Imperial College London

A study of the proof for the full renormalization horseshoe for unimodal maps of higher degree

Master Thesis

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Contents

Chapter 1

Introduction : an approach of universality and renormalization

The full renormalization horseshoe for unimodal maps is a complicated result that first needs to be clearly understood before studying its proof. The renormalization is a general concept used in dynamical systems which is based on the repetition of the essential form of a non-linear map at infinitely many scales. Actually, renormalization explains the phenomenon of universal period doubling for unimodal one-dimensional map. We will see later what these maps are, but we will begin with simple facts about the well known logistic maps family to introduce the concept of renormalization.

1.1 The behaviour of the logistic family

Recall that the logistic map is defined on $[0, 1]$ by

$$f_\mu(x) = \mu x(1 - x)$$

Here we will consider $0 < \mu \leq 4$ so that the unit interval is mapped into itself. We can easily compute the fixed points and study their behaviour (attracting, repelling) as μ grows. A complete study can be found in [*]. It leads to the famous bifurcation diagramm of the logistic family :

- $0 < \mu < 1$: 0 is the only fixed point and is attracting.
- $1 < \mu < 3$: 0 becomes repelling and $x^*_\mu = 1 - \frac{1}{\mu}$, which is no longer negative, is attracting.
- $3 < \mu < 1 + \sqrt{6}$: both fixed points are repelling, but two period two points appear, ie points such that $f^2(x) = x$. Both are attracting.

- $\mu > 1 + \sqrt{6}$: period two points become repelling and attracting period four points appear, and then for μ a little bit larger, period four points become repelling and attracting period eight points appear etc... All these points seem to accumulate to some $\mu_\infty \simeq 3.56995$.

- $\mu_\infty < \mu < 3.83$: the maps moves from chaos to periodic windows many times.

- then, an attractive period three cycle appears at $\mu \simeq 3.83$. If we zoom in one of the three stable branches, a copy of the whole bifurcation diagram appears for $\mu > 3.83$.

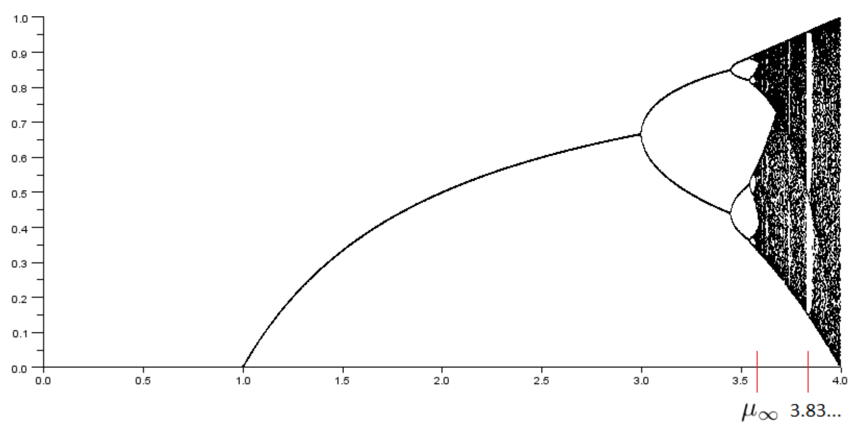

Figure 1.1: Bifurcation diagramm of the logistic family : commons.wikimedia.org

Study of the period doubling window So now we should discuss the period doubling behaviour for $\mu < \mu_\infty$. Thanks to Newton's method, we could compute each value of μ for which the diagramm bifurcates, call them μ_1, μ_2, ..., μ_n, ... and compute the Feigenbaum's constant :

$$\delta = \lim_{n \to \infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

We find $\delta = 4.669201609$. This constant scales the length of the sequence of bifurcation, as it can be seen on the bifurcation diagramm.

What happens when $\mu > 3.83$**?** If we try to solve $f^3_\mu(x) = x$ for μ very close to 3.83, we got 8 solutions. 2 of them are solutions of $f_\mu(x) = x$ and are unstable. 3 of the 6 remaining are unstable. So we get a period three cycle as mentioned above just after $\mu \simeq 3.83$. We could then continue our study, but as the miniature of the bifurcation diagram appears, we know this is just another period doubling phenomenon. So after this period three cycle, we are going to have a period 3×2 cycle, and then a period 3×2^2 cycle... The same mechanism of period doubling appears but now we get 3×2^n period cycles.

1.2 Universality

So far we have seen that a particular family of mappings have a period doubling behaviour scaled by a certain constant δ. In fact, a much more important class of functions (unimodal maps) has the same property of period doubling scaled by Feigenbaum's constant δ.

To see this, let's take a family of regular functions that has the same properties that the logistic one (by properties we mean the same overall shape). Define on $[0, 1]$

$$g_\mu(x) = \mu \sin(\pi x)$$

for $\mu \in [0, 1]$ so that the unit interval is mapped into itself. Using Matlab, one can plothe bifurcation diagram of the family g_μ. The code used to plot it is in the apendix.

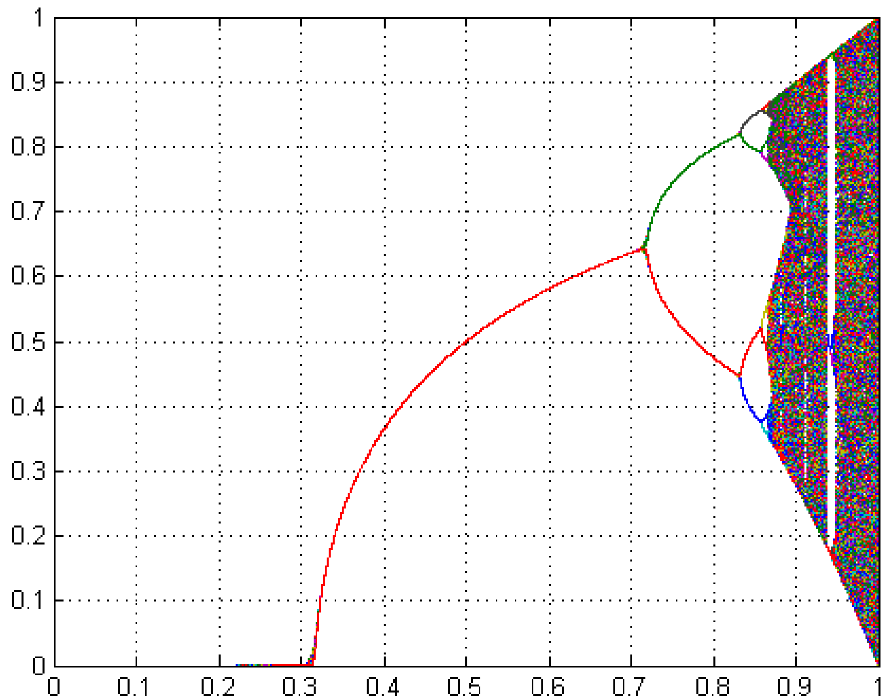

Figure 1.2: Bifurcation diagramm of a random family

Using this diagramm, we can try to approximate the number

$$\delta' = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

and compare it to the Feigenbaum's constant for the logistic family. To get an idea of the value of δ', we can compute the first terms of the sequence

$$\delta'_n = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

We find

$$\delta'_2 = 4.29 \ ; \ \delta'_3 = 4.55 \ ; \ \delta'_4 = 4.61 \ ;$$

We could be more accurate and compute the next terms, but this is not really the point here. It seems that δ'_n converges to the Feigenbaum's constant $\delta = 4.669201609$. In fact, a larger class of maps have the same behaviour, with the same scaling constant δ. In 1975, Mitch Feigenbaum found that, no matter what unimodal map is used, the same convergence scaling constant appears. This is why we are speaking of universality.

Note. *Note not only the map needs to be unimodal, but we also require a strictly quadratic maximum (it will be defined in the next section) ; for instance, the maps* $x \mapsto \mu(1-cx^4)$ *on* $[-1,1]$ *will have a different behaviour because* $f''(0) = 0$ *(we don't have* $f''(0) < 0$*).*

1.3 Renormalization

Now that we have remarked a universal behaviour, we would like to explain it using the concept of renormalization. By universality, we mean the maps that exhibit the same period-doubling behaviour remarked in the two last sections. These maps have the same overall shape as the logistic family. Let us introduce a mathematical setting. We will consider maps from the interval $I = [-1,1]$ that are regular on I, unimodal and with a quadratic maximum ; we will denote this set of maps $\mathcal{U}(I)$.

- By unimodal we mean a continuous map f defined on I that is increasing to the left of an interior point x_{max} of I and decreasing to the right of x_{max}
- By a quadratic maximum we mean that $f''(x_{max}) < 0$

This setting will be redefined accurately later, but or the moment we just need a main idea of the kind of functions we are dealing with. To ilustrate the processus of renormalization, let us introduce a simple map f

$$\forall x \in I,\ f_0(x) = 1 - 1.5x^2$$

The graph of this map is as follows

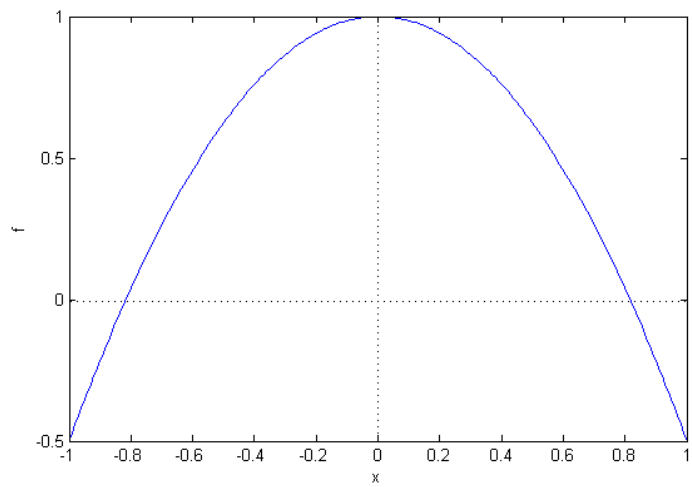

Figure 1.3: f_0, example of a unimodal map

Now, consider a general map $f \in \mathcal{U}(I)$ (f_0 will be used as an example). We set $a = -f(1)$ ($a_0 = 0.5$) and $I_a = [-a, a]$. Then we look at $f^2_{I_a}$ which is f^2 restricted to the interval I_a. For f_0, $f^2_{I_0}$ turns out to be almost $-f$ modulo some rescaling. To see it, let us see the graph of f_0^2 and its behaviour on the interval $[-0.5, 0.5]$: in the red window, f^2 has clearly the same behaviour as $-f$, but we need some rescaling.

In fact, this is not difficult to see that if we applied the change of variable $x \mapsto -ax$ and then multiply the function f^2 by $-\frac{1}{a}$ we get the good rescaling. Then we define a renormalization operator $\mathcal{R}$ acting on function in $\mathcal{U}(I)$:

$$\forall f \in \mathcal{U}(I), \forall x \in I$$
$$\mathcal{R}f(x) = -\frac{1}{a} f^2(-ax)$$

It is important to remark that denoting $b = f(a)$, we need $0 < a < b < 1$ and $f(b) < a$ for $f \in \mathcal{U}(I)$ to be in the domain of $\mathcal{R}$. Note also that for such f, $\mathcal{R}f \in \mathcal{U}(I)$.

In our example,

$$\mathcal{R}f_0(x) = -2f^2(-0.5x)$$

We can plot the graph of $\mathcal{R}f_0$ to make sure we get a shape very close to the one of f on I.

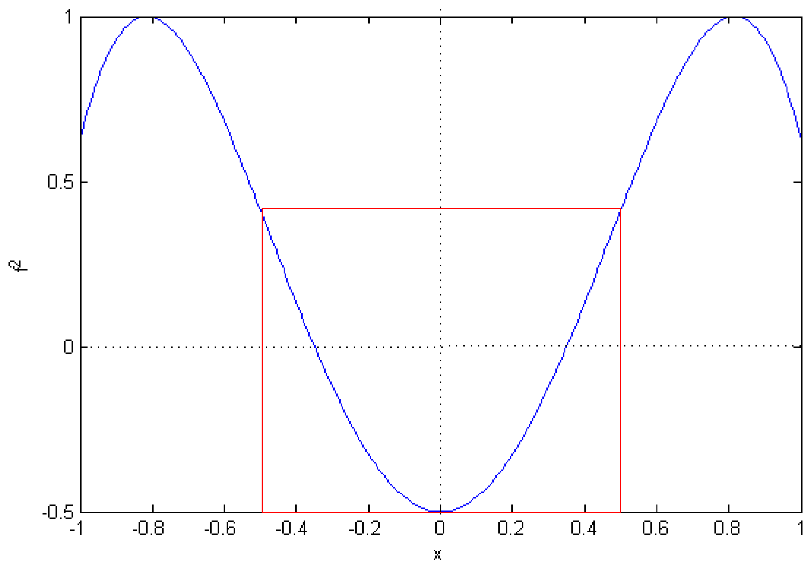

Figure 1.4: Behaviour of f_0^2

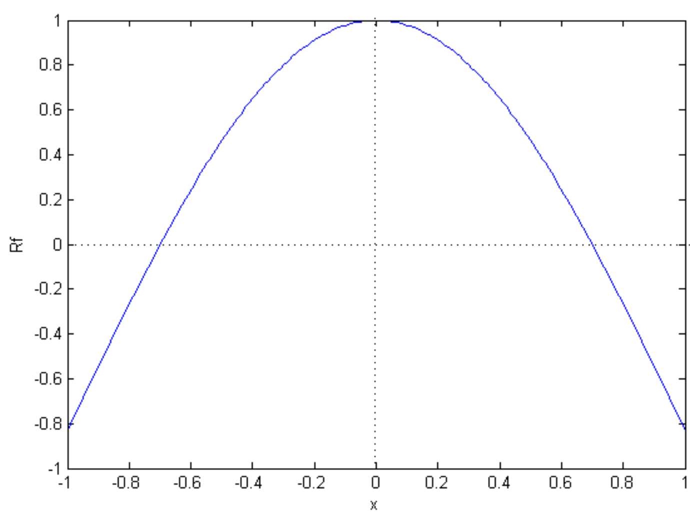

Figure 1.5: Renormalized f_0

So far we have only considered a particular case, ie the period-doubling case applied to a particular interval I. The aim of this project is to prove that this renormalization is possible for a much larger class of functions.

For the moment we only have applied the renormalization operator to some functions which define themselves some dynamical systems (ie for $f \in \mathcal{U}(I)$ and $x_0 \in I$, we study the discrete dynamical system $x_{n+1} = f(x_n)$). Now, we are going to study the dynamics of $\mathcal{R}$ acting on $\mathcal{U}(I)$. Thus the phase space is now an infinite dimensional space $\mathcal{U}$ of functions and we want to study the orbits of the points $f \in \mathcal{U}$ under the action of the renormalization operator $\mathcal{R}$.

Feigenbaum, Coullet and Tresser made some conjectures about this infinite-dimensional dynamical system.

- There exists a Banach space $\mathcal{B} \in \mathcal{U}(I)$ such that $\mathcal{R}_{|\mathcal{B}}$ has a unique fixed point $f^* \in \mathcal{B}$
- The fixed point f^* is hyperbolic, meaning that the derivative of $\mathcal{R}$ at f^*, denoted $\mathcal{D}_{\mathcal{R}}$, has no eigenvalue of modulus one.
- All the eigenvalues of $\mathcal{D}_{\mathcal{R}}$ lie in the open unit disk, except one equal to δ (the Feigenbaum's constant). We can then conjecture the existence of an unstable manifold $\mathcal{W}^U$ of dimension 1 related to the eigenspace relative to δ and a stable manifold $\mathcal{W}^S$ of codimension 1 related to the stable eigenvectors.
- For $n \in \mathbb{N}^*$, we denote Σ_n the set of superstable maps in $\mathcal{B}$ with period 2^n superstable cycle. Then all the Σ_n intersect $\mathcal{W}^U$ transversally and are of codimension 1. Denoting f_n^* the intersection of Σ_n with $\mathcal{W}^U$, the sequence $(f_n^*)_{n\geq 1}$ converges to f^* geometrically, with rate δ. All these things are summed up in the following figure.

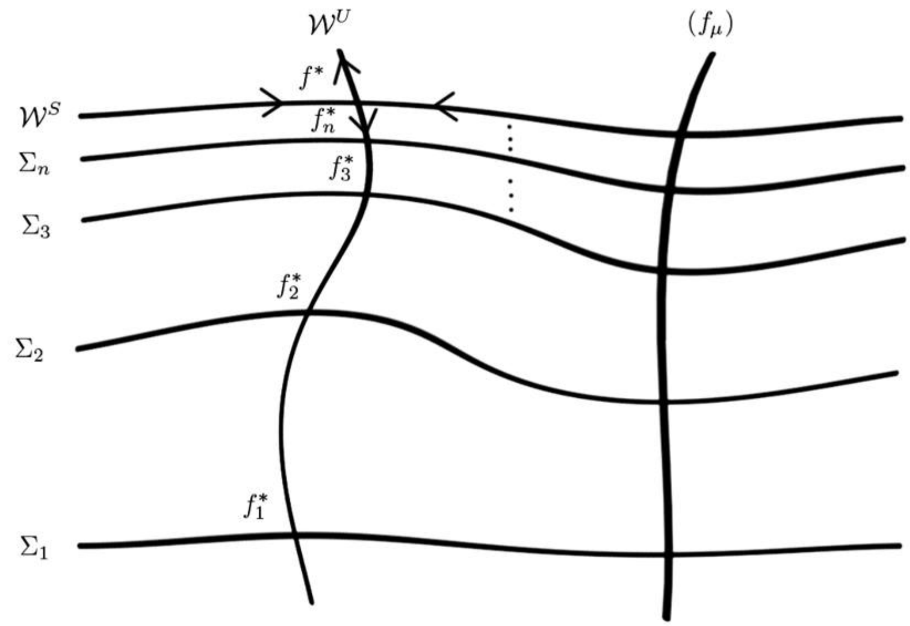

Figure 1.6: General behaviour of the dynamical system related to the renormalization operator

So far we have only considered maps that are "twice" renormalizable, meaning that $\mathcal{R}f \sim f \circ f$. But there exists maps that are n-renormalizable or even infinitely many times renormalizable. Roughly speaking, if we are able to defined a function space where functions are 2,3,...,d renormalizable, then it can be shown that the renormalization operator $\mathcal{R}$ is topologically conjugated to the full shift map σ_{Ω_d} on bi-infinite sequences Ω_d with d symbols. The aim of this paper is to study the proof of this result (we state it rigorously in the next section).

Chapter 2

Statement of the main result

So now, we are going to state the result we want to prove with the proper tools (the good space of functions, the mathematical notions related to chaos and dynamical systems...) and in a generalized way. Remember, Feigenbaum made a conjecture saying that the renormalization process is hyperbolic, given a proper functional space of infinite dimension.

We will consider the space of real unicritical and polynomial-like maps of an arbitrary even degree $d \geq 2$; we will denote it $\mathcal{C}_d^{\mathbb{R}}$ and study the behaviour of the renormalization operator $\mathcal{R}$ on this space. The main point here is that d is arbitrary, as hyperbolicity has been proven for simple cases where d is not arbitrary. Actually, $\mathcal{R}$ has an invariant horseshoe $\mathcal{A}$ and is exponentially contracting on the corresponding hybrid laminations : this will be the keypoint to prove :

Theorem (Main result)**.** *$\mathcal{R}$ has an invariant precompact set $\mathcal{A} \subset \mathcal{I}^{\mathbb{R}}$, called the renormalization horseshoe and $\mathcal{R}_{|\mathcal{A}}$ is topologically conjugate to the two-sided shift in infinitely many symbols. Any germ $f \in \mathcal{I}^{(\mathbb{R})}$ is attracted to some orbit of $\mathcal{A}$ at a uniformly exponential rate. Note that we need a proper metric here, this will be the Caratheodory metric.*

In this theorem, $\mathcal{I}^{\mathbb{R}}$ is the space of real infinitely renormalizable polynomial-like germs and $\mathcal{I}^{(\mathbb{R})}$ is the space of polynomial-like germs that are hybrid equivalent to the real ones. Note that to state this result, we introduced new notions. The mathematical background needed to understand these notions will be presented in the next chapter.

Chapter 3

Basic tools

3.1 Quick reminder on complex analysis

In this section we will especially focus on the complex functions of a complex variable, to make things clear for our further developments. In what follows, z will be a complex variable, naturally decomposed $z = x + iy$, $x, y \in \mathbb{R}$.

3.1.1 Analytic functions

Recall that the derivative of a function f at a point a is given (if the limit exists) by

$$f'(a) = \lim_{z \mapsto a} \frac{f(z) - f(a)}{z - a} \tag{3.1}$$

Definition 1. *In order to define the notion of analytic function, we need to be accurate on the nature of the set where it is defined*

- *A complex function f of a complex variable z is said to be* ***analytic*** *or* ***holomorphic*** *in the region Ω (non-empty connected open set) if f is defined in Ω and possesses a derivative at each point of Ω.*

- *A complex function f of a complex variable z is said to be* ***analytic*** *or* ***holomorphic*** *in the set $A \subset \mathbb{C}$ if it is analytic in some region containing A.*

In the end of this paragraph, we will consider that f is analytic on the whole plane $\mathbb{C}$.

$f'(a)$ must have the same value regardless of the way $z \mapsto a$. This leads us to

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} \tag{3.2}$$

Writing $f(z) = u(z) + iv(z)$, with u, v real functions of the complex variable z, this leads to the famous Cauchy-Rieamnn differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{3.3}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.4}$$

Note that

$$J = |f'(z)|^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{3.5}$$

is the Jacobian of u and v with respect to x and y.

Writing $f(z) = f(x, y)$ as a function of two real variables, one can easily find that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \tag{3.6}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{3.7}$$

Note.

Analytic functions can be characterized by $\dfrac{\partial f}{\partial \bar{z}} = 0$.

