac{1}{4}$ in the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

2) Now, let $\Delta = [0, 1]$ and consider the function

$$\begin{aligned} A : \Delta \times \Delta &\to \mathbb{R} \\ (p, q) &\mapsto pq. \end{aligned}$$

We will say that a convex set $\mathcal{C}$ is *acceptable* if for all q there exists a p such that $A(p, q) \in \mathcal{C}$. For this particular choice of A, we have that $\mathcal{C}$ is acceptable if and only if it contains the point 0.

Since the set $\mathcal{C}$ is a convex subset of $\mathbb{R}$, this restricts its shape. Indeed $\mathcal{C}$ can only be a singleton, the whole real line $\mathbb{R}$, or (potentially unbounded) interval. Notice that $\text{Im } A = [0, 1]$, hence if $\mathcal{C} \cap \text{Im } A = \emptyset$ then $\mathcal{C}$ is automatically not acceptable. In order to reach a contradiction let us assume that $\text{Im } A \cap \mathcal{C} \neq \emptyset$, but $0 \notin \mathcal{C}$. Let $a = \min \mathcal{C} > 0$, then if we take $q = 0$ for example, there are no $p \in [0, 1]$ for which $A(p, q) = pq = 0$ could possible be greater than the positive number a. This means that for $\mathcal{C}$ to be acceptable it must contain 0.

Let us show that this is actually sufficient. If $0 \in \mathcal{C}$ then for any q let $p = 0$, and then $A(p, q) = pq = 0 \in \mathcal{C}$, as we claimed. A similar proof now gives us that if $\mathcal{C}$ is an acceptable set, then there exists $p \in \Delta$ such that for all $q \in \Delta$, we have that $A(p, q) \in \mathcal{C}$. If we fix $p = 0$, then for all $q \in \Delta$ we have that $A(p, q) = pq = 0 \in \mathcal{C}$, since $\mathcal{C}$ is acceptable.

7.6 Blackwell approachability theorem

Exercise 7.6:
1) In Figure 26 we have a diagram illustrating all the various vectors, sets, and

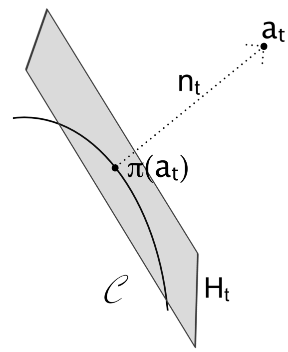

Figure 26: Schematic of the proof of the Blackwell Approachability Theorem

hyperplanes involved in the proof of Blackwell approachability theorem. Note that we only drew a part of the boundary of the convex set $\mathcal{C}$ in Figure 26.

We will now write down an algorithm on how to find p^t.

1. Consider the projection $\pi(a_t)$ on the convex set $\mathcal{C}$, where a_t is a vector representing the time average of the vector-valued payoff $A(p,q)$ up to time t;

2. Compute the vector $n_t = a_t - \pi(a_t)$, normal to $\mathcal{C}$, starting at πa_t and pointing towards a_t;

3. Define the half space $H_t \coloneqq \{a \mid a \cdot n_t \leq \pi(a_t) \cdot n_t\}$ which contains $\mathcal{C}$. Notice that $\{a \cdot n_t = \pi(a_t) \cdot n_t\}$ is a (hyper)plane passing though $\pi(a_t)$, perpendicular to n_t;

4. Rewrite $A(p,q) \cdot n_t$ as $p \cdot A^t q$, where A^t is a matrix depending on t;

5. By the approachability of H_t we have that the min-max theorem holds for $A(p,q) \cdot n_t$, hence there exists a p^t such that $A(p^t,q) \cdot n_t \leq \pi(a_t) \cdot n_t$ for all q.

This algorithm does not give you a way to explicitly compute v^t, but there exist plenty of packages in Python/Julia/Matlab which have been developed for to optimise the construction of such vector.

2) Assume that

$$a_t = \frac{1}{t} \sum_{i=1}^{t} A(p^i, q^i)$$

belongs to the convex set $\mathcal{C}$. As we have seen in the proof of the Blackwell Approachability Theorem

$$a_{t+1} = \frac{t}{t+1} a_t + \frac{1}{t+1} A(p^{t+1}, q^{t+1}).$$

Note that a_{t+1} is the convex combination of the vectors a_t and $A(p^{t+1}, q^{t+1})$. Clearly, if $A(p^{t+1}, q^{t+1}) \in \mathcal{C}$, we have by convexity of $\mathcal{C}$ that a_{t+1} is in $\mathcal{C}$. Unfortunately this is not always the case. Consider the point $m := \partial C \cap \langle a_t, A(p^{t+1}, q^{t+1})\rangle$, and assume that

$$|A(p^{t+1}, q^{t+1}) - m| > \frac{t}{t+1}|A(p^{t+1}, q^{t+1}) - a_t|.$$

Under this assumption we can conclude that $a_{t+1} \in \langle m, A(p^{t+1}, q^{t+1})\rangle$, and therefore outside of $\mathcal{C}$. This counterexample is quite abstract. Try to come up with your own counterexample (maybe let $\mathcal{C}$ be a point...).

7.7 Regret minimisation

Exercise 7.7:
1) Let us look at a specific game to discuss this question. Consider the "battle of the sexes" game given by

$$M = \begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix}$$

and suppose that player 2 decides their play after having seen the strategy chosen by player 1. Let T, and F denote the two strategies for this game, respectively e_1 and e_2. Furthermore, suppose that player 2 holds a grudge against player 1 and so they always play the opposite strategy to 1, i.e. if player 1 plays T, then player 2 replies with F, and vice-versa. This leads to payoff 0 for both players at all times. In particular this leads to at least one of the regrets for player 1 being always strictly positive, which is clearly contradicting the first Hart and Mas-Colell Theorem (Theorem 7.1 in the lecture notes).

Similarly we can show how Theorem 7.2 in the notes will not holds in this situation. Let us disregard the fact that player 2 is definitely not following the no-regret algorithm, and let us focus on what the matrix of frequencies the conjoint actions of player 1 and 2 looks like. Given the choice of strategies from player 2 our probability distributions keeping track of the frequency of actions up to time t will only have weights on the off-diagonal terms

$$P_t = \begin{pmatrix} 0 & 1 - f^t \\ f^t & 0 \end{pmatrix}$$

where $f^t = f^t_B(T)$ the frequency at which player 2 plays T, the first of the available strategies. From Exercise 7.1 we have that P_t does not belong to the CE set for this game, for any t, as we claimed.