3.1.2 Conformal mappings

Let us a consider a function f which is analytic in a region Ω. Now, introduce an arc $\gamma : z(t)$ for $t \in [\alpha, \beta]$ and suppose $\gamma \subset \Omega$. Let us call $\gamma' : w(t) = f(z(t))$ the image of γ through f. We easily find that

$$w'(t) = f'(z(t))z'(t) \tag{3.8}$$

So if we choose a point $z_0 = z(t_0)$ such that $f'(z_0) \neq 0$ and $z'(t_0) \neq 0$, we will have $w'(t_0) \neq 0$, meaning that γ' has a tangent at $w_0 = f(z_0)$ with direction

$$\arg(f'(z_0)) + \arg(z'(t_0))$$

But $\arg(z'(t_0))$ is the direction of the tangent to γ at z_0, so $\arg(f'(z_0))$ is the angle between the tangent to γ at z_0 and the tangent to γ' at w_0 and it is independent of γ. So if we take two curves which are tangent to each other at z_0, they are mapped through f onto tangent curves at w_0. More generally, two curves which form an angle at z_0 will be mapped through f onto curves forming the same angle at w_0. This is why we are speaking of conformal mapping for f. The above justifies the next definition, and a similar study can be done for the anticonformal case.

Definition 2. • *A function which is analytic in a region Ω with $f'(z) \neq 0$ everywhere in Ω is a* ***conformal mapping****.*

- *A function f for which $\bar{f}$ is analytic in a region Ω with $\bar{f}'(z) \neq 0$ everywhere in Ω is an* ***anticonformal mapping****. It reverses the sense of angles*

3.2 Quasiconformal mappings

So far we have considered conformal mappings, but to prove our main result, we need to define the notion of quasiconformal mappings. Roughly speaking, they are a generalization of conformal mappings, and a lot of theorems related to conformal mappings can be extended to quasiconformal mappings. Moreover, a q.c. mapping is easier to use as a tool.

3.2.1 Definition of Grötzsch

The notion of q.c. mappings appeared when Grötzsch tried to find a conformal mapping from a square Q on a rectangle R (not a square) mapping vertices on vertices. There is not such conformal mapping, but Grötzsch built a q.c. mapping to solve this problem.

Let us take the same notation than above, related to a $\mathcal{C}^1$-homeomorphism $w = f(z)$ from one region to another. Then, at a point z_0, one can write

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{3.9}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \tag{3.10}$$

This represents an affine transformation frome the (dx, dy) plane to the (du, dv) plane. Note that we can write it in another way :

$$dw = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \tag{3.11}$$

Remember that the Jacobian was given by (3.5) and together with (3.6) and (3.7) , we are given

$$J = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \tag{3.12}$$

- for sense-preserving mappings, we will have $J > 0$.
- for sense-reversing mappings, we will have $J < 0$.

Note. *This makes sense with our above study of conformal mappings, as* $\frac{\partial f}{\partial \bar{z}} = 0$ *for a conformal mapping.*

So considering sense-preserving, we have $\left|\frac{\partial f}{\partial \bar{z}}\right| < \left|\frac{\partial f}{\partial z}\right|$. Together with (3.11), this gives

$$\left(\left|\frac{\partial f}{\partial z}\right| - \left|\frac{\partial f}{\partial \bar{z}}\right|\right)|dz| \le dw \le \left(\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \bar{z}}\right|\right)|dz| \tag{3.13}$$

The ratio of the major to the minor axis, called the **dilatation** at z is then given by

$$D_f = \frac{\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \bar{z}}\right|}{\left|\frac{\partial f}{\partial z}\right| - \left|\frac{\partial f}{\partial \bar{z}}\right|} \ge 1 \tag{3.14}$$

It is sometimes more convenient to consider

$$d_f = \left|\frac{\partial f}{\partial \bar{z}}\right| / \left|\frac{\partial f}{\partial z}\right| \tag{3.15}$$

These two ratios are linked by the flowing equations

$$D_f = \frac{1 + d_f}{1 - d_f} \tag{3.16}$$

$$d_f = \frac{D_f - 1}{D_f + 1} \tag{3.17}$$

Note. *A mapping is conformal at* z *if and only if* $D_f = 1$ *or* $d_f = 0$.

We can also introduce the **complex dilatation**

$$\mu_f = \frac{\partial f}{\partial \bar{z}} / \frac{\partial f}{\partial z} \tag{3.18}$$

We remark that $|\mu_f| = d_f$.

Definition 3. *A* $\mathcal{C}^1$*-mapping* f *is* ***quasiconformal*** *(q.c.) if* D_f *is bounded. We will say that* f *is* K*-**quasiconformal** (with* $K \ge 1$*) if* $D_f \le K$*. It is equivalent to* $d_f \le \dfrac{K-1}{K+1}$.

Note. *As seen above, a* 1*-q.c. mapping is conformal.*

Proposition 1.
- *If* f *is* K_1*-q.c. and* g *is* K_2*-q.c., then* $f \circ g$ *and* $g \circ f$ *are* $K_1 K_2$*-q.c.*
- *The inverse of a* K*-q.c. homeomorphism is* K*-q.c.*

This point of view should be adequate for our further development, but it could be convenient to have the other definitions that Ahlfors gave in his lectures.

3.2.2 The geometric definition

We are given a homeomorphism f from a region Ω to a region Ω' which is sense-preserving. We introduce the notion of generalized quadrilaterals, which are defined by 4 disctinct points in Ω, linked together by 4 closed and discjoint arcs in Ω. Let $Q(z_1, z_2, z_3, z_4) \subset \Omega$ be such a generalized quadrilateral. By Riemann mapping theorem, we know that there exists a conformal mapping which maps Q onto a rectangle R. Let choose this conformal mapping ϕ such that $\phi(z_1) = 0$, $\phi(z_2) = 1$, $\phi(z_3) = 1 + iM$ and $\phi(z_4) = iM$. Then M is called the modulus (denoted $m(Q)$) of Q.

Definition 4. *f is said to be K-q.c. if the modula of quadrilateras in Ω are K-quasi-invariant, meaning that, given a quadrilateral Q,*

$$m(f(Q)) \leq Km(Q)$$

Note. *if f is $\mathcal{C}^1$, this definition agrees with the previous one. We also have the results of **Proposition** 1 and that a 1-q.c. mapping is conformal.*

3.2.3 The analytic definition

This definition is very close to the definition of Grötzsch, but remember that for the latter, we need a $\mathcal{C}^1$-homeomorphism. We are going to replace this constraint by a weaker one.

Definition 5. *we say that a function f is **absolutely continuous on lines (ACL)** in a region Ω if, given any closed rectangle $R \subset \Omega$ with sides parallel to the x-axis and y-axis, f is absolutely continuous on almost every horizontal and vertical lines of R.*

Note that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist whenever f is ACL.

Definition 6. *We say that a mapping $f : \Omega \mapsto \mathbb{C}$ (Ω is a region) is a K-q.c. mapping ($K \geq 1$) if*

- *f is ACL in Ω*
- *$|\mu_f| = d_f \leq \frac{K-1}{K+1}$ almost everywhere*

Thi definition is equivalent to the geometric one.

Note.
- *If f is q.c. and if $\frac{\partial f}{\partial \bar{z}} = 0$, then f is conformal.*
- *In this configuration, for all $K \geq 1$, the set $\{f | f$ is a K-q.c. homeomorphism$\}$ is compact in the space of ACL functions in Ω. It was not the case in the definition of Grötzsch, as we only considered $\mathcal{C}^1$-homeomorphism.*

3.3 Holomorphic motions

Let $\widehat{\mathbb{C}}$ stand for the Riemann sphere, consisting of $\mathbb{C}$ together with a point at ∞.

Definition 7. *Suppose we are given a connected manifold X and let x_0 be a basepoint in X. Let $E \subset \widehat{\mathbb{C}}$ be a set. We define a* ***holomorphic motion*** *of E over X as a family of injections*

$$h_x : E \to \widehat{\mathbb{C}}$$

such that

- *for all fixed $e \in E$, $x \mapsto h_x(e)$ is holomorphic in x*
- $h_{x_0} = id$

There are some basic results about holomorphic motions that are very useful. In particular, the λ-lemma says that every holomorphic motion of E over X can be extended to a holomorphic motion of $\bar{E}$ over X. The extended λ-lemma is even more powerful as it extends any holomorphic motion of E over X to a holomorphic motion of the whole complex plane over X. One can note that in this definition, we don't know how regular the maps $h_x : E \to \widehat{\mathbb{C}}$ are.These lemmas will also give us an idea of the regularity of these maps.

Theorem 1 (λ-lemma)**.** *Given a holomorphic motion $(h_x)_{x\in X}$ of E over X, $(h_x)_{x\in X}$ admits an extension to a holomorphic motion $(\widetilde{h}_x)_{x\in X}$ of $\bar{E}$ over X. For every $x \in X$, $\widetilde{h}_x : \bar{E} \to \widehat{\mathbb{C}}$ is quasiconformal. Moreover, $(\widetilde{h}_x)_{x\in X}$ is jointly continuous in x and in $e \in \bar{E}$.*

Going further, we obtain the extended λ-lemma :

Theorem 2 (extended λ-lemma)**.** *Given a holomorphic motion $(h_x)_{x\in X}$ of E over X, $(h_x)_{x\in X}$ admits an extension to a holomorphic motion $(\widehat{h}_x)_{x\in X}$ of $\widehat{\mathbb{C}}$ over X. For every $x \in X$, $\widehat{h}_x : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal of dilatation $D_x \leq K_x = \frac{1+|x|}{1-|x|}$. So we have K_x-q.c. mappings here. Moreover, $(\widehat{h}_x)_{x\in X}$ is jointly continuous in x and in $e \in \widehat{\mathbb{C}}$.*

3.4 Julia set

The material that follows is very useful for the study of dynamical systems, and we may need it later.

We define a Riemann surface S as a connected complex analytic manifold of complex dimension one.

Definition 8. *Let U be a complex manifold and $\mathcal{F}$ be a family of holomorphic maps from U to $\widehat{\mathbb{C}}$. $\mathcal{F}$ is said to be a* ***normal*** *family if every sequence of elements of $\mathcal{F}$ has a subsequence converging locally uniformly on compact subsets of U to a limit (which is a holomorphic map).*

In what follows, U is a Riemann surface and $f : U \to U$ is a holomorphic map and we denote by f^n its n-iterate. Let $z_0 \in U$. There are two cases :

- there exists a neighbourhood N_0 of z_0 such that the family $(f^n_{|N_0})$ of iterates restricted to N_0 forms a normal family. We say that z_0 is **regular**.
- there is no such neighbourhood.

Definition 9. *With this, we can define the notions of Fatou an Julia sets :*

- *The set of regular points in U is the **Fatou set** of f, denoted $\Omega(f)$.*
- *The **Julia set** is the complement of the Fatou set in U, denoted $J(f)$.*

Note that the Julia and Fatou sets are invariant under f. The components of the Fatou set are preperiodic, meaning that if C is a component of $\Omega(f)$, then there exist $i > j > 0$ such that $f^i(C) = f^j(C)$. We can classify the periodic component of $\Omega(f)$.

Theorem 3. *Suppose C is a p-periodic component of the Fatou set. Then C is of one of the following types :*

1. *there is an attracting periodic point z^* in C such that, for all $z \in C$:*

$$\lim_{n \mapsto \infty} f^{np}(z) = z^*$$

   *C is called an **attracting basin**.*

2. *there is a parabolic periodic point (by parabolic, we mean that the multiplier is equal to 1) $\widehat{z} \in \partial C$ such that, for all $z \in C$:*

$$\lim_{n \mapsto \infty} f^{np}(z) = \widehat{z}$$

   *C is called an **parabolic basin**.*

3. *C is a disk, and f acts as an irrational rotation on it. C is called a **Siegel disk***

4. *C is an annulus, and f acts as an irrational rotation on it. C is called an **Herman ring***

Definition 10. *The complement of the attracting basin of infinity is called the **filled** Julia set, denoted $K(f)$. The Julia set is the boundary of $K(f)$: $J(f) = \partial K(f)$.*

The corresponding Julia set is $J(f) = \partial K(f)$.

3.5 The modulus of an annulus

3.5.1 Facts about Riemann surfaces

Suppose we are given two Riemann surfaces S and S' ; we say that they are **conformally isomorphic** if there exists a holomorphic homeomorphism between them, with holomorphic inverse. The next result states that simply connected Riemann surfaces can be sorted in three main classes.

Theorem 4. *If we are given any simply connected Riemann surface S, then S is conformally isomorphic either to*

1. *the whole plane $\mathbb{C}$.*
2. *the unit disc $\mathbb{D}$.*
3. *the Riemann sphere $\widehat{\mathbb{C}}$.*

It could be useful to link any arbitrary Riemann surface to a simply connected one, as the latter can be studied as one of the three cases of the above theorem. In fact, it can be shown that any arbitrary Riemann surface S is conformally isomorphic to a quotient $\widetilde{S}/G$. $\widetilde{S}$ is a simply connected Riemann surface, called the universal covering of S, wich is thus conformally isomorphic to $\mathbb{C}$, $\mathbb{D}$ or $\widehat{\mathbb{C}}$. G is a group of conformal automorphisms of $\widetilde{S}$. Except the identity element, the elements of G do not have fixed points in $\widetilde{S}$. G can be identified with the fundamenatl group of S, denoted $\pi_1(S)$

Note. *The punctured disc $\mathbb{D}\setminus\{0\}$, the punctured plane $\mathbb{C}\setminus\{0\}$ and any annulus $A_R = \{z : 1 < |z| < R\}$ have a fundamental group $\pi_1 \cong \mathbb{Z}$. Every Riemann surface S with $\pi_1(S) \cong \mathbb{Z}$ is isomopric either to $\mathbb{D} \setminus \{0\}$, $\mathbb{C} \setminus \{0\}$ or $A_R = \{z : 1 < |z| < R\}$ for some $R > 1$.*

3.5.2 definition of the modulus for a doubly-connected Riemann surface

Given a Riemann surface S with $\pi_1(S) \cong \mathbb{Z}$ and which is also isomorphic to A_R for some $R > 1$, we define the **modulus** of S as follows :

$$\text{mod }(S) = \frac{\log(R)}{2\pi} \tag{3.19}$$

3.6 Polynomial-like maps

3.6.1 Definition

Polynomial-like maps of even degree are the mappings that interest us for the main result, but we need to introduce some background to define and understand what they are.

Note that if we consider a normalized polynomial P of degree $d > 1$, then when the modulus of z is large, $P(z) \sim z^d$. So if we choose a large enough disc U, and denote its preimage $U' = P^{-1}(U)$, the compact closure of U' will lie in $U : \bar{U}' \subset U$. Polynomial-like maps generalize this concept.

First we are going to define what is the degree for a continuous map between two manifolds U and U' in $\mathbb{C}$. Roughly speaking, the degree is the number of time that U wraps around U' under the continuous map.

Definition 11. *Let $f : U \to V$ be a map between two discs. If f is holomorphic and if for all compact set $K \subset V$ we have that $f^{-1}(K)$ is compact, then f is said to be a* ***proper map****. Then for all $x \in V$, $f^{-1}(x)$ is a finite set and we define the* ***degree*** *of f to be the cardinal of this set, counting multiplicity.*

Definition 12. *Let U and U' be open sets isomorphic to discs and U' be relatively compact in U. A* ***polynomial-like map*** *of degree d is a proper analytic map $f : U' \to U$ of degree d.*

- The filled Julia set of a polynomial-like map is defined to be

$$K(f) = \bigcap_{n \geq 1} f^{-n}(U) \tag{3.20}$$

- Given a p.l.-map $f : U \to V$, $V \setminus U$ is called the **fundamental annulus** of f.

Unicritical polynomial-like maps In the proof we study, we only consider unciritical p.l.-maps. We will say that a p.l.-map f of degree d is **unicritical** if it has a unique critical point of local degree d. Note that we can normalize unicritical p.l.-maps so that 0 is the unique critical point and $f(z) = z^d + c + \mathcal{O}(z^{d+1})$ in a neighbourhood of 0.

3.6.2 Connected filled Julia set

The following theorem was first observed and proved for polynomial maps. It was then generalized for p.l.-maps.

Theorem 5. *Given a p.l.-map f, its filled Julia set $K(f)$ is connected if and only if all the critical points of f belong to $K(f)$. If there is no critical points in $K(f)$, $K(f)$ is a Cantor set.*

Remember that 0 is the unique critical point in our case. This gives the following :

Corollary 1. *Given an unicritical p.l.-map f with 0 for unique critical point, its filled Julia set $K(f)$ is connected if and only if $0 \in K(f)$. Otherwise, $K(f)$ is a Cantor set.*

We then have a dichotomy between the p.l.-maps with connected filled Julia set and those for which the filled Julia set is a Cantor set. In what follows, we assume that p.l.-maps are unicritical with critical point 0.

3.6.3 Polynomial-like germs

We are going to gather the p.l.-maps with the same connected filled Julia set and the same behaviour near 0 in a same class. That is why we introduce the notion of germs.

Definition 13. *Given a p.l.-map f with connected filled Julia set $K(f)$, the equivalence class of p.l.-maps g such that $K(f) = K(g)$ and $f = g$ in a neighbourhood of 0 is a* ***polynomial-like germ*** *(associated to f, denoted $[f]$). We will denote by $\mathcal{C}$ the set of all polynomial-like germs. The associated modulus of $[f]$ is given by*

$$\text{mod } [f] = \sup_{g \in [f]} \quad \text{mod } (V \setminus U) \tag{3.21}$$

where the supremum is taken over all p.l.-mappings $g : U \to V$ in $[f]$.

Note that the unicritical polynomial $P_c : z \to z^d + c$ is such that $[P_c]$ belongs to $\mathcal{C}$ if and only if c belongs to the Multibrot set $\mathcal{M}$ where

$$\mathcal{M} = \left\{ c \in \mathbb{C} : \sup_{n \in \mathbb{N}^*} |P_c^n(0)| < \infty \right\} \tag{3.22}$$

We will denote by $\mathcal{C}^{\mathbb{R}}$ the set of germs in $\mathcal{C}$ that preserves the real line $\mathbb{R}$. Note that such germs are such that U and U' are $\mathbb{R}$-symmetric in the definition of p.l.-maps above (ie the x-axis is an axis of symmetry for U and U' in the complex plane)

3.6.4 Hybrid classes

Definition 14.
- *We say that two p.l.-maps $f : U \to U'$ and $g : V \to V'$ are* ***topologically equivalent*** *or* ***conjugate*** *if there exists a homeomorphism $h : N(K(f)) \to N(K(g))$ (where $N(A)$ denotes a niegbourhood of A) such that $h \circ f = g \circ h$.*
- *If h is quasiconformal, we say that f and g are* ***quasi-conformally equivalent.***
- *If h is quasiconformal and such that $\frac{\partial h}{\partial \bar{z}} = 0$ on $K(f)$, we say that f and g are* ***hybrid equivalent.***
- *We say that two p.l.-germs $[f]$ and $[g]$ are* ***hybrid equivalent*** *if f and g are hybrid equivalent.*

The following theorem states that every p.l.-map is hybrid equivalent to a polynomial map (the result follows for the corresponding germs).

Theorem (Straightening theorem).

1. *Every p.l.-map $f : U \to U'$ of degree d is hybrid equivalent to a polynomial map P of degree d. P is unique (up to conjugation by an affine map) if $K(f)$ is connected.*
2. *Every p.l.-germ $[f] \in \mathcal{C}$ is hybrid equivalent to some polynomial germ $[P_c]$ with $c \in \mathcal{M}$.*

In what follows, we will denote by $\mathcal{H}_c$ the **hybrid class** of $[P_c]$, ie all the p.l.-germs that are hybrid equivalent to $[P_c]$.

3.7 Teichmüller spaces

3.7.1 Elementary homotopy theory

Let X and Y be two topological spaces and $f : X \to Y$ and $g : X \to Y$ be two continuous maps.

Definition 15.

- *We say that f and g are **homotopic** if there exists a continouous map $F : X \times [0,1] \to Y$ such that $F(.,0) = f$ and $F(.,1) = g$.*
- *We say that f is null-homotopic if it is homotopic to some constant function $c : X \to \{y_0\}$.*

To define Teichmüller spaces we only need the notion of homotopic maps. But later we will have to use the notion of contractible spaces. That is why we go further in our theory of homotopy here.

Definition 16. *Two topological spaces X and Y are called **homotopic** if there exist $f : X \to Y$ and $g : Y \to X$ such that*

- *$f \circ g$ is homotopic to id_Y*
- *$g \circ f$ is homotopic to id_X*

Evidently, the homotopy-relation is an equivalence relation. Now let us have a look at the simplest objects of homotopy theory.

Definition 17. *We say that a topological space X is **contractible** if it is homotopic to a point (singleton space). Equivalently, X is contractible if id_X is null-homotopic.*

Proposition 2. *Any convex subspace of a topological space is contractible.*

Proof : let X' be a convex subspace of a topological space X. Let $x'_0 \in X'$. Define

$$h \quad : \quad X' \times [0,1] \to X' \tag{3.23}$$
$$h(x',t) \quad = \quad tx'_0 + (1-t)(x' - x'_0) \tag{3.24}$$

h is a well defined homotopy of X' onto X' (by convexity of X') such that $h(.,0) = id_{X'}$ and $h(.,1) = c_0$ where c_0 is the constant map (mapping every $x' \in X'$ to x'_0), meaning that $id_{X'}$ is null-homotopic.

3.7.2 Definition

Definition 18. *Given a Riemann surface S of finite type (ie conformally equivalent to a compact Riemann surface minus a finite number of punctured points) and two q.c.-mappings $f : S \to S_1$ and $g : S \to S_2$, we say that f and g are equivalent ($(S_1, f) \sim (S_2, g)$) in the sense of Teichmüller if $f \circ g^{-1}$ is homotopic to a conformal mapping between S_2 and S_1. The equivalence classes are the points of the* ***Teichmüller space****, denoted $T(S)$.*

Remember that every Riemann surface S is isomorphic to a quotient $\widetilde{S}/G$. Suppose $\widetilde{S}$ is conformally isomorphic to the upper half plane $\mathbb{H}$. We can then define an analytic projection

$$p : \mathbb{H} \to \mathbb{H}/G = S \tag{3.25}$$

Now consider the subgroups that are conjugated to G, for instance $G' = BGB^{-1}$ where B belongs to the group of conformal automorphisms of $\mathbb{H}$. Then B is a one-to-one conformal mapping of $S' = \mathbb{H}/G'$ on $S = \mathbb{H}/G$. Obviously we can define some p' just like above. We can study the converse problem. Consider a mapping $g : S' \to S$. It induces a mapping $\widetilde{g} : \widetilde{S'} \to \widetilde{S}$. $\widetilde{g}$ will naturally satisfy

$$p \circ \widetilde{g} = g \circ p' \tag{3.26}$$

If g is given conformal, $\widetilde{g}$ is also conformal and $\widetilde{g} = B$ and $G' = BGB^{-1}$.

Now if we are given g which is not conformal, for $B' \in G'$, we still have that $B = \widetilde{g} \circ B' \circ \widetilde{g}^{-1}$ is an element of G. Keeping the notations of Ahlfors, $\widetilde{g}$ defines an isomorphism θ such that

$$\theta(B') = \widetilde{g} \circ B' \circ \widetilde{g}^{-1} \tag{3.27}$$

This isomorphism is not unique, $\widetilde{g}$ can be replaced by $A \circ \widetilde{g} \circ A'$ for some $A \in G$ and $A' \in G'$. It then defines another isomophism $\widehat{\theta}$ which is equivalent to θ. Ahlfors proves the following lemma in his lectures :

Lemma 1. *Given two mappings, they determine equivalent isomorphisms if and only if they are homotopic.*

Two isomorphisms will correspond to the same Teichmüller point if and only if they are conjugated, ie they differ by an inner automorphism.

3.7.3 Beltrami differentials

Going back to our definition of Teichmüller spaces, if we are given a sense-preserving q.c.-mapping $f : S \to S_0$, it induces a q.c.-mapping $\widetilde{f} : \mathbb{H} \to \mathbb{H}$ and thus an isomorphism θ. f satisfies

$$\widetilde{f} \circ B' = B \circ \widetilde{f} \tag{3.28}$$

Obviously, every such map $\widetilde{f}$ also induces a map $f : S \to S_0$. Let us derive (3.28) :

$$(\frac{\partial f}{\partial z} \circ B')dB' = (dB \circ f)\frac{\partial f}{\partial z} \tag{3.29}$$

$$(\frac{\partial f}{\partial \bar{z}} \circ B')\overline{dB'} = (dB \circ f)\frac{\partial f}{\partial \bar{z}} \tag{3.30}$$

Remember we introduced the notion of complex dilatation (3.18). Clearly with (3.29) and (3.30) we get

$$\mu_f = (\mu_f \circ B')\overline{dB'}/dB' \tag{3.31}$$

More generally, if we are given a measurable and essentially bounded function μ satisfying, for all $B' \in G'$,

$$\mu(B'z) = \mu(z)\frac{B'(z)}{\overline{B'(z)}} \tag{3.32}$$

we say that μ is a **Beltrami differential.**

Note. *By **essentially bounded**, we mean that there exists a constant $C \geq 0$ for which the set $|\mu(z)| > C$ is of measure zero.*

The above definition is due to Ahlfors. For more simplicity, we will use the following notations : given a quasiconformal map f, one can associate its Bletrami differential defined by

$$\mu_f = \frac{\bar{\partial} f}{\partial f}\frac{d\bar{z}}{dz}$$

with the property that the norm of μ_f is no more than 1: $||\mu_f||_\infty < 1$. More generally, we will identify the Beltrami differential of $f : \mathbb{C} \to \mathbb{C}$ with the function $\frac{\bar{\partial} f}{\partial f}$.

Note. *This coincides with our definition of the complex dilatation that we introduced in the definition of quasiconformal maps. The coefficient*

$$Dil_f = \frac{1 + ||\mu_f||_\infty}{1 - ||\mu_f||_\infty}$$

also coincides with the definition of quasiconformal mappings : if $Dil_f \leq K$, f is K-quasiconformal.

3.7.4 Beltrami equation

Now, consider the inverse problem : given a Beltrami differential μ with $||\mu||_\infty < 1$, is there a quasiconformal map f such that

$$\mu = \frac{\bar{\partial} f}{\partial f} \tag{3.33}$$

This equation is called a **Beltrami equation** and in fact, the following powerful result ensures that (3.33) has a unique (up to the choice of three fixed points in $\widehat{\mathbb{C}}$) quasiconformal solution :

Theorem 6 (Measurable Riemann mapping theorem)**.** *Given a Beltrami differential μ defined on $\widehat{\mathbb{C}}$ with $||\mu||_\infty < 1$, there exists a quasiconformal map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ solving the Beltrami equation*

$$\mu = \frac{\bar{\partial} f}{\partial f}$$

f is unique up to the choice of three fixed points in $\widehat{\mathbb{C}}$ for f and depends holomorphically on μ.

This is natural to compose a quasiconformal homeomorphism and a holomorphic map to get more general maps. Such maps are called **quasiregular**. We can compose Beltrami differential with these class of function. Given a quasiregular function g and a Beltrami differential μ, we denote by $(g)_*\mu$ their composition (in fact, this is the Beltrami differential of $f \circ g$ where f is the solution of the Beltrami equation $\mu = \frac{\bar{\partial} f}{\partial f}$). μ is said to be f-**invariant** if

$$(f)_*\mu = \mu \text{ a.e.}$$

3.8 Beltrami paths, hybrid leaves

3.8.1 Beltrami paths

Definition 19. *If we are given a path of p.l.-germs $[f_\lambda] \in \mathcal{C}$ with $\lambda \in \mathbb{D}$ and a holomorphic motion $(h_\lambda)_{\lambda \in \mathbb{D}}$ of $\mathbb{C}$ over $\mathbb{D}$ for which $\lambda_0 = 0$ is the basepoint, both of them such that $[f_0]$ and $[f_\lambda]$ are hybrid equivalent through the conjugacy h_λ near the filled-Julia set $K(f_0)$, then we say that $[f_\lambda] \in \mathcal{C}$, $\lambda \in \mathbb{D}$ is a **Beltrami path**. The pair $([f_\lambda], h_\lambda)$ is called a **guided Beltrami path**.*

There is an important equivalence between Beltrami paths and Beltrami differentials. Namely, the guided Beltrami paths with a fixed initial point, say f_0, are in a one-to-one correspondance with particular families of holomorphic Beltrami differentials μ_λ on $\mathbb{C}$. These families are such that

- $\mu_0 \equiv 0$
- the differentials μ_λ vanish on $K(f_0)$.
- the differentials μ_λ are f_0-invariant near $K(f_0)$.

In what follows, we might use both of these points of view.

3.8.2 Hybrid leaves

Note. *The polynoms P_c and $P_{\varepsilon c}$ with $\varepsilon = \exp \frac{i2\pi}{d}$ are affinely equivalent, so they belong to the same hybrid class : c is not uniquely defined when the degree d is larger than 2. The multibrot set $\mathcal{M}$ is then symmetric up to a rotation of order $d-1$.*

The hybrid class $\mathcal{H}_c$ is the union of its path connected components. They are called **hybrid leaves** and denoted $\widehat{\mathcal{H}}_c$. $\mathcal{H}_c$ is in fact the union of the hybrid leaves $\widehat{\mathcal{H}}_{\varepsilon^k c}$ for $0 \leq k \leq d-1$.

3.9 Contractibility of the set of expanding circle maps

3.9.1 Definitons and notations

We will denote by $\mathbb{T}$ the unit circle. Consider a real analytic map $g : \mathbb{T} \to \mathbb{T}$

Definition 20. *We say that g is **expanding** if for all $z \in \mathbb{T}$, $|Df^{\circ n}(z)| > 1$ for some $n \geq 1$ (where Df is the derivative of f). We will denote by $\mathcal{E}_d$ the space of real analytic expanding maps of degree d with $g(1) = 1$.*

This is quite obvious that every $g \in \mathcal{E}_d$ can be extended to a holomorphic map $U \to V$ of degree d where $\mathbb{T} \subset U \subset V$ are annuli ; all these extensions will be called **annuli representatives** of g. For more simplicity, we will keep the same notation for the extensions of g. We can define the modulus of g as follows :

$$\text{mod } (g) = \sup \quad \text{mod } (V \setminus (U \cup \mathbb{D})) \tag{3.34}$$

the supremum is taken over all annuli representatives of g. We will say that the sequence $g_n \in \mathcal{E}_d$ converges to some $g \in \mathcal{E}_d$ if there exists a neighbourhood X of $\mathbb{T}$ such that every g_n ca n be extended to a holomorphic map on X with $g_n \to g$ uniformly on X.

Remember that we can lift every $g \in \mathcal{E}_d$ to $\mathbb{R}$. The corresponding **lift** $\widetilde{g}$ is such that

$$\widetilde{g} \quad : \quad \mathbb{R} \to \mathbb{R} \tag{3.35}$$

$$\widetilde{g}(x) \quad = \quad dx + \phi(x) \tag{3.36}$$

Evidently, ϕ is an analytic 1-periodic function with $\phi(0) = 0$. In our further developments, $\mathcal{E} = \mathcal{E}_d$ (we assume we are dealing with p.l.-maps of degree d) and $\mathcal{E}_{\mathbb{R}}$ will denote the subspace of maps g in $\mathcal{E}$ such that $g(\bar{z}) = \overline{g(z)}$ ($\mathbb{R}$-symmetric maps). The following result will be important in order to prove that Beau bounds for real maps implies Beau bounds for complex maps in our particular situation.

Theorem 7. *The space $\mathcal{E}$ and its subspace $\mathcal{E}_{\mathbb{R}}$ are both contractible.*

3.9.2 Proof of contractibility

As suggested above, the proof will be done with the lifts $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ and denote by $\widetilde{\mathcal{E}}$ the corresponding space. We let $\widetilde{\mathcal{E}}_1$ stand for the set of of functions $g \in \widetilde{\mathcal{E}}$ such that for all $x \in \mathbb{R}$, $|g'(x)| > 1$.

Lemma 2. *$\widetilde{\mathcal{E}}_1$ is a convex set.*

Proof. take $\lambda \in [0,1]$ and $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{\mathcal{E}}_1$. Define $\widetilde{h} = \lambda \widetilde{g}_1 + (1-\lambda)\widetilde{g}_2$. Then, for all $x \in \mathbb{R}$,

$$|\widetilde{h}'(x)| \leq \lambda |\widetilde{g}_1'(x)| + (1-\lambda)|\widetilde{g}_2'(x)| < \lambda.1 + (1-\lambda).1 = 1$$

Then $\widetilde{h} \in \widetilde{\mathcal{E}}_1$ and $\widetilde{\mathcal{E}}_1$ is convex. □

It is well known that a map $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ is said to preserve the Lebesgue measure if, denoting the latest by λ_*, we have that for any measurable set $A \subset \mathbb{R}$:

$$\lambda_*(\widetilde{g}^{-1}(A)) = \lambda_*(A)$$

But denoting $\widetilde{\mathcal{E}}_*$ the space of maps preserving the Lebesgue measure, we have $\widetilde{\mathcal{E}}_* \subset \widetilde{\mathcal{E}}_1$. But now, by convexity of $\widetilde{\mathcal{E}}_1$ and by Proposition 2, $\widetilde{\mathcal{E}}_1$ is contractible, ie homotopic to a point $\widetilde{g}^* \in \widetilde{\mathcal{E}}_1$ and one can choose $\widetilde{g}^*$ such that $\widetilde{g}^* \in \widetilde{\mathcal{E}}_*$. Let $r_t : \widetilde{\mathcal{E}}_1 \to \widetilde{\mathcal{E}}_1$, $t \in [0,1]$ be an homotopy such that $r_0 \equiv Id$ and $r_1 \equiv constant = \widetilde{g}^*$. Then restricting this homotopy to $\widetilde{\mathcal{E}}_*$, $\widetilde{\mathcal{E}}_*$ becomes itself contractible.

Now we are going to build a projection $\Pi : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}_*$. We assume that any $g \in \widetilde{\mathcal{E}}$ has an absolutely continuous invariant measure, that we will denote $d\mu = \rho d\theta$. $\rho(\theta) > 0$ is a real analytic density. Now, consider a real analytic circle diffeomorphism $h(t) = \int_0^t \rho(\theta) d\theta$ such that $h(d\mu) = d\theta$. Then the map $\widetilde{G} = \Pi(\widetilde{g}) = h \circ \widetilde{g} \circ h^{-1}$ preserves the Lebesgue measure because $\widetilde{G}(d\theta) = d\theta$. So $G \in \widetilde{\mathcal{E}}_*$. We have built the desired projection. Moreover, the space of densities ρ is easily a convex space (take $\lambda\rho_1 + (1-\lambda)\rho_2 ...$) so it is contractible. But the space H of diffeomorphism h constructed above is identified with this space of densities, so it is also contractible. it follows that we can build a continuous map

$$F : \widetilde{\mathcal{E}} \times [0,1] \to \widetilde{\mathcal{E}}$$

such that, for all $\widetilde{g} \in \widetilde{\mathcal{E}}$ and $\widetilde{g}^* \in \widetilde{\mathcal{E}}_*$,

$$F(\widetilde{g}, 0) = \widetilde{g},, \; F(\widetilde{g}, 1) \in \widetilde{\mathcal{E}}_* \text{ and } F(\widetilde{g}^*, t) = \widetilde{g}^*$$

One way to construct such a F is to choose, for each $\widetilde{g} \in \widetilde{\mathcal{E}}$, the homotopty $s_t : H \to H$ such that $s_0 = Id$ and $s_1 = h_{\widetilde{g}}$, where $h_{\widetilde{g}}$ is the diffeomorphism such that $\widetilde{G} = \Pi(\widetilde{g}) = h_{\widetilde{g}} \circ \widetilde{g} \circ h_{\widetilde{g}}^{-1}$ preserves the Lebesgue measure. Then we take

$$F(\widetilde{g}, t) = s_t(Id) \circ \widetilde{g} \circ (s_t(Id))^{-1}$$

Such a F is called a **deformation retraction**. It follows that $\widetilde{\mathcal{E}}_*$ is a deformation retract for $\widetilde{\mathcal{E}}$, so $\widetilde{\mathcal{E}}$ is itself contractible. The result follows for $\mathcal{E}$ and thus for $\mathcal{E}_{\mathbb{R}}$.

3.10 External map

3.10.1 externally equivalence, external map

Douady and Hubbard defined the notion of external map for a p.l.-mapping in their work on the dynamics of p.l.-mappings. We will begin with the externally equivalence between two p.l.-mappings.

Definition 21. *Given two p.l.-mappings $f : U \to U'$ and $g : V \to V'$, with $K(f)$ and $K(g)$ connected, we say that f and g are* ***externally equivalent*** *($f \sim_{ext} g$) if there exists open sets U_0, U'_0, V_0 and V'_0 such that*

$$K(f) \subset U_0 \subset U'_0 \subset U' \tag{3.37}$$

$$K(g) \subset V_0 \subset V'_0 \subset V' \tag{3.38}$$

with

$$f^{-1}(U'_1) = U_1 \tag{3.39}$$

$$g^{-1}(V'_1) = V_1 \tag{3.40}$$

together with the existence of an analytic isomorphism $\phi : U'_1 \setminus K(f) \to V'_1 \setminus K(g)$ such that $\phi \circ f = g \circ \phi$.

Now, for a given p.l.-mapping $f : U \to U'$ of degree d ($[f] \in \mathcal{C}$), there exists a unique, up to conjugation by a rotation, expanding circle endomorphism $h_f : \mathbb{T} \to \mathbb{T}$ of degree d which is externally equivalent to f. We can normalize it so that $h_f \in \mathcal{E}$. h_f is called the **external map** of f and we can define a projection $\pi : \mathcal{C} \to \mathcal{E}$ which associates to each germ $[f] \in \mathcal{C}$ its external map h_f. We also have $\mod [f] = \mod [h_f]$. Douady and Hubbard show how to construct the external map for a connected $K(f)$ and in the general case. Here is a proof in the case of a connected filled Julia set.

Proof: Let $\log R = \mod (U' \setminus K(f))$ and $\Omega'_+ = \{z : 1 < |z| < R\}$. We denote by ρ the reflection with respect to the unit circle : for all $z \in \mathbb{C}$, $\rho(z) = \bar{z}^{-1}$. Let $\Omega'_- = \rho(\Omega_+)$ and $\Omega' = \Omega'_+ \cup \Omega'_- \cup \partial\mathbb{D}$.
Now let

$$\varphi : U' \setminus K(f) \to \Omega'_+$$

be an isomorphism such that for every sequence $\{z_n\}_{n \geq 0}$ with

$$\lim_{n\to\infty} d(z_n, K(f)) = 0,$$

we have :

$$\lim_{n\to\infty} |\varphi(z_n)| = 1$$

Define $\Omega_+ = \varphi(U \setminus K(f))$, $\Omega_- = \rho(\Omega_+)$ and $\Omega = \Omega_+ \cup \Omega_- \cup \partial\mathbb{D}$. We construc a map

$$h_+ : \Omega_+ \to \Omega'_+$$

such that

$$h_+ = \varphi \circ f \circ \varphi^{-1}$$

Thanks to the Schwarz reflection principle stated in the appendix, h_+ can be extended to a holomorphic map $h : \Omega \to \Omega'$. If we denote by h_f the restriction of h to $\mathbb{T}$, we get the external map of f. Indeed, denoting by $\widetilde{h} : \widetilde{\Omega} \to \widetilde{\Omega}'$ the lift to the universal coverings we see that $\widetilde{h}$ is an isomorphism and using Schwarz lemma on $\widetilde{h}^{-1}$, $\widetilde{h}^{-1}$ is striclty contracting for the hyperbolic metric because $\widetilde{\Omega} \subset \widetilde{\Omega}'$; thus h is expanding. Moreover, $\mod [f] = \mod [h_f]$, for the mapping φ establishes a one-to-one correspondence between the fundamental annuli of f and h_f.

3.10.2 Matings

Her we would like to construct a mating between any $c \in \mathcal{M}$ and $g \in \mathcal{E}$. It would consist of a map $\mathcal{M} \times \mathcal{E} \to \mathcal{C}$ continuous both in c and g. Note that the inverse would be a map $(\pi, \chi) : \mathcal{C} \to \mathcal{E} \times \mathcal{M}$ giving the external map of a $[f] \in \mathcal{C}$ and its straightening $\chi(f) \in \mathcal{M}$.

Construction of the mating Suppose we are given $c \in \mathcal{M}$ and the corresponding polynomial P_c and an expanding map $g \in \mathcal{E}$.

- First, we easily see that the set of non escaping points for $P_0 = z^d$ is $\mathbb{D}$. Thus $K(P_0) = \overline{\mathbb{D}}$. Given any $c \in \mathcal{M}$, we can construct a holomorphic one-to-one function on the complement of $K(P_0)$:

  $$\xi_c : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(P_c)$$

  such that

  $$\xi_c \circ P_0 = P_c \circ \xi_c$$

  and that ξ_c is tangent to the identity at ∞.

- Now, define $g_0 : z \mapsto z^d$. We link g_o to g with a continious path g_t for $t \in [0, 1]$ ($g_1 = g$). We assume that we can construct continuous quasiconformal maps $h_t : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \mathbb{D}$ that conjugate any g_t to g_0 near the unit circle $\mathbb{T}$, ie

  $$h_t \circ g_0 = g_t \circ h_t$$

  Suppose the h_t have continuously depending Beltrami differentials. In that case, we can define the Beltrami differentials of the functions $h_t \circ \xi_c^{-1}$. We will call μ_t the extension of these Beltrami differentials to the whole complex plane $\mathbb{C}$ such that $\mu_t \equiv 0$ on $K(P_c)$. μ_t is easily invariant under P_c on a Jordan disc containing $K(P_c)$.

- Now, denote by ϕ_t the solution of the Beltrami equation

  $$\frac{\overline{\partial}\phi_t}{\partial \phi_t} = \mu_t$$

Define

$$f_t = \phi_t \circ P_c \circ \phi_t^{-1}$$

By invariance of μ, f_t is a holomorphic map in a neighbourhood of $K(f_t)$ (because $K(f_t) = \phi_t(K(P_c))$). We would like $f = f_1$ to define a germ $[f] \in \mathcal{H}_c$, that is why we require $t \mapsto \phi_t$ to be continuous with $\phi_0 = Id$.

- We have construct $[f] \in \mathcal{H}_c$ starting from $c \in \mathcal{M}$ and $g \in \mathcal{E}$. But we used a particular construction. We have to show that $[f]$ does not depend on this construction, but only on c and g. We also need to check that g is indeed the external map of $[f]$.

Lemma 3. *For every $t \in [0,1]$, the map g_t is an external map of f_t.*

Proof : Construct an external map for f_t as in the previous section : we set

$$\psi_t : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(f_t)$$

to be the family of holomorphic one-to-one mappings normalized such that $\psi_0 = \xi_c$ and

$$\widehat{g}_t = \psi_t^{-1} \circ f_t \circ \psi_t$$

As in the construction of the external map for any p.l.-mapping, we can extend $\widehat{g}_t$ analytically accross $\mathbb{T}$ by use of the Schwarz reflection principle. Note that we require $\widehat{g}_t$ to be such that $\widehat{g}_t(1) = 1$ in order to be an external map for f_t. Now, set

$$\zeta_t : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \mathbb{D}$$

to be the quasiconformal map such that

$$\zeta_t = \psi_t^{-1} \circ \phi_t \circ \xi_c$$

Then

$$\zeta_t \circ g_0 = \psi_t^{-1} \circ \phi_t \circ \xi_c \circ g_0$$

and because $f_t \circ \phi_t = \phi_t \circ P_c$ and $\xi_c \circ P_0 = P_c \circ \xi_c$,

$$\widehat{g}_t \circ \zeta_t = \psi_t^{-1} \circ f_t \circ \psi_t \circ \psi_t^{-1} \circ \phi_t \circ \xi_c = \psi_t^{-1} \circ \phi_t \circ P_c \circ \xi_c = \psi_t^{-1} \circ \phi_t \circ \xi_c \circ g_0$$

So ζ_t is conjugating g_0 to $\widehat{g}_t$ and its Beltrami differential coincides with the one of h_t. Then $\sigma_t = \zeta_t \circ h_t^{-1}$ is a rotation that conjugates g_t with $\widehat{g}_t$:

$$\sigma_t \circ g_t = \widehat{g}_t \circ \sigma_t$$

But $g_t \in \mathcal{E}$ so $g_t(1) = 1$, and using the previous equality, this gives $\widehat{g}_t \circ \sigma_t(1) = \sigma_t(1)$. As $\sigma_0 = Id$, by continuity we get $\sigma_t(1) = 1$. But σ_t is a rotation fixing 1 so $\sigma_t = Id$ and in the end,

$$\widehat{g}_t = g_t$$

So g_t is an external map for f_t.

The mating $[f]$ does not depend on the above construction First, suppose that the path g_t is chosen. We will begin by showing that with g_t given, f_t does not depend on the choice of the family of conjugacies h_t. To prove this, let h'_t ba another family of continuous quasiconformal conjugacies with $\rho_t = h_t^{-1} \circ h'_t$ commuting with g_0 near $\mathbb{T}$:

$$\rho_t \circ g_0 = g_0 \circ \rho_t$$

Now let us have a look at $\varrho_t = \xi_c \circ \rho_t \circ \xi_c^{-1}$. Then

$$P_c \circ \varrho_t = P_c \circ \xi_c \circ \rho_t \circ \xi_c^{-1} = \xi_c \circ P_0 \circ \rho_t \circ \xi_c^{-1} = \xi_c \circ \rho_t \circ P_0 \circ \xi_c^{-1} = \xi_c \circ \rho_t \circ \xi_c^{-1} \circ P_c$$

Thus $P_c \circ \varrho_t = \varrho_t \circ P_c$. Next, we extend ϱ_t to the whole plane $\mathbb{C}$ setting $\varrho_t = Id$ on $K(P_c)$. It can be shown, using the pullback argument, that ϱ_t is a quasiconformal homeomorphism. We will assume it for the end of the proof.

Now, if we denote by $(g)_*\mu_f$ the Beltrami differential of $f \circ g$ (where μ_f is the Beltrami differential of f- this the notation introduced in the section where we defined Beltrami differentials), we can build μ'_t as the Beltrami differential of $h'_t \circ \xi_c$ and extend to the whole plane setting $\mu'_t \equiv 0$ on $K(P_c)$, as it was done in our construction of the mating. But denoting by ν'_t the Beltrami differentials of h'_t, this also gives $\mu'_t = (\xi_c)_*\nu'_t$ and as ν'_t is the Beltrami differential of $h'_t = h_t \circ \rho_t$, we get $\nu'_t = (\rho_t)_*\nu_t$, where ν_t is the Beltrami differential of h_t. Putting everything together,

$$\mu'_t = (\xi_c)_*((\rho_t)_*\nu_t) = (\xi_c \circ \rho_t \circ \xi_c^{-1})_*\mu_t = (\varrho_t)_*\mu_t$$

But as ϕ_t solves the Beltrami equation for μ_t, we get that μ'_t is the Beltrami differential of $\phi'_t = \phi_t \circ \varrho_t$. Then defining the family of mappings $f'_t = \phi'_t \circ P_c \circ \phi_t'^{-1}$, as it was done in our initial construction, we get that $[f'] = [f'_1]$ is the mating of P_c and g_t with quasiconformal conjugacies h'_t. As ϱ_t is commuting with P_c near $K(P_c)$, we get that

$$f'_t = \phi'_t \circ P_c \circ \phi_t'^{-1} = \phi_t \circ \varrho_t \circ P_c \circ \varrho_t^{-1} \circ \phi_t^{-1} = \phi_t \circ P_c \circ \varrho_t \circ \varrho_t^{-1} \circ \phi_t^{-1} = \phi_t \circ P_c \circ \phi_t^{-1} = f_t$$

This proves that the mating does not depend on the family of conjugacies h_t.

Now, it remains to show that the endpoint of the family f_t, ie $f = f_1$, does not depend on the path that links g_0 to g. Suppose that g'_t is another path connecting g_0 to g. Then, as the space $\mathcal{E}$ is simply connected, we can choose an homotopy g_t^s for $s \in [0,1]$ connecting $g_t^0 = g_t$ to $g_t^1 = g'_t$ for all $t \in [0,1]$. We can then construct, for every fixed $s \in [0,1]$, the mating of P_c and the path g_t^s, denoted $[f_t^s] \in \mathcal{C}$. But we can choose the corresponding hybrid conjugacies ϕ_t^s (so that $f_t^s = \phi_t^s \circ P_c \circ (\phi_t^s)^{-1}$) in such a way that they depend continuously on t and s. Then for every s, g is an external map for f_1^s. If we denote by

$$\psi^s : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(f_1^s)$$

to be the holomorphic one-to-one mapping conjugating f_1^s to g, we can choose these conjugacies ψ^s so that (ψ^s) is a continuous family. Now define $\varrho^s : \mathbb{C} \to \mathbb{C}$ such that

$$\varrho^s = ((\psi^s)^{-1} \circ \phi_1^s)^{-1} \circ ((\psi^0)^{-1} \circ \phi_1^0)$$

outside $K(P_c)$ and $\varrho^s = Id$ on $K(P_c)$. Then after some computation, ϱ^s commutes with $P_c$in anouter neighborhood of $K(P_c)$. Using again an argument of pullback, we can show that ϱ^s is a quasiconformal homeomorphism. The hybrid conjugacy between f_1^0 and f_1^s then takes the form

$$\tau^s = \phi_1^s \circ \varrho^s \circ (\phi_1^0)^{-1}$$

We see that

$$\tau^s = \phi_1^s \circ (\phi_1^s)^{-1} \circ \psi^s \circ (\psi^0)^{-1} \circ \phi_1^0 \circ (\phi_1^0)^{-1} = \psi^s \circ (\psi^0)^{-1}$$

But the ψ^s are holomorphic and one-to-one, which means that tau^s is an analytic conformal mapping outside $K(f_1^0)$. Thus τ^s is a hybrid conjugacy and it is conformal outside $K(f_1^0)$, so it is affine. Moreover, $\tau^0 = Id$. The germs $[f_1^s]$ are normalized such that τ^s is tangent to $z \mapsto e^{2\pi ik/(d-1)}$ for some $k \in \mathbb{Z}/(d-1)\mathbb{Z}$. But if $k \neq 0$ for some s, it contradicts the continuity of $s \mapsto \tau^s$. Then $\tau^s = Id$ for all $s \in [0,1]$. In particular, the initial and final points coincide : $f_1^0 = f_1^1$. We conclude by saying that we have construct a well defined mapping (called the inter mating) $\mathcal{E} \times \mathcal{M} \to \mathcal{C}$. We will denote the mating of P_c for $c \in \mathcal{M}$ and $g \in \mathcal{E}$ by $f = i_c(g) \in \mathcal{C}$

The intern mating is a homeomorphism Here we are going to give an outline of the proof of the following : the intern mating

$$\begin{cases} \mathcal{E} \times \mathcal{M} \to \mathcal{C} \\ (g, c) \mapsto i_c(g) \end{cases}$$

is a continuous and bijective.

- We begin with the continuity ; the continuity with respect to g, uniformly with respect to c is directly given by the construction of the mating. So we just need to show that for a given $g \in \mathcal{E}$, $c \mapsto i_c(g)$ is continuous. We take a sequence $c_n \in \mathcal{M}$ such that $c_n \to c$, as in the initial construction, we connect g_0 to g with a path g_t. In that way, we build a sequence of paths $f_{t,n} = i_{c_n}(g_t)$ and it is reasonable to say that taking a subsequence, $(f_{t,n})$ converges uniformly to a path f_t. Then the path f_t lies into $\widehat{\mathcal{H}}_c$, this means that f_t and P_c are hybrid conjugate. The proof is a little bit technical. Now, if we let

$$\varphi_{t,n} : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(f_{t,n})$$

be the map that conjugates each $f_{t,n}$ to its external map g_t and taking the limit as $n \to \infty$, we define a φ_t and g_t is the external map of f_t through the conjugacy φ_t, ie $f_t = i_c(g_t)$. We conclude by saying that

$$\lim_{n\to\infty} i_{c_n}(g) = \lim_{n\to\infty} f_{1,n} = f_1 = i_c(g)$$

which is the definition of the continuity of $c \mapsto i_c(g)$.

- Now we look at the bijectivity as follows : we remark that a polynomial P_c has a single preimage (P_0, c) through the mating. Now if we connect P_c to any map $f \in \widehat{\mathcal{H}}_c$ with a path, since $i_c^{-1}(P_c) = P_0$, each $i_c : \mathcal{E} \to \widehat{\mathcal{H}}_c$ is a bijection. But in that case, $\widehat{\mathcal{H}}_c$ contains a single polynomial (P_c) and all the hybrid leaves are distinct. Thus we have a bijective mating because $\mathcal{C}$ is the union of these hybrid leaves.

- we conclude by saying that the mating is a homeomorphism $\mathcal{E} \times \mathcal{M} \to \mathcal{C}$.

The inverse of the mating will be denoted $(\pi, \chi) : \mathcal{C} \to \mathcal{E} \times \mathcal{M}$. $\pi(f)$ will give the canonical external map in $\mathcal{E}$ of any $[f] \in \mathcal{C}$ and $\chi(f) \in \mathcal{M}$ will give the canonical straightening of $[f]$.

Note. *1. A function f is said to be equivariant with respect to complex conjugation if we have $\bar{f} \circ Id = f \circ \overline{Id}$. The construction of the mating is equivariant with repect to complex conjugation, ie the external map, the straightening and the mating are equivariant with respect to complex conjugation.*

2. $i_{c_1} \circ i_{c_2}^{-1}$ transforms Beltrami paths in $\widehat{\mathcal{H}}_{c_2}$ into Beltrami paths in $\widehat{\mathcal{H}}_{c_1}$

So now we can regroup the above proofs into a final result :

Theorem 8. *The inverse of the mating constructed above gives a canonical choice of the straightening $\chi(f) \in \mathcal{M}$ and the external map $\pi(f) \in \mathcal{E}$ associated to every germ $[f] \in \mathcal{C}$. Moreover*

- *$(\pi, \chi) : \mathcal{C} \to \mathcal{E} \times \mathcal{M}$ is a homeomorphism.*

- *Given $c \in \mathcal{M}$, the associated hybrid leave $\widehat{\mathcal{H}}_c$ is the preimage of the straightening c : $\widehat{\mathcal{H}}_c = \chi^{-1}(c)$. Then π restrict to a homeomorphism*

$$\pi_c : \widehat{\mathcal{H}}_c \to \mathcal{E}$$

We denote by i_c its inverse and we get the canonical mating.

Chapter 4

Carathéodory metric and Schwarz lemma

4.1 Schwarz lemma and hyperbolic metric

4.1.1 Schwarz lemma

We now state the famous Schwarz lemma, saying that a holomorphic map of $\mathbb{D}$ either contracts $\mathbb{D}$ or is a rotation.

Theorem 9 (Schwarz lemma)**.** *Given a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$, we have that $|f'(0)| \leq 1$. Then we have two cases :*

- *if the equality holds, ie $|f'(0)| = 1$, then $f(z) = az$ with $|a| = 1$, meaning that f is a rotation about the origin.*
- *if the inequality is strict, ie $|f'(0)| < 1$, then $|f(z)| < |z|$ for all $z \in \mathbb{D}^*$.*

Proof : The proof is based on the maximum modulus principle. We define

$$g(z) = \begin{cases} \dfrac{f(z)}{z} \text{ if } z \neq 0 \\ f'(0) \text{ if } z = 0 \end{cases}$$

Then $g : \mathbb{D} \to \mathbb{C}$ defines a holomorphic function.

- Note that if $|z| = r$ for some $r < 1$, then $|g(z)| \leq \frac{1}{r}$. By the maximum modulus principle, we have that, for all $z \in D(0, r)$, $|g(z)| \leq \frac{1}{r}$. Taking the limit as $r \to 1$, we get $|g(z)| \leq 1$ for all $z \in \mathbb{D}$.
- The case of equality is equivalent to $|g(z)| = 1$ for some $z \in \mathbb{D}$. By the maximum modulus principle, g is constant. So $g(z) = a$ for some $|a| = 1$: $f(z) = az$ is a rotation.

- If we exclude the previous case, we have that $\left|\frac{f(z)}{z}\right| < 1$ for all $z \neq 0$ and $|f'(0)| < 1$.

4.1.2 Hyperbolic metric on $\mathbb{D}$

In what follows, we will denote by $|dz|$ the common norm in $\mathbb{C}$. Let S be any Riemann surface.

Definition 22. *Given a positive and smooth function ρ defined on S, we define a* ***conformal metric*** *$\rho|dz|$ on S. We will also denote the metric ρ.*

We then define the Poincaré/hyperbolic metric on $\mathbb{D}$:

Definition 23 (Hyperbolic metric on $\mathbb{D}$)**.** *The conformal metric with $\rho_{\mathbb{D}}(z) = \dfrac{2}{1-|z|^2}$ on $\mathbb{D}$ is defined to be the* ***hyperbolic metric*** *on $\mathbb{D}$.*

This hyperbolic metric induces a distance in $\mathbb{D}$: let $z, w \in \mathbb{D}$, and join z and w with a smooth path γ in $\mathbb{D}$. We define the hyperbolic length of γ by

$$l(\gamma) = \int_\gamma \rho_{\mathbb{D}}(z)|dz| \tag{4.1}$$

We can then define the **hyperbolic distance** between z and w by taking the infinimum over all paths γ joining z and w and lying in $\mathbb{D}$.

$$d_{\mathbb{D}}(z, w) = \inf_\gamma l(\gamma) \tag{4.2}$$

Definition 24. *Here we define the notion of isometry, which is important because we will see soon that it is linked to automorphisms of the disk.*

- *We say that a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is an* ***isometry*** *with respect to the hyperbolic metric if for all $z \in \mathbb{D}$,*

$$\rho_{\mathbb{D}}(f(z))|f'(z)| = \rho_{\mathbb{D}}(z) \tag{4.3}$$

- *we say that f is an* ***isometry*** *with respect to the hyperbolic distance if for all $z, w \in \mathbb{D}$,*

$$d_{\mathbb{D}}(f(z), f(w)) = d_{\mathbb{D}}(z, w) \tag{4.4}$$

We let $Aut(S)$ stand for the group of all conformal automorphisms of a given Riemann surface S. It can be shown that $Aut(\mathbb{D})$ consists of Möbius maps of the form

$$z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z} \tag{4.5}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. In fact, the group of automorphisms and the group of isometries coincide.

Theorem 10. *Given a holomorphic map $f : \mathbb{D} \to \mathbb{D}$, the following statements are equivalent :*

1. *$f \in Aut(\mathbb{D})$*
2. *f is an isometry w.r.t. the hyperbolic metric.*
3. *f is an isometry w.r.t. the hyperbolic distance.*

For more simplicity, in the two last cases we will just say that f is an isometry.

Proof :

- 1. implies 2. because $f(z) = e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. In particular,

$$f'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2} \tag{4.6}$$

  Thus we have

$$\rho_{\mathbb{D}}(f(z))|f'(z)| = \frac{2}{1-|f(z)|^2}|f'(z)| = \frac{2}{1-|z|^2} = \rho_{\mathbb{D}}(z) \tag{4.7}$$

- We now show that 2. implies 1. Given an isometry f of the hyperbolic metric, we can choose $g \in Aut(\mathbb{D})$ such that $g(f(0)) = 0$. Consider $h = g \circ f$. Then by assumption we have that

$$\rho_{\mathbb{D}}(h(0))|h'(0)| = 2|h'(0)| = \rho_{\mathbb{D}}(0) = 2$$

  Thus $|h'(0)| = 1$ and $h(0) = 0$; by Schwarz lemma, h is a rotation that fixes 0, meaning that $h \in Aut(\mathbb{D})$. Then $f = g^{-1} \circ h \in Aut(\mathbb{D})$ by group properties.

- To prove that 1. implies 3., we take $f \in Aut(\mathbb{D})$ - so f is also an isometry with respect to the hyperbolic metric- and we look at $l(f \circ \gamma)$ for any path γ lying in $\mathbb{D}$:

$$l(f \circ \gamma) = \int_{f\circ\gamma} \rho_{\mathbb{D}}(w)|dw| = \int_{\gamma} \rho_{\mathbb{D}}(f(z))|f'(z)||dz| = l(\gamma)$$

  Consequently, for all $z, w \in \mathbb{D}$, we have that $d_{\mathbb{D}}(f(z), f(w)) \leq d_{\mathbb{D}}(z, w)$. But $f \in Aut(\mathbb{D})$ so we can apply the same argument for f^{-1}, leading to $d_{\mathbb{D}}(z, w) \leq d_{\mathbb{D}}(f(z), f(w))$, ie f is an isometry with respect to the hyperbolic distance.

- Finally, we show that 3. implies 1. Let f be a holomorphic isometry with respect to the hyperbolic distance. We choose $g \in Aut(\mathbb{D})$ such that $g(f(0)) = 0$. We consider $h = g \circ f$. h is a holomorphic isometry such that $h(0) = 0$ and using the assumption,

$$d_{\mathbb{D}}(h(0), h(z)) = d_{\mathbb{D}}(0, h(z)) = d_{\mathbb{D}}(0, z)$$

  This means that $|h(z)| = |z|$; by Schwarz lemma, h is a rotation about the origin so $h \in Aut(\mathbb{D})$ and $f = g^{-1} \circ h \in Aut(\mathbb{D})$.

We have an explicit expression for the hyperbolic metric, but it could be useful to have an explicit form for the hyperbolic distance too. The next result will not be proved.

Theorem 11. *Given $z, w \in \mathbb{D}$, the explicit form of the hyperbolic distance is*

$$d_{\mathbb{D}}(z, w) = 2 \tanh^{-1} \left| \frac{z - w}{1 - z\bar{w}} \right| \tag{4.8}$$

Another useful result on the hyperbolic metric is that it is the only Riemannian metric that is invariant under the action of $Aut(\mathbb{D})$, up to a constant. This enables us to give a stronger form of the Schwarz lemma. Note that we can also define a distance $\delta_{\mathbb{D}}$ such that

$$\delta_{\mathbb{D}}(z, w) = \left| \frac{z - w}{1 - z\bar{w}} \right| = \frac{\exp d_{\mathbb{D}}(z, w) - 1}{\exp d_{\mathbb{D}}(z, w) + 1} \tag{4.9}$$

$\delta_{\mathbb{D}}$ induces the unique invariant metric under $Aut(\mathbb{D})$ such that $\delta_{\mathbb{D}}(0, z) = |z|$. We call $\delta_{\mathbb{D}}$ the **pseudo-hyperbolic** distance. We will now state the stronger form of the Schwarz lemma with respect to these two metrics.

Theorem 12 (The Schwarz-Pick lemma)**.** *Suppose we are given some holomorphic function $f : \mathbb{D} \to \mathbb{D}$. Then either*

- *f contracts the hyperbolic distance and the pseudo-hyperbolic distance : for all $z, w \in \mathbb{D}$*

$$d_{\mathbb{D}}(f(z), f(w)) < d_{\mathbb{D}}(z, w) \tag{4.10}$$
$$\rho_{\mathbb{D}}(f(z))|f'(z)| < \rho_{\mathbb{D}}(z) \tag{4.11}$$
$$\delta_{\mathbb{D}}(f(z), f(w)) < \delta_{\mathbb{D}}(z, w) \tag{4.12}$$

- *f is an isometry : $f \in Aut(\mathbb{D})$ and for all $z, w \in \mathbb{D}$,*

$$d_{\mathbb{D}}(f(z), f(w)) = d_{\mathbb{D}}(z, w) \tag{4.13}$$
$$\rho_{\mathbb{D}}(f(z))|f'(z)| = \rho_{\mathbb{D}}(z) \tag{4.14}$$
$$\delta_{\mathbb{D}}(f(z), f(w)) = \delta_{\mathbb{D}}(z, w) \tag{4.15}$$

Proof : for the case of an isometry, this is the same result than theorem 10. Suppose f is not an isometry and take $z, w \in \mathbb{D}$. Choose $g, h \in Aut(\mathbb{D})$ such that $g(z) = 0$ and $h(f(z)) = 0$. Consider $u = h \circ f \circ g^{-1}$. Then u is holomorphic and fixes 0 ; it is not an isometry (as g and h are isometries, if u was an isometry, then f would be an isometry too, but we supposed the contrary). By Schwarz lemma, for all $z \in \mathbb{D}$

$$d_{\mathbb{D}}(0, u(z)) < d_{\mathbb{D}}(0, z),\ \delta_{\mathbb{D}}(0, u(z)) < \delta_{\mathbb{D}}(0, z) \text{ and } |u'(0)| < 1$$

But $ug = hf$, and as g, h are ismoetries, we have :

$$d_{\mathbb{D}}(f(z), f(w)) = d_{\mathbb{D}}(h\circ f(z), h\circ f(w)) = d_{\mathbb{D}}(u\circ g(z), u\circ g(w)) = d_{\mathbb{D}}(0, u\circ g(w)) < d_{\mathbb{D}}(0, g(w))$$

But

$$d_{\mathbb{D}}(0, g(w)) = d_{\mathbb{D}}(g(z), g(w)) = d_{\mathbb{D}}(z, w)$$

Finally,

$$d_{\mathbb{D}}(f(z), f(w)) < d_{\mathbb{D}}(z, w)$$

Obviously, we can do the same for $\delta_{\mathbb{D}}$. To obtain the "metric" inequality, we only need to derive the equality $ug = hf$.

4.2 Hyperbloic Riemann surfaces

Remember our brief study of Riemann surfaces : every Riemann surface S is conformally isomorphic to a quotient $\widetilde{S}/G$, $\widetilde{S}$ being a simply connected Riemann surface called the universal covering of S. By uniformization theorem, it is conformally isomorphic either to $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{D}$.

Definition 25. *We say that a Riemann surface S is **hyperbolic** if its universal covering is conformally isomorphic to $\mathbb{D}$, or, equivalently, to the upper half plane $\mathbb{H}$.*

As we have done for the unit disk, we can define a hyperbolic metric ρ_S on any hyperbolic Riemann surface and build its corresponding hyperbolic distance d_S. Here we are going to extend the Schwarz lemma in a natural way : instead of considering holomorphic maps $f : \mathbb{D} \to \mathbb{D}$ we will consider $f : S \to T$ where S and T are two hyperbolic Riemann surfaces.

Theorem 13 (Schwarz-Pick). *Given a holomorphic map $f : S \to T$ between two hyperbolic Riemann surfaces S and T, then f contracts the hyperbolic distance : for $z, w \in S$, we have that*

$$d_T(f(z), f(w)) \leq d_S(z, w) \tag{4.16}$$

If equality holds for some $z \neq w$, then f is a conformal isomorphism between S and T.

4.3 Path holomorphic structure and Carathéodory pseudo-metric

4.3.1 Path holomorphic maps

Arthur Avila and Mikhail Lyubich state the Schwarz lemma in a different way than the previous section. They introduce path holomorphic configurations. If we are given a space X, and a family $H(X)$ of maps $\gamma : \mathbb{D} \to X$, we say that $H(X)$ is a **path holomorphic structure** if the following stands :

- $H(X)$ contains the constant maps
- for all $\gamma \in H(X)$ and holomorphic map $\psi : \mathbb{D} \to \mathbb{D}$, $\gamma \circ \psi \in H(X)$

The elements of $H(X)$ are called **holomorphic paths**, and X is a **path holomorphic space**.

Now we are interested in maps between two path holomorphic spaces. Our goal is to state the Schwarz lemma for such a configuration. Given two path holomorphic spaces X, Y, we say that a map $\phi : X \to Y$ is **path holomorphic** if for every holomorphic path $\gamma : \mathbb{D} \to X$, the composition $\phi \circ \gamma : \mathbb{D} \to Y$ is itself a holomorphic path (with respect to Y). We will denote by $H(X, Y)$ the space of path holomorphic maps from X to Y.

Note.

- *A simple example of a path holomorphic space is a complex Banach manifold. In this configuration, path holomorphic is just the same as holomorphic in the usual sense.*
- *If $Y \subset X$ is a subset of a path holomorphic space X, Y can be viewed as a path holomorphic space : if we denote by $I : Y \to X$ the natural inclusion of Y into X, then $\gamma : \mathbb{D} \to Y \in H(Y)$ if $I \circ \gamma \in H(X)$.*

4.3.2 Carathéodory pseudo-metric

Given a path holomorphic space X, we define the fllowing pseudo-metric, called the **Carathéodory pseudo-metric** : given any $x, y \in X$,

$$\delta_X(x, y) = \sup_{\phi \in H(X,\mathbb{D})} \delta_{\mathbb{D}}(\phi(x), \phi(y)) \tag{4.17}$$

where $\delta_{\mathbb{D}}$ is the distance defined in (4.9). A path holomorphic space X is called **Carathéodory hyperbolic** if the pseudo-metric δ_X is a metric. This is true if and only if bounded path holomorphic maps on X separate points (for $x \neq x'$ in X, there is at least a bounded path holomorphic map ϕ such that $\phi(x) \neq \phi(y)$).

All this background enables us to state a first (weak) form of the Schwarz lemma :

Theorem 14 (weak form of the Schwarz lemma)**.** *Given a path holomorphic map $\phi : X \to Y$ between two Carathéodory hyperbolic spaces X and Y, ϕ is weakly contracting : for any $x, x' \in X$,*

$$\delta_Y(\phi(x), \phi(x')) \leq \delta_X(x, x') \tag{4.18}$$

In particular, using the Schwarz lemma, any subset $Y \subset X$ of a Carathéodory hyperbolic space is itself Carathéodory hyperbolic.

Lemma 4. *Suppose we are given a complex banach space denoted $\mathcal{B}$. Let $\mathcal{B}_1$ stand for the unit ball of $\mathcal{B}$. Then*

- *$\mathcal{B}_1$ is Carathéodory hyperbolic with $\delta_{\mathcal{B}_1}(x, 0) = ||x||$ for every $x \in \mathcal{B}_1$.*
- *A path olomorphic space X is Carathéodory hyperbolic if and only if there exists a holomorphic injection $X \to \mathcal{B}_1$ for some Banach ball $\mathcal{B}_1$.*

If $Y \subset X$ is a subset of a path holomorphic space X, we denote by $diam_X(Y)$ the diameter of Y with respect to the pseudo-metric in X. We say that the subset Y is **small** if $diam_X(Y) < 1$. The next result intuitively says that smaller is the subset in X, stronger the Carathéodory defined on it will be.

Lemma 5. *If we are given a small $Y \subset X$ where X is a path holomorphic space, then for all $x, y \in Y$,*

$$\delta_X(x, y) \leq diam_X(Y)\delta_Y(x, y) \tag{4.19}$$

Proof :

Take $r > diam_X(Y)$, then for all $\phi \in H(X, \mathbb{D})$ normalized so that there exists some x_0 with $\phi(x_0) = 0$, we have that $\phi(Y) \subset D(0, r)$. Now for all such ϕ, consider $\psi = \frac{1}{r}\phi_{|Y}$. Obviously, $\psi \in H(Y, \mathbb{D})$. Then for all $x, y \in Y$, as the set of such functions ψ is included in $H(Y, \mathbb{D})$, we have

$$\delta_Y(y, y') = \sup_{\psi' \in H(Y,\mathbb{D})} \delta_{\mathbb{D}}(\psi'(x), \psi'(y)) \geq \sup_{\phi \in H(X,\mathbb{D})} \delta_{\mathbb{D}}(\psi(x), \psi(y)) =$$

But

$$\sup_{\phi \in H(X,\mathbb{D})} \delta_{\mathbb{D}}(\psi(x), \psi(y)) = \frac{1}{r} \sup_{\phi \in H(X,\mathbb{D})} \delta_{\mathbb{D}}(\phi_{|Y}(x), \phi_{|Y}(y)) = \frac{1}{r} \sup_{\phi \in H(X,\mathbb{D})} \delta_{\mathbb{D}}(\phi(x), \phi(y))$$

And $\sup_{\phi \in H(X,\mathbb{D})} \delta_{\mathbb{D}}(\phi(x), \phi(y)) = \delta_X(x, y)$, so finally we get

$$\delta_Y(y, y') \geq \frac{1}{r}\delta_X(x, y)$$

Now take $r \to diam_X(Y)$, this gives the result :

$$\delta_X(x,y) \leq diam_X(Y).\delta_Y(x,y)$$

This enables us to give a much stronger form of the Schwarz lemma, in the sense that the contraction becomes a $diam_X(Y)$-Lipschitz function. This will obviously be a key-point for the proof of the exponential contraction.

Theorem 15 (strong form of the Schwarz lemma)**.** *Suppose we are given a path holomorphic map $\phi : X \to Y$ where $\phi(X)$ is small in the sense defined above. Then ϕ is strongly contracting : for all $x, x' \in X$,*

$$\delta_Y(\phi(x), \phi(x')) \leq diam_Y(\phi(X)).\delta_X(x, x') \tag{4.20}$$

Proof :
We have that $\phi(X) \subset Y$, so we can consider $\phi_0 = \phi : X \to \phi(X)$. If we apply the weak form of the Schwarz lemma to ϕ_0, we get, for all $x, x' \in X$:

$$\delta_{\phi(X)}(\phi_0(x), \phi_0(x')) \leq \delta_X(x, x') \tag{4.21}$$

But applying the previous lemma to $\phi(X) \subset Y$, with $\phi(X)$ small, we have :

$$\delta_Y(\phi(x), \phi(x')) \leq diam_Y(\phi(X)).\delta_{\phi(X)}(\phi_0(x), \phi_0(x')) \tag{4.22}$$

Putting (4.21) and (4.22) together, we get the result :

$$\delta_Y(\phi(x), \phi(x')) \leq diam_Y(\phi(X)).\delta_X(x, x')$$

4.4 Link with the Hybrid leaves configuration

Here we are going to use the theoretical things introduced above to our situation in the previous chapter. Remember the notion of hybrid leaves $\widehat{\mathcal{H}}_c$: they are the connected components of $\mathcal{H}_c$, the hybrid class of the polynomial P_c with $c \in \mathcal{M}$. We would like to have a path holomorphic structure on $\widehat{H}_c$ to apply the Schwarz lemma stated above.

Definition 26. *Suppose we are given a continuous family of germs $(f_\lambda : U_\lambda \to V_\lambda) \in \widehat{\mathcal{H}}_c$ where $\lambda \in \mathbb{D}$ and a holomorphic motion $(h_\lambda)_{\lambda \in \mathbb{D}}$ of $\mathbb{C}$ over $\mathbb{D}$ with basepoint 0, both of them such that :*

- $h_\lambda(K(f_0)) = K(f_\lambda)$
- $\bar{\partial} h_\lambda = 0$ *a.e. on* $K(f_0)$.
- $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ *on* $K(f_0)$.

*Then $(f_\lambda : U_\lambda \to V_\lambda) \in \widehat{\mathcal{H}}_c$ is said to be a **holomorphic path** in $\widehat{\mathcal{H}}_c$.*

Recall the definition of a Beltrami paths : in particular, they are holomorphic paths but the two notions coincide locally. Now, remember that when we constructed the mating

$$\begin{cases} \mathcal{E} \times \mathcal{M} \to \mathcal{C} \\ (g, c) \mapsto i_c(g) \end{cases}$$

in the end we said that given $c_1, c_2 \in \mathcal{M}$, $i_{c_1} \circ i_{c_2}^{-1}$ transforms Beltrami paths in $\widehat{\mathcal{H}}_{c_2}$ into Beltrami paths in $\widehat{\mathcal{H}}_{c_1}$. With this statement and the above definition of holomorphic path, given a holomorphic path $(f_\lambda : U_\lambda \to V_\lambda) \in \widehat{\mathcal{H}}_{c_2}$, $i_{c_1} \circ i_{c_2}^{-1}$ will take $(f_\lambda : U_\lambda \to V_\lambda) \in \widehat{\mathcal{H}}_{c_2}$ into a holomorphic path into $\widehat{\mathcal{H}}_{c_1}$.
Thus $\widehat{\mathcal{H}}_c$ inherits a path holomorphic structure. The next step is to get a Carathéodory hyperbolic space in order to use the Schwarz lemma as in the theoretical way of the last section. We have the following :

Theorem 16. *Let $c \in \mathcal{M}$. Then the hybrid leaf $\widehat{\mathcal{H}}_c$ is Carathódory hyperbolic.*

Proof By th previous discussion, if we prove the Carathódory hyperbolic property for a hybrid leaf $\widehat{\mathcal{H}}_c$ for some $c \in \mathcal{M}$, every other leaf $\widehat{\mathcal{H}}_{c'}$ for some $c' \in \mathcal{M}$ with $c' \neq c$ is also Carathódory hyperbolic because $i_{c'} \circ i_c^{-1}$ will take any holomorphic path in $\widehat{\mathcal{H}}_c$ to a holomorphic path in $\widehat{\mathcal{H}}_{c'}$ and thus transfers the Carathédory hyperbolicity structure. So if the hybrid leaf $\widehat{\mathcal{H}}_0$ is Carathódory hyperbolic, the result follows. To prove that $\widehat{\mathcal{H}}_0$ is Carathéodory hyperbolic, we can use lemma4 ; this means proving that $\widehat{\mathcal{H}}_0$ holomorphically injects in a Banach ball. To achieve this, we need two results of complex analysis. We won't prove these results.

The first result is the Böttcher theorem. Suppose we are given a germ $[f]$ (any germ, not necessarily $[f] \in \mathcal{C}$) such that its representatives $f : \mathbb{C} \to \mathbb{C}$ fix 0 (0 is a superattracting fixed point) and can be written

$$f(z) = az^d + \mathcal{O}(z^{d+1})$$

Denote by $f_0 : z \mapsto az^k$.

Theorem 17 (Böttcher theorem)**.** *If a germ $[f]$ is as above, there exists a germ of analytic maps $\phi : \mathbb{C} \to \mathbb{C}$ fixing 0, tangent to the identity map at 0 such that $\phi \circ f = f_0 \circ \phi$ in some neighborhood of 0 :*

$$\phi(f(z)) = a(\phi(z))^d$$

*ϕ is called the **Böttcher coordinate** of $[f]$.*

In our case, $f \in \widehat{\mathcal{H}}_0$ so its representatives can be written

$$f(z) = z^d + \mathcal{O}(z^{d+1})$$

and the Bötcher theorem gives the existence of a germ $[\phi]$ of analytic maps tangent to the identity at 0 such that

$$\phi(f(z)) = (\phi(z))^d$$

in a neighborhood of 0. Note that in particular, $D\phi(0) = 1$ and $f_0 = P_0$. One can restrict this ϕ to an analytic one-to-one map (we keep the same notations) from $intK(f)$ and onto $intK(P_0) = \mathbb{D}$:

$$\phi : intK(f) \to \mathbb{D}$$

Here, we need a second result called the Köebe-1/4 theorem :

Theorem 18 (Köebe-1/4 theorem)**.** *Let f be an univalent map on $\mathbb{D}$, that is a one-to-one and analytic function ; Suppose $f(0) = 0$ and $f'(0) = 1$. Then the disc $D(0, 1/4)$ centred at the origin and of radius $1/4$ is contained in the image of $\mathbb{D}$, $f(\mathbb{D})$:*

$$D(0, 1/4) = \mathbb{D}_{1/4} \subset f(\mathbb{D})$$

Note that our map ϕ is one to one and onto, so we can apply the Köebe-1/4 theorem to $\phi^{-1} : \mathbb{D} \to K(f)$: indeed, $\phi^{-1}(0) = 0$ and $(\phi^{-1})'(0) = 1$; The Köebe-1/4 theorem applied to ϕ^{-1} gives

$$D(0, 1/4) \subset \phi^{-1}(\mathbb{D}) = K(f)$$

Now, we have the necesarry ingredients to prove that $\widehat{\mathcal{H}}_0$ holomorphically injects in a Banach ball. In fact, we are going to prove that $\widehat{\mathcal{H}}_0$ holomorphically injects into $\mathcal{B}_{\mathbb{D}_r}$ for some $0 < r < 1/4$, where $\mathcal{B}_U$ stands for the Banach space of holomorphic functions in U and continuous in $\overline{U}$. By the principle of analytic continuation, the restiction operator

$$\begin{array}{rcl} I_r : \widehat{\mathcal{H}}_0 & \to & \mathcal{B}_{\mathbb{D}_r} \\ f & \mapsto & f_{|\mathbb{D}_r} \end{array}$$

is injective. One can also show that it is a bounded operator, ie for $f \in \widehat{\mathcal{H}}_0$, $f_{|\mathbb{D}_r}$ can be bounded in terms of r. This proves that there exists a Banach ball $\mathcal{B}_{\mathbb{D}_r}(r)$ such that $I_r(\widehat{\mathcal{H}}_0) \subset \mathcal{B}_{\mathbb{D}_r}(r)$. The most important thing to prove is that I_r is path holomorphic. If we are given a holomorphic path in $\widehat{\mathcal{H}}_0$, we need to prove that its image through I_r is still a holomorphic path in $\mathcal{B}_{\mathbb{D}_r}$. But a holomorphic path $(f_\lambda) \subset \mathcal{B}_{\mathbb{D}_r}$ is just a family of maps such that

$$\begin{array}{rcl} \mathbb{D} \times \mathbb{D}_r & \to & \mathbb{C} \\ (\lambda, z) & \mapsto & f_\lambda(z) \end{array}$$

is holomorphic in $(\lambda, z) \in \mathbb{D} \times \mathbb{D}_r$. So if we prove the following lemma, we are done

Lemma 6. *If (f_λ) is a holomorphic path in $\widehat{\mathcal{H}}_0$, then*

$$\begin{array}{rcl} \mathbb{D} \times \mathbb{D}_r & \to & \mathbb{C} \\ (\lambda, z) & \mapsto & f_\lambda(z) \end{array}$$

is holomorphic in $(\lambda, z) \in \mathbb{D} \times \mathbb{D}_{1/4}$.

Proof. Take a holomorphic path (f_λ) in $\widehat{\mathcal{H}}_0$. By definition, there exists a holomorphic motion $h_\lambda : \mathbb{C} \to \mathbb{C}$, with $\lambda \in \mathbb{D}$ and 0 as basepoint, such that $h_\lambda(K(f_0)) = K(f_\lambda)$, $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ on $K(f_0)$ and holomophic on $intK(f_0)$ (because $\bar{\partial} h_\lambda \equiv 0$ on $intK(f_0)$). Then

$$\begin{array}{ccc} \mathbb{D} \times intK(f_0) & \stackrel{\kappa}{\longrightarrow} & \cup_{\lambda\in\mathbb{D}} \left(\{\lambda\} \times intK(f_\lambda)\right) \\ (\lambda, z) & \mapsto & (\lambda, h_\lambda(z)) \end{array}$$

is easily holomorphic on $\mathbb{D} \times intK(f_0)$. But then

$$(\lambda, h_\lambda(z)) \stackrel{\eta}{\longmapsto} f_\lambda(z) = h_\lambda \circ f_0 \circ h_\lambda^{-1}(z)$$

is also holomorphic. So

$$(\lambda, z) \stackrel{\eta\circ\kappa}{\longmapsto} f_\lambda(z)$$

is holomorphic on $(\mathbb{D} \times D(0, 1/4)) \subset \mathbb{D} \times intK(f_0)$. □

We conclude that $\widehat{\mathcal{H}}_0$ holomorphically injects in a Banach ball. Then it is carathódory hyperbolic, and as the Carathéodory hyperbolicity of one hybrid leaf implies the Carathéodory hyperbolicity of any other hybrid leaf, every hybrid leaf $\widehat{\mathcal{H}}_c$ is carathódory hyperbolic.

Chapter 5

Renormalization and bounds

Now, we are going to introduce the renormalization operator $\mathcal{R}$. Remember that our goal is to prove exponential contraction of this operator along hybrid leaves.

5.1 Renormalizable maps, renormalization operator

Mac Mullen defined the notion of renormalizable quadratic-like maps (ie p.l.-maps with $d = 2$) as follows ; given a quadratic-like map f, we say that f is renormalizable if there exists $n \geq 2$ and two open topological discs $U, V \subset \mathbb{C}$ such that $f^n : U \to V$ is itself a quadratic-like map with connected Julia set. Mac Mullen calls (U, V) a choice of the renormalization.

Then Mac Mullen proves that two renormalizations of f will have the same filled Julia set. Suppose we denote by $K_n(f)$ this filled Julia set. Then for $j = 1, ..., n$ we defined the little filled Julia sets as the images $K_n^j = f^j(K_n(f))$. We have the following result, which will be important in our further developments :

Theorem 19. *The little filled Julia sets do not touch, except at a repelling fixed point of f^n.*

This result can be generalized in the case p.l.-maps. Now, let us define the notion of renormalizable p.l.-maps :

Definition 27. *Let $f : U \to V$ be unicritical p.l.-map of degree d. f is called* ***renormalizable*** *with period $p > 1$ if the following three conditions hold :*

1. *There exists a topological disc $W \subset \mathbb{C}$ with $0 \in W$ such that the restriction $g = f^p_{|W} : W \to W'$ is a p.l.-map of degree d. g is called the* ***pre-renormalization*** *of f.*
2. *The little filled Julia set $K(g)$ is connected.*

3. $K(g)$ does not touch the other little Julia sets $K_j(g) = f^j(K(g))$ for $j = 1, ..., p-1$, except perhaps at one of its β-fixed points.

We will say that a p.l.-germ $[f] \in \mathcal{C}$ is ***renormalizable*** *if it has a renormalizable representative $g \in [f]$.*

Now we can define the renormalization operator $\mathcal{R}$. Given a renormalizable f, take a pre-renormalization g with the smaller possible period. We normalize g so that g behaves near 0 like the p.l.-maps we define before :

$$\mathcal{R}f(z) = \lambda^{-1}g(\lambda z) = z^d + c + \mathcal{O}(z^{d+1})$$

We are suppose to work with $\mathcal{R}$ along the hybrid leaves, so we need to check if $\mathcal{R}$ maps hybrid leaves into hybrid leaves.

Lemma 7. *$\mathcal{R}$ takes Beltrami paths to Beltrami paths.*

Proof : let $([f_\lambda], h_\lambda)$ be a Beltrami path in some renormalizable hybrid leaf. For $[f_0]$, we let $f_0 : U_0 \to V_0$ be a representative of the p.l.-germ $[f_0]$. We assume that V_0 will be small enough so that f_0 let the Beltrami differentials μ_λ of h_λ invariant for all $\lambda \in \mathbb{D}$. Suppose $g_0 = f_0^p : W_0 \to W_0'$ is a pre-renormalization of f_0. Then all μ_λ are invariant through g_0. But then $g_\lambda = h_\lambda \circ g_0 \circ h_\lambda^{-1}$ is a pre-renormalization for f_λ. If we denote by a_λ the affine map that normalize g_λ, it is in fact such that $g_\lambda \circ a_\lambda = a_\lambda \circ \mathcal{R}f_\lambda$, ie it conjugates g_λ and $\mathcal{R}f_\lambda$. But then the pair $(\mathcal{R}f_\lambda, a_\lambda \circ h_\lambda \circ a_\lambda^{-1})$ becomes a Beltrami path.

Proposition 3. *$\mathcal{R}$ maps hybrid leaves into hybrid leaves*

Proof. Using the above lemma, as the hybrid leaves are the connected component of the hybrid class, the result follows. □

So it enables the renormalization orperator to be applied more than one time on a given map. This is why a map can be n times renormalizable : in fact, it is renormalized a first time with a period p_1 say. But then we renormalize $\mathcal{R}f$ with a period p_2 ; we get $\mathcal{R}^2 f$ that we can renormalize again etc... until we reach $\mathcal{R}^n f$ wich can't be renormalized if the map is only n time renormalizable. It is natural to generalize this concept to infinitely many times renormalizable maps. In that case, we get a sequence of periods $(p_n)_{n\geq 1}$. In particular, if $(p_n)_{n\geq 1}$ is a bounded sequence, f is said to have a **bounded combinatorics**. The notion of **infinitely renormalizable** germ then follows ; it means that the germ possesses an infinitely renormalizable representative.

For our further developments, we will use the following notations ; $\mathcal{I} \subset \mathcal{C}$ is the subspace of infinitely renormalizable germs. $\mathcal{C}^{(\mathbb{R})}$ will denote the space of p.l.-germs that are hybrid equivalent to real p.l.-germs in $\mathcal{C}^{\mathbb{R}}$. We can then denote by $\mathcal{I}^{(\mathbb{R})} = \mathcal{I} \cap \mathcal{C}^{(\mathbb{R})}$ the set of infinitely renormlizable germs that are hybrid equivalent to the real ones.

5.2 A priori bounds

In our further developments, given any $\varepsilon > 0$, we will denote by $\mathcal{C}(\varepsilon)$ the set of all $[f] \in \mathcal{C}$ with $\mod [f] \geq \varepsilon$. More generally, for a set $A \subset \mathcal{C}$, we will denote by $A(\varepsilon)$ the subset of germs in A such that $\mod [f] \geq \varepsilon$. Mac Mullen has shown that $\mathcal{C}(\varepsilon)$ is compact, proving that any sequence $f_n : U_n \to V_n$ so that the euclidean diameter of $K(f_n)$ is one has a convergent subsequence. We also have that for any compact subset $\mathcal{K} \subset \mathcal{C}$, $\mathcal{K}$ is contained in $\mathcal{C}(\varepsilon)$ for some $\varepsilon \geq 0$.

Definition 28 (a priori bounds)**.** *An infinitely renormalizable germ $f \in \mathcal{C}$ is said to have* ***a priori bounds*** *if for some $\varepsilon > 0$, its renormalizations $\mathcal{R}^n f$, $n = 1, 2, ...$ are such that*

$$\mod \mathcal{R}^n f \geq \varepsilon$$

meaning that $\mathcal{R}^n f \in \mathcal{C}(\varepsilon)$ for $n = 1, 2, ...$

Now we are going to state a result that gives an idea of the size of the little Julia sets for an infinitely renormalizable map with a priori bounds. Remember, if g is the pre-renormalization of f with period q, we denoted by $K_j(g) = f^j(K(g))$ the little Julia sets for $j = 0, ..., q-1$. In fact, if we denote by $(f_n)_{n\geq 1}$ the sequence of pre-renormalizations and $(q_n)_{n\geq 1}$ the sequence of total periods, the diameters of these little Julia sets exponentially decay with n :

Proposition 4. *Given $f \in \mathcal{C}$ an infinitely renormalizable p.l.-map that has a priori bounds, we denote by $(f_n)_{n\geq 1}$ its sequence of pre-renormalizations and $(q_n)_{n\geq 1}$ the corresponding sequence of total periods. Then there exist real constants $C > 0$ and $\lambda < 1$ such that*

$$\max_{m\in(\mathbb{Z}/q_n\mathbb{Z})} diam K_m(f_n) \leq C\lambda^n \tag{5.1}$$

C and λ only depend on the a priori bounds.

Avila and Lyubich prove this result showing the existence of a $\delta > 0$ only depending on the a priori bounds such that for all $n_1 < n_2$ and $m_1 \in \mathbb{Z}/q_{n_1}\mathbb{Z}$, $m_2 \in \mathbb{Z}/q_{n_2}\mathbb{Z}$ with $K_{m_2}(f_{n_2}) \subset K_{m_1}(f_{n_1})$ the Carathéodory distance between $K_{m_1}(f_{n_1})$ and $K_{m_2}(f_{n_2})$ is at least $\delta.diam(K_{m_1}(f_{n_1}))$. But repeating this a second time for a certain $n_3 > n_2$ with $m_3 \in \mathbb{Z}/q_{n_3}\mathbb{Z}$ such that $K_{m_3}(f_{n_3}) \subset K_{m_2}(f_{n_2})$, we will find that the Carathéodory distance between $K_{m_2}(f_{n_2})$ and $K_{m_3}(f_{n_3})$ is at least $\delta.diam(K_{m_2}(f_{n_2}))$. Repeating this a certain amount of time, we clearly get the existence of an integer $k > 0$ such that for all $n' > n+k$,

$$diam(K_{m'}(f_{n'})) < \frac{1}{2}.diam(K_m(f_n))$$

k depends only on the a priori bounds and is bounded ; the exponential decay follows, with $\lambda \sim (1/2)^{1/k}$.

5.3 Beau bounds

5.3.1 Definition

Definition 29. *We say that a family of infinitely renormalizable maps $\mathcal{F}$ with a priori bounds has* ***beau bounds*** *if there exists ε_0 such that for any $\delta > 0$, there exists an integer n_δ such that for all $f \in \mathcal{F}$ with $\mod f \geq \delta$, and $n \geq n_\delta$:*

$$\mod \mathcal{R}^n f \geq \varepsilon_0 \tag{5.2}$$

Why is it stronger than a priori bounds? A priori bounds for a a family $\mathcal{F}$ of infinitely renormalizable maps provides for every $f \in \mathcal{F}$ a constant $\varepsilon \geq 0$ such that $\mod \mathcal{R}^n f \geq \varepsilon$. ε depends on the infinitely renormalizable germ. Beau bounds gives a uniform bound ε_0 for every maps in the family, provided the pre-renormalization f_n are such that $n \geq n_\delta$, where n_δ depends only on the constant δ, that is to say only on the family of maps (because $\mod f \geq \delta$ in this family) and not on every map of this family.

Kahn and Lyubich have proves that a certain class of p.l.-maps have beau bounds, using a theory of "decorations" and "molecules". However, beau bounds for general complex maps is a very difficult result to prove. Though, this gives the exponential contraction that we need.

5.3.2 (C, ε)-closeness between germs in $\mathcal{C}$

Here we are going to state a notion that will be useful to prove intermediate results.

Definition 30. *Suppose we are given two p.l.-germs $[f_1], [f_2] \in \mathcal{C}$ and two real constants $C \geq 1$ and $\varepsilon > 0$. Then $[f_1]$ and $[f_2]$ are said to be (C, ε)-**close** if they have p.l.-representatives, $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ respectively, such that*

- $\mod (V_1 \setminus U_1) > \varepsilon$ *and* $\mod (V_2 \setminus U_2) > \varepsilon$.
- *there exists a quasiconformal homeomorphism*

$$h : \mathbb{C} \setminus U_1 \to \mathbb{C} \setminus U_2$$

  such that $Dil_h < C$ and $h \circ f_1 = f_2 \circ h$ on ∂U_1

Here are some results about this property between germs. The first result states that the quasiconformal homeomorphism h can be extended to a hybrid conjugacy in the case where the germs are hybrid equivalent.

Proposition 5. *Suppose that two given germs $[f_1], [f_2] \in \mathcal{C}$ are (C, ε)-close with quasiconformal homeomorphism h such that $Dil_h < C$. Moreover, suppose that $[f_1]$ and $[f_2]$ are hybrid equivalent. Then h can be extended to a hybrid conjugacy $\widetilde{h}$ between $[f_1]$ and $[f_2]$. The dilatation $Dil_{\widetilde{h}}$ is also bounded by C.*

The proof of this result is based on the pull-back argument. The next result links the compact set $\mathcal{C}(\varepsilon)$ to the notion of (C, ε)-closeness.

Proposition 6. *Suppose $\varepsilon_0 > 0$ and take two germs $[f_1], [f_2] \in \mathcal{C}(\varepsilon_0)$. Then for every ε such that $0 < \varepsilon < \varepsilon_0$, there exists a constant $C > 1$ such that $[f_1]$ and $[f_2]$ are (C, ε)-close.*

The next result is a stronger form of the previous one. It states that if we take two close germs, the constant C can be taken closer to 1. Thus, in order to state it, this is more convenient to take two sequences of germs $([f_1]_n)_{n\geq 1}$ and $([f_2]_n)_{n\geq 1}$ converging to the same limit : roughly speaking, as n grows, $[f_1]_n$ and $[f_2]_n$ will be closer and closer.

Proposition 7. *Given two converging sequences of p.l.-germs $([f_1]_n)_{n\geq 1}, ([f_2]_n)_{n\geq 1} \in \mathcal{C}^{\mathbb{N}}$, both converging to the same limit, there exists $\varepsilon > 0$ and a sequence of real numbers $(C_n)_{n\geq 1}$, $C_n > 1$, such that*

- $\lim_{n\to\infty} C_n = 1$
- *$[f_1]_n$ and $[f_2]_n$ are (C_n, ε)-close (for n sufficiently large)*

5.3.3 Beau bounds imply exponential contraction

As we have proved that $\widehat{\mathcal{H}}_c$ is Carathódory hyperbolic for any $c \in \mathcal{M}$, we would like to use the Schwarz lemma along hybrid leaves to prove exponential contraction. But we need to make the assumption of beau bounds to prove this result. We will begin with a lemma stating that the diameter of the set of functions in $\widehat{\mathcal{H}}_c$ with $\mod f \geq \varepsilon$ with respect to the pseudo metric in $\widehat{\mathcal{H}}_c(\delta)$ for some $0 < \delta < \varepsilon$ is less than one.

Lemma 8. *Given some $\varepsilon > 0$, there exist two constants (depending on ε) δ, $0 < \delta < \varepsilon$ and $\gamma < 1$ such that :*

$$\forall c \in \mathcal{M},\ diam_{\widehat{\mathcal{H}}_c(\delta)} \widehat{\mathcal{H}}_c(\varepsilon) < \gamma$$

Proof. • Suppose $c \in \mathcal{M}$. If we take $[f_1], [f_2] \in \widehat{\mathcal{H}}_c(\varepsilon)$, then by proposition 6, for all $\varepsilon' < \varepsilon$ there exists $C > 1$ such that $[f_1]$ and $[f_2]$ are (C, ε')-close. But we can take $\varepsilon' = \varepsilon/2$ for instance. So $[f_1]$ and $[f_2]$ are $(C, \varepsilon/2)$-close and we can denote h the quasiconformal homeomorphism of the definition of (C, ε)-closeness.

But as $[f_1], [f_2] \in \widehat{\mathcal{H}}_c(\varepsilon)$, they are hybrid conjugate : by proposition 5, we can extend h to a hybrid conjugacy $h : \mathbb{C} \to \mathbb{C}$ with $Dil_h < C$. Dealing with the beltrami differential $\mu = \frac{\bar{\partial} h}{\partial h}$, this is equivalent to the existence of a constant $r = r(\varepsilon) < 1$ such that $||\mu||_\infty \leq r(\varepsilon)$.

- Now, consider a Beltrami path

$$\begin{array}{rcl} \mathbb{D}_\rho & \to & \widehat{\mathbb{H}}_c \\ \Phi : \lambda & \mapsto & f_\lambda = h_{\lambda\mu} \circ f_1 \circ h_{\lambda\mu}^{-1} \end{array}$$

where $h_{\lambda\mu}$ is the solution of the Beltrami equation $\dfrac{\bar{\partial} h}{\partial h} = \lambda\mu$ and $\rho = \frac{1+r}{2r}$. Note that $\rho > 1$ and $\phi(0) = f_{\lambda=0} = f_1$ and $\phi(1) = f_{\lambda=1} = f_2$. So $h_{\lambda\mu}$ hybrid conjugates f_1 and f_λ. In particular, if $h_{\lambda\mu}$ has a dilatation $Dil_{h_{\lambda\mu}} \leq K$, the fundamental annulus of f_λ will have modulus $\mod f_1/K \geq \varepsilon/2K$ (because $\mod f_1 \geq \varepsilon/2$). It is enough to show that K is a function of r. In particular,

$$||\lambda\mu||_\infty = |\lambda|.||\mu||_\infty \leq \rho.r = \frac{1+r}{2}$$

Thus, because $x \mapsto \dfrac{1+x}{1-x}$ is non decreasing for $x > 0$,

$$Dil_{h_{\lambda\mu}} = \frac{1 + ||\lambda\mu||_\infty}{1 - ||\lambda\mu||_\infty} \leq \frac{1 + \frac{1+r}{2}}{1 - \frac{1+r}{2}} = \frac{3+r}{1-r} = K$$

Thus, denoting by $\delta = \varepsilon/2K$, we have $\mod f_\lambda \geq \delta$ and because $h_{\lambda\mu}$ hybrid conjugates $f_1 \in \widehat{\mathcal{H}}_c$ to $f_{\lambda\mu}$, we have that $f_{\lambda\mu} \in \widehat{\mathcal{H}}_c(\delta)$; but the beltrami path is a path holomorphic map, so by the weak form of the Schwarz lemma we have that

$$\delta_{\widehat{\mathcal{H}}_c(\delta)}(f_1, f_2) = \delta_{\widehat{\mathcal{H}}_c(\delta)}(\Phi(0), \Phi(1)) \leq \delta_{\mathbb{D}_\rho}(0,1) = \rho^{-1} = \gamma < 1$$

We conclude with

$$diam_{\widehat{\mathcal{H}}_c(\delta)} \widehat{\mathcal{H}}_c(\varepsilon) < \gamma < 1$$

□

Now, consider a family $\mathcal{F} \in \mathcal{C}$ of infinitely renormalizable maps, which is forward invariant with $\mathcal{R}$ (ie $\mathcal{R}(\mathcal{F}) = \mathcal{F}$) with beau bounds. We set $\varepsilon_0 > 0$ to be the beau bound for $\mathcal{F}$: for all $\delta > 0$, there exists an integer n_δ such that, for all $f \in \mathcal{F}$ with $\mod f \geq \delta$ and for all $n \geq n_\delta$,

$$\mod (\mathcal{R}^n f) \geq \varepsilon_0$$

We shall also assume that $\mathcal{F}$ is a union of hybrid leaves ; remember that $\mathcal{R}$ maps hybrid leaves into hybrid leaves, so it makes sense to consider such an union. We will denote by c_n the straightening of $\mathcal{R}^n(\widehat{\mathcal{H}}_c)$ for every integer n.

By lemma 8, one can choose $\delta_0 < \varepsilon_0$ such that for all $c \in \mathcal{M}$,

$$diam_{\widehat{\mathcal{H}}_c(\delta_0)} \widehat{\mathcal{H}}_c(\varepsilon_0) < \gamma$$

for some $\gamma < 1$. Set $\lambda = \gamma^{1/n_{\delta_0}}$. For all $f_1, f_2 \in \widehat{\mathcal{H}}_c \subset \mathcal{F}$ and all integer n,

$$\delta_{H_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \leq \delta_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \tag{5.3}$$

(5.3) is true because $H_{c_n}(\delta_0) \subset H_{c_n}$ and by lemma 5, subsets have stronger Carathéodory metric. Now applying the weeak form of the Schwarz lemma to $\mathcal{R}^n : \widehat{\mathcal{H}}_c(\delta) \to \widehat{\mathcal{H}}_{c_n}$, we get

$$\delta_{\widehat{\mathcal{H}}_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \leq \delta_{\widehat{\mathcal{H}}_c(\delta)}(f_1, f_2) \tag{5.4}$$

But combined together, (5.3) and (5.4) give

$$\delta_{\widehat{\mathcal{H}}_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \leq \min\left\{\delta_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}(\mathcal{R}^n f_1, \mathcal{R}^n f_2),\, \delta_{\widehat{\mathcal{H}}_c(\delta)}(f_1, f_2)\right\} \tag{5.5}$$

Now, we are going to bound the term $\delta_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}(\mathcal{R}^n f_1, \mathcal{R}^n f_2)$;

- For $n \geq n_\delta$, by definition of complex bound, we have that

  $$\mathcal{R}^n(\widehat{\mathcal{H}}_c(\delta)) \subset \widehat{\mathcal{H}}_{c_n}(\varepsilon_0)$$

  Now, applying the strong form of the Schwarz lemma to

  $$\mathcal{R}^n : \widehat{\mathcal{H}}_c(\delta) \to \widehat{\mathcal{H}}_{c_n}(\varepsilon_0)$$

  together with the fact that $diam_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}\left(\widehat{\mathcal{H}}_{c_n}(\varepsilon_0)\right) \leq \lambda^{n_{\delta_0}}$, we obtain :

  $$\begin{aligned}
  & \delta_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) && (5.6)\\
  \leq\ & diam_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}\left\{\mathcal{R}^n(\widehat{\mathcal{H}}_c(\delta))\right\}.\delta_{\widehat{\mathcal{H}}_c(\delta)}(f_1, f_2) && (5.7)\\
  \leq\ & diam_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}\left(\widehat{\mathcal{H}}_{c_n}(\varepsilon_0)\right).\delta_{\widehat{\mathcal{H}}_c(\delta)}(f_1, f_2) && (5.8)\\
  \leq\ & \lambda^{n_{\delta_0}}.\delta_{\widehat{\mathcal{H}}_c(\delta)}(f_1, f_2) && (5.9)
  \end{aligned}$$

- Now, for $n \geq n_\delta + n_{\delta_0}$, one can do exactly the same as above, but this time applied to

  $$\mathcal{R}^{n_{\delta_0}} : \widehat{\mathcal{H}}_{c_{n-n_{\delta_0}}}(\delta_0) \to \widehat{\mathcal{H}}_{c_n}(\varepsilon_0)$$

  together with

  $$\mathcal{R}^{n_{\delta_0}}(\widehat{\mathcal{H}}_{c_{n-n_{\delta_0}}}(\delta_0)) \subset \widehat{\mathcal{H}}_{c_n}(\varepsilon_0)$$

  this gives, with the strong form of the Schwarz lemma :

  $$\begin{aligned}
  & \delta_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) && (5.10)\\
  \leq\ & diam_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}\left\{\mathcal{R}^{n_{\delta_0}}(\widehat{\mathcal{H}}_{c_{n-n_{\delta_0}}}(\delta_0))\right\}.\delta_{\widehat{\mathcal{H}}_{c_{n-n_{\delta_0}}}(\delta_0)}(\mathcal{R}^{n-n_{\delta_0}} f_1, \mathcal{R}^{n-n_{\delta_0}} f_2) &&\\
  \leq\ & diam_{\widehat{\mathcal{H}}_{c_n}(\delta_0)}\left(\widehat{\mathcal{H}}_{c_n}(\varepsilon_0)\right).\delta_{\widehat{\mathcal{H}}_{c_{n-n_{\delta_0}}}(\delta_0)}(\mathcal{R}^{n-n_{\delta_0}} f_1, \mathcal{R}^{n-n_{\delta_0}} f_2) &&\\
  \leq\ & \lambda^{n_{\delta_0}}.\delta_{\widehat{\mathcal{H}}_{c_{n-n_{\delta_0}}}(\delta_0)}(\mathcal{R}^{n-n_{\delta_0}} f_1, \mathcal{R}^{n-n_{\delta_0}} f_2) && (5.11)
  \end{aligned}$$

Now, suppose n is large enough ; denoting k the first integer such that $n-k.n_{\delta_0} \leq n_\delta + n_{\delta_0}$ and combining (5.5), (5.9) and (5.11), we obtain :

$$\delta_{\widehat{\mathcal{H}}_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \leq \lambda^{kn_{\delta_0}}.C(\delta).\delta_{\widehat{\mathcal{H}}_{c(\delta)}}(f_1, f_2) \tag{5.12}$$

Where $C(\delta)$ depends only on δ. But for a large n, $k \sim \lfloor \frac{n}{n_{\delta_0}} \rfloor$. Eventually,

$$\delta_{\widehat{\mathcal{H}}_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \leq \lambda^n.C(\delta).\delta_{\widehat{\mathcal{H}}_{c(\delta)}}(f_1, f_2) \tag{5.13}$$

For more simplicity, defining $C = C(\delta).\delta_{\widehat{\mathcal{H}}_{c(\delta)}}(f_1, f_2)$, we get the exponential contraction :

$$\delta_{\widehat{\mathcal{H}}_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \leq C\lambda^n \tag{5.14}$$

The following result sums up the conclusions of the above proof :

Theorem 20. *Suppose we are given a family $\mathcal{F} \subset \mathcal{C}$ of infinitely renormalizable maps with beau bounds. Moreover, suppose that $\mathcal{F}$ is a forward invariant union of hybrid leaves. Then there exists $\lambda < 1$ such that for all $f_1, f_2 \in \widehat{\mathcal{H}}_c \subset \mathcal{F}$ (we need to start in the same leaf),*

$$\delta_{\widehat{\mathcal{H}}_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) \leq C\lambda^n \tag{5.15}$$

$C > 0$ only depends on $\mod f_1$ and $\mod f_2$.

Note. *Here the exponential contraction of the renormalization operator along hybrid leaves has been proved. However, the beau bounds for complex maps are assumed, but this is a difficult result to prove. Avila and Lyubich based their proof on beau bounds for real maps, which is a more reasonable result to prove.*

5.4 Beau bounds for real maps

The next result was proved by Levin and Van Strien and also by Lyubich and Yampolsky. It is much more easier to prove than beau bounds for complex maps. Remember that we denoted by $\mathcal{C}^{\mathbb{R}}$ the p.l.-germs of maps preserving the real line.

Theorem 21 (Beau bounds for real maps). *Maps in germs of $\mathcal{C}^{\mathbb{R}}$ have beau bounds ; namely, there exists $\varepsilon_0 > 0$ such that : for all $\delta > 0$, there exist $\varepsilon(\delta) > 0$ and $N(\delta)$ such that, for all representative f of $[f] \in \mathcal{C}^{\mathbb{R}}$, we have :*

- *for every $n \geq 0$, $\mathcal{R}^n f \in \mathcal{C}(\varepsilon(\delta))$.*
- *for every $n \geq N(\delta)$, $\mathcal{R}^n f \in \mathcal{C}(\varepsilon_0)$.*

This result is not sufficient to provel contraction just like the way we did with beau bounds for complex maps. Avila and Lyubich introduced some tools that bring this result to thel contraction. In the next chapter, we are going to study these tools and show how they give the contraction. Concerning the space of infinitely renormalizable maps hybrid equivalent to real p.l.-maps such that $\mod f \geq \delta$, denoted $\mathcal{I}^{(\mathbb{R})}(\delta)$ we have the following result :

Theorem 22. *For all $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that if $[f] \in \mathcal{I}^{(\mathbb{R})}(\delta)$, then for all $n \geq 0$:*

$$\mathcal{R}^n f \in \mathcal{C}(\varepsilon)$$

Proof. Given $[f] \in \mathcal{I}^{(\mathbb{R})}(\delta)$, let $c = \chi(f)$ be the straightening. Then f is hybrid equivalent to $g = P_c : z \mapsto z^d + c$. Let g_n be the n-th pre-renormalization of g and q_n the corrseponding periods.

But $[f] \in \mathcal{C}(\delta)$ and $[g] \in \mathcal{C}(\delta)$ so by proposition 6, there exists a constant $C > 1$ such that $[f]$ and $[g]$ are $(C, \delta/2)$-close. But then, by definition of (C, ε)-closeness, there exist p.l.-representatives $f : U \to V$ and $g : U' \to V'$ with $\mod f > \delta/2$ and $\mod g > \delta/2$ and a quasiconformal homeomorphism $h : \mathbb{C} \setminus U \to \mathbb{C} \setminus U'$ with $Dil_h < C$ such that $h \circ f = g \circ h$ on ∂U. But because $c = \chi(f)$, $[f]$ and $[g]$ are in the same hybrid leaf $\widehat{\mathcal{H}}_c$ and then are hybrid equivalent. By proposition 5, we can extend h to a hybrid conjugacy $h : \mathbb{C} \to \mathbb{C}$ between f and g, with $h \circ f = g \circ h$ in U.

Then, using the a priori bound for real maps on g (ie the existence of $\eta > 0$ such that for all $n \geq 0$, $\mod g_n \geq \eta$), Avila and Lyubich show that each germ $[g_n]$ has a p.l.-representative $g_n : U_n \to V_n$ with $\mod g_n > \eta'$ and that $g^k(U_n) \subset U'$ for all $0 \leq k \leq q_n - 1$. Consequently,

$$h^{-1} \circ g_n \circ h : h^{-1}(U_n) \to h^{-1}(V_n)$$

is a representative of the n-th pre-renormalization of f (perhaps not normalized). Bu remember, h has a diliatation bounded by C. In the end, we obtain

$$\mod (h^{-1} \circ g_n \circ h) = \mod (h^{-1}(V_n) \setminus h^{-1}(U_n)) > \eta'/C = \varepsilon(\delta)$$

that is to say $\mod (\mathcal{R}^n f) > \varepsilon(\delta)$ or $\mathcal{R}^n f \in \mathcal{C}(\varepsilon(\delta))$ for all $n \geq 0$. □

Chapter 6

Proof of contraction with Beau bounds for real maps

6.1 Cocycles

6.1.1 Definitions

Here is a brief background on cocycles which is an algebraic notion.

Definition 31. *We say that a pair $(\mathcal{S}, *)$, where $\mathcal{S}$ is a non-empty set and $* : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ an intern operation on $\mathcal{S}$ (denoted $*(a, b) = a*b$), is a* ***semigroup*** *if $*$ is associative, ie for all $a, b, c \in \mathcal{S}$:*

$$(a * b) * c = a * (b * c)$$

We will denote by Q the set of pairs of positive integers (m, n) such that $n > m$:

$$Q = \{(m, n) \in \mathbb{N}^2 : n > m\}$$

Definition 32. *Given a semigroup $(\mathcal{S}, *)$, a $\mathcal{S}$-**cocycle** is a map*

$$\begin{array}{rcl} G : Q & \longrightarrow & \mathcal{S} \\ (m, n) & \mapsto & G^{(m,n)} \end{array}$$

with the following property : for all $l < m < n$,

$$G^{(m,n)} * G^{(l,m)} = G^{(l,n)} \tag{6.1}$$

Take $m < n$. Then if $m + 1 < n$, by (6.1), we can write

$$G^{(m,n)} = G^{(m+1,n)} * G^{(m,m+1)}$$

if $m + 2 < n$, we can continue this transformation :

$$G^{(m,n)} = G^{(m+2,n)} * G^{(m+1,m+2)} * G^{(m,m+1)}$$

By induction, this gives, denoting $F_n = G^{(n,n+1)}$:

$$G^{(m,n)} = F_{n-1} * \ldots * F_m \tag{6.2}$$

Now, if we take a sequence $(F'_n)_{n\geq 0} \in \mathcal{S}^{\mathbb{N}}$, we can define a cocycle G' using the formula (6.2). This gives a correspondence between any sequence in $\mathcal{S}$ and a $\mathcal{S}$-cocycle.

6.1.2 Cocycle setting for our configuration

Our goal is to reduce the proof of contraction to the contraction in a setting with cocycles. In order to do this, we need to choose a semigroup and to explain our reduction.

Proposition 8. *Consider the set of path holomorphic maps from $\widehat{\mathcal{H}}_0$ to $\widehat{\mathcal{H}}_0$, denoted $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$, paired with the composition $\circ$. Then $(H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0), \circ)$ is a semigroup. $(H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0), \circ)$ is a sub-semigroup (maps f in $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ such that $f(\widehat{\mathcal{H}}_0^{\mathbb{R}}) \subset \widehat{\mathcal{H}}_0^{\mathbb{R}}$ where $\widehat{\mathcal{H}}_0^{\mathbb{R}}$ is the subset of real p.l.-germs in $\widehat{\mathcal{H}}_0$).*

Proof.

- First, we need to check that $\circ$ is an intern operation in $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$. Indeed, given a holomorphic path $\gamma \in H(\widehat{\mathcal{H}}_0)$ and $f_1, f_2 \in H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$, as f_2 is path holomorphic, we have that $f_2 \circ \gamma$ is a holomorphic path in $H(\widehat{\mathcal{H}}_0)$. But f_1 is also path holomorphic so $f_1 \circ (f_2 \circ \gamma)$ is again a holomorphic path in $H(\widehat{\mathcal{H}}_0)$. Thus

$$\circ : H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0) \times H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0) \to H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$$

  is a well defined intern operation in $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$.

- The composition is associative, so the associativity is obvious.
- For $(H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0), \circ)$, the proof follows from the definition of $H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$.

□

Now, suppose that $[f] \in \mathcal{C}$. Rememnber that we defined the mating $i_c : \mathcal{E} \times \mathcal{M} \to \mathcal{C}$ between a circle map in $\mathcal{E}$ and a straightening in $\mathcal{M}$. We also defined the inverse of this mating. In particular, $\pi(f) = g$ gives the external map of $[f]$. Then we can defined a map

$$\begin{aligned} \Pi : \mathcal{C} &\to \widehat{\mathcal{H}}_0 \\ f &\mapsto i_0 \circ \pi(f) \end{aligned}$$

It naturally restricts to a path holomorphic homeomorphism $\Pi_c : \widehat{\mathcal{H}}_c \to \widehat{\mathcal{H}}_0$ for every $c \in \mathcal{M}$, also preserving the modulus. This map enables us to build a cocycle adapted to the semigroup $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$: for all $c \in \mathcal{M}$ such that the hybrid leaf $\widehat{\mathcal{H}}_c$ is renormalizable, we defined this cocycle as follows, for all $(m, n) \in Q$:

$$G_c : (m, n) \mapsto G_c^{(m,n)}(\Pi_c(f)) = \Pi_c(\mathcal{R}^{n-m}(\mathcal{R}^m f))$$

The property of cocycles is easily checked. We will denote by $\mathcal{G}$ this family of cocycles restricted to real symmetric hybrid leaves. Now, we are going to prove that this setting satisfies two strong hypothesis.

We know that $[P_0] \in \mathcal{C}^{\mathbb{R}}$ so the corresponding hybrid leaf $\widehat{\mathcal{H}}_0$ is such that $\widehat{\mathcal{H}}_0 \subset \mathcal{C}^{(\mathbb{R})}$. But $\widehat{\mathcal{H}}_0$ is also infinitely renormalizable so $\widehat{\mathcal{H}}_0 \subset \mathcal{I}$. In the end, for $\delta > 0$, we get $\widehat{\mathcal{H}}_0(\delta) \subset \mathcal{I}^{(\mathbb{R})}(\delta)$. Thus the theorem 22 can be applied to our configuration and we obtain the following :

(H1) : for every $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that : if $f \in \widehat{\mathcal{H}}_0(\delta)$, then for every $G \in \mathcal{G}$ and $(m, n) \in Q$, $G^{(m,n)}(f) \in \widehat{\mathcal{H}}_0(\varepsilon(\delta))$.

For the second hypothesis, we need an intermediate result. It will not be proved, but Lyubich and Avila provide a proof in their paper.

Lemma 9. *Given two sequences $(f_{1,n})_{n\geq 0}, (f_{2,n})_{n\geq 0} \in \mathcal{I}^{\mathbb{N}}$ with the same straightening $\chi(f_{1,n}) = \chi(f_{2,n})$ and converging to the same limit f. Suppose $(k_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$ is a sequence of integers such that $\lim_{n\to\infty} k_n = \infty$. Then we have :*

$$\lim_{n\to\infty} \inf \quad \text{mod } (\mathcal{R}^{k_n} f_{1,n}) = \lim_{n\to\infty} \inf \quad \text{mod } (\mathcal{R}^{k_n} f_{2,n})$$

To prove this result, Avila and Lyubich show that for every $\varepsilon > 0$, if

$$\lim_{n\to\infty} \inf \quad \text{mod } (\mathcal{R}^{k_n} f_{1,n}) > \varepsilon$$

then

$$\lim_{n\to\infty} \inf \quad \text{mod } (\mathcal{R}^{k_n} f_{2,n}) > \varepsilon$$

Let us show that it is indeed equivalent ; the only difficult implication in what we are just written above implying the equality of the limits. Then suppose the two limits are differents : $\lim_{n\to\infty} \inf \quad \text{mod } (\mathcal{R}^{k_n} f_{1,n}) < \lim_{n\to\infty} \inf$ mod $(\mathcal{R}^{k_n} f_{2,n})$. Then we can choose ε_0 such that

$$\lim_{n\to\infty} \inf \quad \text{mod } (\mathcal{R}^{k_n} f_{1,n}) < \varepsilon_0 < \lim_{n\to\infty} \inf \quad \text{mod } (\mathcal{R}^{k_n} f_{2,n})$$

Then it invalidates that for all $\varepsilon > 0$ if a limit is bigger than ε, the other must too. Avila and Lyubich then use proposition 7 and proposition 4 to prove this lemma. Now, let us state the second hypothesis :

H2 : there exists ε_0 such that for all $\delta > 0$ there exist an integer $N(\delta)$ and a positive real $\eta(\delta)$ such that if $[f_1] \in \widehat{\mathcal{H}}_0^{\mathbb{R}}(\delta)$ and $[f_2] \in \widehat{\mathcal{H}}_0(\delta)$ with $\delta_{\widehat{\mathcal{H}}_0}(f_1, f_2) \leq \eta(\delta)$ then for all $G \in \mathcal{G}$ and $(m, n) \in Q$ with $n - m \geq N(\delta)$, $G^{(m,n)}(f_2) \in \widehat{\mathcal{H}}(\varepsilon_0)$

Now, suppose this second hypothesis is not satisfied by our cocycles setting. Let ε_0 be the beau bound for real maps. Suppose **H2** is wrong for $\varepsilon_0/2$. Then there exists $\delta > 0$ such that, for all integer N and positive real number η such that if $f_1 \in \widehat{\mathcal{H}}_0^{\mathbb{R}}(\delta)$ and $f_2 \in \widehat{\mathcal{H}}_0(\delta)$ with $\delta_{\widehat{\mathcal{H}}_0}(f_1, f_2) \leq \eta$ there is a $G \in \mathcal{G}$ and $(n,m) \in Q$ with $n - m \geq N$ such that $G^{(m,n)}(f_2) \notin \widehat{\mathcal{H}}_0(\varepsilon_0)$. But as it is true for all N and η, we can construct two sequences $(f_{1,n})_{n\geq 1} \in (\widehat{\mathcal{H}}_0^{\mathbb{R}}(\delta))^{\mathbb{N}}$ and $(f_{2,n})_{n\geq 1} \in (\widehat{\mathcal{H}}_0(\delta))^{\mathbb{N}}$ such that $\lim_{n\to\infty} \delta_{\widehat{\mathcal{H}}_0}(f_{1,n}, f_{2,n}) = 0$ and a sequence of integers $(k_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mathcal{R}^{k_n} f_{2,n} \notin \widehat{\mathcal{H}}_0(\varepsilon_0/2)$$

Using subsequences, we can assume that these two sequences converge to the same limit. But by lemma 9, we must have

$$\liminf_{n\to\infty} \mod (\mathcal{R}^{k_n} f_{1,n}) = \liminf_{n\to\infty} \mod (\mathcal{R}^{k_n} f_{2,n})$$

Giving

$$\liminf_{n\to\infty} \mod (\mathcal{R}^{k_n} f_{1,n}) \leq \varepsilon_0/2$$

This last inequality contradicts the beau bounds for real maps because we must have

$$\liminf_{n\to\infty} \mod (\mathcal{R}^{k_n} f_{1,n}) \geq \varepsilon_0$$

The main purpose is now to prove that **H1** and **H2** imply contraction and Beau bounds for complex maps in the real hybrid class.

6.2 Retractions and Banach setting

6.2.1 Background on functional analysis

Roughly speaking, a retraction is a continuous map from a space X onto a subset $Y \subset X$ leaving each point of Y fixed. Here is a more accurate definition

Definition 33. *Given a topological space X and closed subspace $Y \subset X$, we say that a continuous map $P : X \to Y$ is a **retraction** if for all $y \in Y$, $P(y) = y$ ($P_{|Y} = Id$ and $P(X) \subset Y$). Y is called a **retract** for X.*

Proposition 9. *If $P : X \to Y$ is a retraction, then $P^2 = P$.*

Proof. Take $x \in X$. Then $P(x) \in Y$. But $P_{|Y} = Id$ so $P(P(x)) = P^2(x) = P(x)$. □

Some notions of functional analysis are also needed to continue our proof. Remember that a subset in a topological space is said to be **precompact** if its closure is compact.

Definition 34. *Let B and B' be two Banach spaces and $T : H_1 \to H_2$ be an operator.*

- *T is said to be* ***compact*** *if it maps bounded sets into precompact sets : $\overline{T(B_1)}$ is a compact set, where B_1 is the unit ball in B. Equivalently, T is compact if for all bounded sequence $(x_n)_{n\geq 1} \in B^{\mathbb{N}}$, $(Tx_n)_{n\geq 1}$ has a convergent subsequence in B'. Denote $\mathcal{K}(B)$ the space of compact operators.*

- *T is said to be of* ***finite-rank*** *if $\dim T(B) < \infty$. In that case, $r = \dim T(B)$ is the* ***rank*** *of the operator and $K_r(B)$ denotes the space of operators of rank r and $K(B) = \cup_{r\geq 0}(K_r(B))$ the space of all finite-rank operators.*

It is well known that every finite-rank operator is compact and that $\overline{K(B)} = \mathcal{K}(B)$ in the case where B is a Hilbert space. In particular, every compact operator is the limit of a sequence of finite-rank operators. The study of the spectrum of a compact set shows that for all non-zero eigenvalue of T where T is compact, the corresponding eigenspace is finite dimensional. Now suppose that we have a compact operator $T : B \to B$ such that $T^2 = T$.

Proposition 10. *Given T above, $T(B)$ is finite dimensional. It follows that T is fo finite rank.*

Proof. $X^2 - X$ annihilates T, so 0 and 1 are potential eignenvalues for T. But by the previous discussion, as T is compact, the eigenspace corresponding to the eigenvalue 1 must be finite dimensional. It clearly follows that $T(B)$ is finite dimensional. Its dimension is given by $Tr(T)$. □

All this material is required to state and prove the following result :

Theorem 23. *Given a complex Banach space $\mathcal{B}$ and a holomorphic map $P : U \to \mathcal{B}$ where U is an open set of $\mathcal{B}$ such that $P(0) = 0$, we assume that the derivative of P at 0 is a compact operator and that $P^2 = P$ near 0. Then for any open ball B of $\mathcal{B}$ around 0 (sufficiently small), $P(B)$ is a complex finite dimensional manifold. We have the same result for a "real" configuration.*

Proof.

- The derivative of P at 0, $DP(0)$, is of finite rank : it is a compact operator and because $P^2 = P$ near 0, we have that $DP(0)^2 = DP(0)$. $DP(0)$ satisfies the assumptions of proposition 10, and then it is finite rank.

- Next, we show that $f = Id - DP(0) - P$ is a diffeomorphism near 0. Indeed,

$$\begin{aligned} Df(0) &= Id - DP(0)^2 - DP(0) = Id - 2DP(0) \\ Df(0)^2 &= (Id - 2DP(0)) \circ (Id - 2DP(0)) = Id \end{aligned}$$

- Moreover, we have

$$f \circ P = P - DP(0) \circ P - P^2 = DP(0) \circ P$$
$$DP(0) \circ f = DP(0) - DP(0)^2 - DP(0) \circ P = DP(0) \circ P$$

this gives $f \circ P = DP(0) \circ f$; with this last equality, we get

$$P(B) = f^{-1}(DP(0)(h(B)))$$

But because $DP(0)$ is of finite rank and f a diffeomorphism, $P(B)$ is an open subset of a finite dimensional space.

□

As precised above, a Banach structure is required to use this result. That is why Avila and Lyubich introduce the notion of Banach slices.

6.2.2 Banach slices

Definition 35. *If $f \in \widehat{\mathcal{H}}_0$, an open quasidisk W is said to be f-**admissible** if the two following properties hold :*

- *The filled Julia set is a subset of W : $K(f) \subset W$*
- *f extends holomorphically to W, and continuously to ∂W.*

Remember that $\mathcal{B}_W$ stands for the Banach space of holomorphic functions in W. $\mathcal{B}_W^*$ will stand for the Banach space of functions $w \in \mathcal{B}_W$ such that :

- $w(z) = \mathcal{O}(z^{d+1})$ near 0
- w extends continuously to ∂W.

Using the metric of $\mathcal{B}_W$, the definition of balls in $\mathcal{B}_W^*$ is straightforward. $B_{W,r}^*$ will stand for the ball in $\mathcal{B}_W^*$ centred at $w = 0$ of radius r. Now, given a p.l.-map $f \in \widehat{\mathcal{H}}_0(\varepsilon)$, a f-admissible quasidisk W and a radius r_0, we would like to build a map from B_{W,r_0}^* to $\widehat{\mathcal{H}}_0$ giving, for $w \in B_{W,r_0}^*$ the function $f + w$. Such a constrcution will work be cause of the topology in $\widehat{\mathcal{H}}_0$:
For every $\varepsilon > 0$, $f \in \widehat{\mathcal{H}}_0 \varepsilon$, sufficiently small quasidisk that is f-admissible and $r > 0$, one can find a neighborhood $\mathcal{N}$ of f in $\widehat{\mathcal{H}}_0(\varepsilon)$ such that, for all $\widetilde{f} \in \mathcal{N}$,

- W is $\widetilde{f}$-admissible.
- The map $f - \widetilde{f}$ belongs to $B_{W,r}^*$.

This follows that given f and a f-admissible quasidisk W, one can find $\varepsilon_0 > 0$ and r_0 such that for all $w \in B_{W,r_0}^*$, $f + w$ can be restricted to a p.l.-map $\widetilde{f} : U \to V$ with $\mod \widetilde{f} \geq \varepsilon_0$ (ie $\widetilde{f} \in \widehat{\mathcal{H}}_0(\varepsilon_0)$). The map

$$\begin{array}{rcl} \mathcal{J}_{f,W,r_0} : B_{W,r_0}^* & \to & \widehat{\mathcal{H}}_0 \\ w & \mapsto & \widetilde{f} = f + w \end{array}$$

is then well defined, continuous and injective.

Theorem 24. *Given f, a sufficiently small quasidisk W that is f-admissible and r_0 constructed as above and a continuous map*

$$\begin{array}{ccc} \mathbb{D} & \to & B^*_{W,r_0} \\ \lambda & \mapsto & w_\lambda \end{array}$$

the two following statements are equivalent :

- $\lambda \mapsto w_\lambda$ *is holomorphic*
- $(f_\lambda)_{\lambda\in\mathbb{D}} = (\mathcal{J}_{f,W,r_0}(w_\lambda))_{\lambda\in\mathbb{D}}$ *is a holomorphic path in* $\widehat{\mathcal{H}}_0$.

Proof.

- we begin with the first inclusion. Suppose that $\lambda \mapsto w_\lambda$ is holomorphic. As $w_\lambda \in B^*_{W,r_0}$, $\mathcal{J}_{f,W,r_0}(w_\lambda)$ is well defined and as $\mathcal{J}_{f,W,r_0}$ is continious, it follows that $\lambda \mapsto f_\lambda$ is continuous. Now take a radius R such that $0 < R < 1/4$. We have already seen that if $f \in \widehat{\mathcal{H}}_0$, then $\mathbb{D}_{1/4} \subset K(f)$. It is now natural to consider the restriction operator

$$\begin{array}{rcl} I_R : \mathcal{B}^*_W & \to & \mathcal{B}_{\mathbb{D}_R} \\ f & \mapsto & f_{|\mathbb{D}_R} \end{array}$$

  which is holomorphic. But for f_λ, we have $I_R(f_\lambda) = I_R(f+w_\lambda) = I_R(f)+I_R(w_\lambda)$. $\lambda \mapsto I_R(f)$ is a constant and $\lambda \mapsto I_R(w_\lambda)$ is holomorphic as the composition of two holomorphic functions, $\lambda \mapsto w_\lambda$ and I_R. By lemma 33 (see appendix C), $(f_\lambda)_{\lambda\in\mathbb{D}}$ is a holomorphic path in $\widehat{\mathcal{H}}_0$.

- Now, assume that $(f_\lambda)_{\lambda\in\mathbb{D}}$ is a holomorphic path in $\widehat{\mathcal{H}}_0$. To prove holomorphicity of $\lambda \mapsto w_\lambda$, we will prove the "weak" holomorphicity of w_λ. Actually they are equivalent thanks to the following result, which is well known in complex analysis :

  Theorem 25. *Suppose that $F : U \to \mathcal{B}$ is map from an open set of $\mathbb{C}$ to a Banach space $\mathcal{B}$. The two following statements are equivalent :*

  - *For every bounded linear functional $L : \mathcal{B} \to \mathbb{C}$, $z \mapsto L(F(z))$ is holomorphic in U. (weak holomorphicity)*
  - *F is holomorphic in U.*

  In our case, the result to prove is that for every bounded linear functional $L : \mathcal{B}^*_W \to \mathbb{C}$, $\lambda \mapsto L(w_\lambda)$ is holomorphic. As $(f_\lambda)_{\lambda\in\mathbb{D}}$ is a holomorphic path and by Köebe-1/4 theorem (just like the way we did for Carathéodory hyperbolicity of hybrid leaves, $(\lambda, z) \mapsto f_\lambda(z)$ is holomorphic in $\mathbb{D} \times \mathbb{D}_{1/4}$. It follows that $(\lambda, z) \mapsto w_\lambda(z) = f_\lambda(z) - f(z)$ is holomorphic in $\mathbb{D} \times \mathbb{D}_{1/4}$. But as $z \mapsto w_\lambda(z)$ is holomorphic in W for all fixed λ, Hartog's theorem (theorem 34) gives that $(\lambda, z) \mapsto w_\lambda(z)$ is holomorphic in $\mathbb{D} \times W$. Remember that $w_\lambda \in \mathcal{B}^*_W$ means also that w_λ is continuous on ∂W. Then for a fixed $z \in \overline{W}$, the function $\lambda \mapsto f_\lambda(z)$ is holomorphic. The Riesz representation theorem gives the final argument of this proof ;

Theorem 26 (Riesz representation theorem)**.** *For every bounded linear functional $L : \mathcal{B}_W^*$, there exists a finite complex measure μ_L supported on $\overline{W}$, such that, for every $w \in \mathcal{B}_W^*$,*

$$L(w) = \int_{\overline{W}} w(z) d\mu_L(z)$$

But as $\lambda \mapsto w_\lambda(z)$ is holomorphic for every fixed $z \in \overline{W}$, $\int_{\overline{W}} w_\lambda(z) d\mu_L(z)$ is also holomorphic in λ, ie $\lambda \mapsto L(w_\lambda)$ is holomorphic for every bounded linear functional L. It follows, from theorem 25, that $\lambda \mapsto w_\lambda$ is holomorphic.

□

6.2.3 A result on retractions of $H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$

Using the Banach slices setting which adapt our configuration to theorem 23, we will prove the following :

Theorem 27. *Given a retraction $P \in H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ and assuming that : there exists a compact subset $\mathcal{K}$ of $\widehat{\mathcal{H}}_0$ such that if $(f_n)_{n\geq 1} \in \widehat{\mathcal{H}}_0^{\mathbb{N}}$ is a converging sequence with limit $f \in \widehat{\mathcal{H}}_0^{\mathbb{R}}$ then the sequence $(P(f_n))_{n\geq n_0} \in \mathcal{K}^{\mathbb{N}}$ for n_0 large enough;*
then the retraction P is constant.

In order to prove this result, let $P^{\mathbb{R}}$ be the restriction of P to $\widehat{\mathcal{H}}_0^{\mathbb{R}}$ and $\mathcal{Z}^{\mathbb{R}} = Im(P^{\mathbb{R}})$. By the assumption (existence of the compact set $\mathcal{K}$), $\mathcal{Z}^{\mathbb{R}} \subset \mathcal{K}$. It gives $P^{\mathbb{R}}(\mathcal{Z}^{\mathbb{R}}) = \mathcal{Z}^{\mathbb{R}} \subset P^{\mathbb{R}}(\mathcal{K})$, thus $\mathcal{Z}^{\mathbb{R}} = P^{\mathbb{R}}(\mathcal{K})$. But $P(\mathcal{K})$, as the image of a compact through a continuous function, is compact so $\mathcal{Z}^{\mathbb{R}}$ is compact too.

Lemma 10. *$\mathcal{Z}^{\mathbb{R}}$ is finite dimensional manifold.*

Proof. We take $\varepsilon > 0$ such that $\mathcal{K} \subset \widehat{\mathcal{H}}_0(\varepsilon)$. Suppose $f \in \mathcal{Z}^{\mathbb{R}}$ and consider a neighborhood $\mathcal{U}$ of f in $\widehat{\mathcal{H}}(\varepsilon/2)$; we choose $\mathcal{U}$ such that for all $\widetilde{f} \in \overline{\mathcal{U}}$, $\widetilde{f}$ is defined on some f-admissible quasidisk W. In particular, $\overline{U}$ belongs to a certain ball centred in f with some radius $r > 0$. The corresponding ball in $\mathcal{B}_W^*$ is $B_{W,r}^*$. This inclusion can be seen as a map

$$\begin{array}{rcl} J : \overline{\mathcal{U}} & \to & B_{W,r}^* \\ \widetilde{f} & \mapsto & \widetilde{f} - f_{|W} \end{array}$$

$\overline{\mathcal{U}}$, as a closed subset of the compact set $\widehat{\mathcal{H}}_0(\varepsilon/2)$, is itself compact. As J is continuous, $J(\overline{\mathcal{U}})$ is compact too. By the assumption, one can take a $\rho > 0$ such that for $w \in B_{W,\rho}^*$, $P(\mathcal{J}_{f,W,\rho}(w)) \in \mathcal{K}$; in particular, $\mod w \geq \varepsilon$. But then, by continuity of P (and because $\mathcal{U} \subset \widehat{\mathcal{H}}_0(\varepsilon/2)$), we cant take ρ even smaller in order to have $P(\mathcal{J}_{f,W,\rho}(w)) \in \mathcal{U}$. If we denote by

$$P_f = J \circ P \circ \mathcal{J}_{f,W,\rho} : B_{W,\rho}^* \to B_{W,r}^*$$

P_f is easily a retraction (we have $P_f^2 = P_f$ and $P_f(0) = 0$). By compacity of $J(\overline{\mathcal{U}})$, it is easily a compact operator. Considering its "real" part $P_f^{\mathbb{R}} : B_{W,\rho}^{*,\mathbb{R}} \to B_{W,r}^{*,\mathbb{R}}$, $P_f^{\mathbb{R}}$ is a real analytic and compact retraction : applying theorem 23 to it, one gets that $Im(P_f^{\mathbb{R}})$ ia a real analytic finite dimensional manifold (submanifold of $B_{W,\rho}^*$). $\mathcal{Z}^{\mathbb{R}}$ is compact and has the same topology as $Im(P_f^{\mathbb{R}})$, thus it is also a finite dimensional manifold near f. But the choice of $f \in \mathcal{Z}^{\mathbb{R}}$ was completely arbitrary, so $\mathcal{Z}^{\mathbb{R}}$ is finite dimensional. □

Lemma 11. *$\mathcal{Z}^{\mathbb{R}}$ is a single point.*

Proof. By homeomorphicity of the mating $i_c : \mathcal{E} \to \widehat{\mathcal{H}}_c$, $\widehat{\mathcal{H}}_0^{\mathbb{R}}$ is homeomorph to $\mathcal{E}^{\mathbb{R}}$ through i_0. But by theorem 7, $\mathcal{E}^{\mathbb{R}}$ is contractible. It follows that $\widehat{\mathcal{H}}_0^{\mathbb{R}}$ is contractible. But $\mathcal{Z}^{\mathbb{R}} = Im(P_{|\widehat{\mathcal{H}}_0^{\mathbb{R}}})$ is clearly a retract of $\widehat{\mathcal{H}}_0^{\mathbb{R}}$, so it is also contractible. Indeed, take any homotopy

$$h : \widehat{\mathcal{H}}_0^{\mathbb{R}} \times [0,1] \to \{f_0\}$$

contracting $\widehat{\mathcal{H}}_0^{\mathbb{R}}$ to a point f_0 of $\mathcal{Z}^{\mathbb{R}}$. Then

$$P \circ h : \mathcal{Z}^{\mathbb{R}} \times [0,1] \to \{f_0\}$$

is an homotopy that contracts $\mathcal{Z}^{\mathbb{R}}$ to one of its points f_0. We assume that a contractible finite-dimensional manifold that is compact is a point (otherwise, we would need to introduce basic theory about homotopy groups, which is not our purpose here). This argument concludes the proof : $\mathcal{Z}^{\mathbb{R}}$ is compact, contractible, and finite dimensional by lemma 10 ; thus it reduces to a single point. □

Lemma 12. *$\mathcal{Z} = Im(P)$ is a single point.*

Proof. We are going to prove that $\mathcal{Z}$ has an isolated point. Indeed, $\widehat{\mathcal{H}}_0$ is connected (connected component of the hybrid class), and $\mathcal{Z}$ is the image of $\widehat{\mathcal{H}}_0$ through the (continuous) retraction P. Then, as the image of a connected set through a continuous map, $\mathcal{Z}$ is connected. But a connected set with an isolated point is obviously only composed of this isolated point. We will take the same notations as in the proof of lemma 10.
By lemma 11, considering that f is the only point in $\mathcal{Z}^{\mathbb{R}}$, $P_f(B_{W,\rho}^{*,\mathbb{R}}) = J(\{f\}) = 0$. But by analytic continuity, as P_f is holomorphic, P_f reduces to the constant function $w \mapsto 0$ on $B_{W,\rho}^* : P_f(B_{W,\rho}^*) = 0$.
Now, consider a small neighborhood $\mathcal{N}$ of $f \in \mathcal{Z}^{\mathbb{R}}$ in $\mathcal{Z}$ such that $J(\mathcal{N}) \subset B_{W,\rho}^*$. Then $\mathcal{N} \subset \mathcal{U}$. Suppose this is wrong : as $\mathcal{N}$ is chosen small, $f \in \overline{\mathcal{Z} \setminus U}$. Then we can choose a sequence $(f_n)_{n\geq 1}$ in $\mathcal{Z} \setminus U$ converging to f (in particular, $P(f_n) = f_n$). But by the assumption of the theorem, $(f_n)_{n\geq 1}$ converges to an element of $\widehat{\mathcal{H}}_0^{\mathbb{R}}$, so for n sufficienlty large, $P(f_n) = f_n \in \mathcal{K} \subset \widehat{\mathcal{H}}_0(\varepsilon) \subset \widehat{\mathcal{H}}_0(\varepsilon/2)$. So $\mod f_n \geq \varepsilon/2$ and the maps f_n belongs to $\widehat{\mathcal{H}}_0(\varepsilon/2)$ and thus are in $\mathcal{U}$. In the end, $\mathcal{N} \subset \mathcal{U} \subset B_{W,r}^*$ and $\mathcal{N} = P(\mathcal{N})$ because $N \subset Im\mathcal{Z}$ so

$$J(\mathcal{N}) = J(P(\mathcal{N})) \subset P_f(B_{W,\rho}^*) = \{0\}$$

But $J : \mathcal{N} \to \{0\}$ is injective, meaning that $\mathcal{N}$ consists only of f. f is indeed an isolated point, and then $\mathcal{Z}$ consists only of this single point. □

proof of theorem 27. By lemma 12, $Im(P)$ is a single point, so P is a constant function. □

6.3 Tame spaces

6.3.1 Definitions and results

Avila and Lyubich introduce the term "tame spaces" in their work in order to use the almost periodic cocycles and, in the end, prove contraction in the case of beau bounds for real maps. To present this notion, let us consider very basic objects. Let X be a topological space. X is assumed to be sequential, meaning that the limit of sequences is well defined ; continuity, compactness, etc are then defined sequentially. Suppose there exists a continuous metric on X

$$\delta : X \times X \to \mathbb{R}_+$$

The two following definitions are two basics topological notions that are required to define tame spaces.

Definition 36. *A set O in a topological space X is said to be* ***relatively open*** *with respect to a subset $K \subset X$ if $O \cap K$ is open in K for the relative topology on K.*

Definition 37. *A* ***filtration*** *on X is a family of sets $(X_i)_{i\in I}$ such that $X_i \subset X$ and $X_i \subset X_{i+1}$ for all $i \in I$.*

Definition 38. *Let X be a topological space as above. Suppose*

- *there is a filtration of compact sets $(X_i)_{i\in I}$ such that $\cup_{i\in I} X_i = X$*
- *every compact set in X is in fact contained in a X_i for some $i \in I$.*
- *a set O is open in X if and only if it is relatively open with respect to any compact subset K of X.*

Then X is a ***tame space.***

Now, consider families of functions between two tame spaces : let X_1, X_2 be two tame spaces with respective metrics δ_1 and δ_2 and $(F_\sigma)_{\sigma\in\Sigma}$ be a family of maps $F_\sigma : X_1 \to X_2$.

Definition 39.

1. *If for all compact subset K_1 of X_1, there exists another compact subset K_2 of X_2 such that, for all $\sigma \in \Sigma$, $F_\sigma(K_1) \subset K_2$, then $(F_\sigma)_{\sigma\in\Sigma}$ is called* ***equicompact***
2. *Let $(F_n)_{n\geq 1}$ be a sequence of maps from X_1 to X_2. Suppose*

- $(F_n)_{n\geq 1}$ *is equicompact*
- *there exists a continuous map* F *from* X_1 *to* X_2 *such that, for all compact set* K_1 *in* X_1

$$\lim_{n\to\infty} \sup_{x_1\in K_1} \delta_2(F_n(x_1), F(x_1)) = 0$$

then $(F_n)_{n\geq 1}$ *is said to be* ***uniformly converging to*** F ***on compact sets.***

3. *Let* $(G_\sigma)_{\sigma\in\Sigma}$ *be a family of cocycles. Suppose that for all compact subset* K_1 *of* X_1 *and any* $\gamma > 0$*, there exist a compact subset* K_2 *of* X_2 *and an integer* $N(\gamma)$ *such that, for all* $\sigma \in \Sigma$ *and* $(n,m) \in Q$ *with* $n - m \geq N(\gamma)$*,*

$$G_\sigma^{(m,n)}(K_1) \subset K_2 \tag{6.3}$$

$$diam(G_\sigma^{(m,n)}(K_1)) < \gamma \tag{6.4}$$

Then $(G_\sigma)_{\sigma\in\Sigma}$ *is said to be* ***uniformly contracting on compact sets***

Proposition 11. *If* X *is a tame space, the set* $\mathcal{S}_X$ *of all continuous weak contractions of* X *is a topological semigroup (with respect to the composition). The idempotent elements of* $\mathcal{S}_X$ *are the retractions of* X*.*

Proof. The composition is associative. Moreover, the composition of two continuous functions is continuous and the composition of two contractions is again a contraction. This proves that $(\mathcal{S}_X, \circ)$ is indeed a semigroup. Retractions are obviously idempotent elements. If an element $P \in \mathcal{S}_X$ is idempotent, then $P^2 = P$ and P is continuous, which is the definition of a retraction. □

We will assume the following result (the material concerning almost periodic cocycles is required to prove it, see appendix).

Lemma 13. *Suppose we are given a tame space* X*. Considering the corresponding semigroup* $\mathcal{S}_X$ *defined above, let* $\mathcal{G} = (G_\sigma)_{\sigma\in\Sigma}$ *be a family of* $\mathcal{S}_X$*-cocycles. Suppose that* $\mathcal{G}$ *is uniformly almost periodic (see appendix). Then if all the idempotent elments of* $\omega(\mathcal{G})$ *(ie retractions) are constants,* $\mathcal{G}$ *is uniformly contracting on compact sets.*

6.3.2 $\widehat{\mathcal{H}}_0$ as a tame space, consequences

$\widehat{\mathcal{H}}_0$ is equipped with the continuous Carathéodory metric. We know that for every $\varepsilon > 0$, $\widehat{\mathcal{H}}_0(\varepsilon)$ is compact and that for $\varepsilon_1 > \varepsilon_2$, $\widehat{\mathcal{H}}_0(\varepsilon_1) \subset \widehat{\mathcal{H}}_0(\varepsilon_2)$. Then defining $X_i = \widehat{\mathcal{H}}_0(1/2^i)$, we have, for all $i \in \mathbb{N}$, $X_i \subset \widehat{\mathcal{H}}_0$, X_i is compact and $X_i \subset X_{i+1}$. Moreover, $\cup_{i\in\mathbb{N}} X_i = \widehat{\mathcal{H}}_0$: there exists a filtration of compact sets for $\widehat{\mathcal{H}}_0$. Consequently, $\widehat{\mathcal{H}}_0$ satisfies the first property of tame spaces. Moreover, remember that for every compact set $\mathcal{K} \subset \mathcal{C}$, there exists a $\varepsilon > 0$ such that $\mathcal{K} \subset \mathcal{C}(\varepsilon)$. Restricting to the case of $\widehat{\mathcal{H}}_0$, this gives that any compact set $\mathcal{K} \in \widehat{\mathcal{H}}_0$ is contained in $\widehat{\mathcal{H}}_0(\varepsilon)$ for some $\varepsilon > 0$. But choosing i such that $1/2^i < \varepsilon$,

$\mathcal{K} \subset X_i$. This proves that $\widehat{\mathcal{H}}_0$ satisfies the second property of tame spaces. The third property of tame spaces follows because of the induced topology on compact sets of $\widehat{\mathcal{H}}_0$. Therefore $\widehat{\mathcal{H}}_0$ is a tame space.

Remember that in our first study of cocycles, we considered cocycles taking values in the semigroup of path holomorphic maps $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$. In fact, this semigroup is a sub-semigroup of the semigroup of continuous weak contractions $\mathcal{S}_{\widehat{\mathcal{H}}_0}$. Indeed, $\widehat{\mathcal{H}}_0$ equipped with its Carathéodory metric is Carathéodory hyperbolic and path holomorphic maps from Carathódory hyperbolic spaces are weakly contracting, thanks to the weak form of the Schwarz lemma. This proves that $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0) \subset \mathcal{S}_{\widehat{\mathcal{H}}_0}$.

The following lemma will be useful to prove that,if our cocycle setting satisfies the property **H1**, then macroscopic contraction follows along hybrid leaves.

Lemma 14. *The sub-semigroup $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ is closed in the sense of the uniform convergence on compact sets : if $(f_n)_{n\geq 1} \in (H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0))^{\mathbb{N}}$ is a sequence that converges uniformly on compact sets to f, then $f \in H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$.*

Proof. Suppose $\gamma : \mathbb{D} \to \widehat{\mathcal{H}}_0$ is a holomorphic path. Then, as $f_n \in H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ for all $n \geq 1$, $f_n \circ \gamma$ is also a holomorphic path. Take $0 < R < 1/4$. Our purpose is to use lemma 33 : proving that $(I_R(f \circ \gamma)$ is a holomorphic path in $\mathcal{B}_{\mathbb{D}_R}$ gives that $f \circ \gamma$ is holomorphic path in $\widehat{\mathcal{H}}_0$ by lemma 33. Thus, the only thing to prove is that $(I_R(f \circ \gamma)$ is indeed a holomorphic path in $\mathcal{B}_{\mathbb{D}_R}$.

Define $\Gamma_n = I_R \circ f_n \circ \gamma$. Then $(\Gamma_n)_{n\geq 1}$, by uniform convergence of $(f_n)_{n\geq 1}$ on compact sets to f, is itself uniformly converging on compact sets to $\Gamma = I_R \circ f \circ \gamma$. But $f_n \circ \gamma$ is a holomorphic path in $\widehat{\mathcal{H}}_0$ by assumption, so using the other sense of the equivalence in lemma 33, we get that $I_R \circ f_n \circ \gamma$ is a holomorphic path in $\mathcal{B}_{\mathbb{D}_R}$. But path holomorphicity in $\mathcal{B}_{\mathbb{D}_R}$ is equivalent to holomorphicity in the usual sense in $\mathcal{B}_{\mathbb{D}_R}$. Consequently, as the limit of a uniformly converging sequence of holomorphic maps on compact sets is itself holomorphic, Γ is holomorphic in $\mathcal{B}_{\mathbb{D}_R}$ and is then a holomorphic path in $\mathcal{B}_{\mathbb{D}_R}$. Consequently, $I_R \circ f$ is path holomorphic and by lemma 33, f is also path holomorphic. □

6.4 Proof of Beau bounds and macroscopic contraction fro complex maps in the real hybrid classes

Remember that our cocycle setting (the family $\mathcal{G}$ defined in the section of cocycles) satisfies the properties **H1** and **H2**. The aim of this section is to prove that **H1** and **H2** lead to the two following conclusions :

C1 : for every $\delta > 0$ and $\gamma > 0$, there exists an integer $N(\delta, \gamma)$ such that, for $f_1, f_2 \in \widehat{\mathcal{H}}_0(\delta)$ and for every $G \in \mathcal{G}$ and $(m,n) \in Q$ with $n - m \geq N(\delta, \gamma)$ we have macroscopic contraction :

$$\delta_{\widehat{\mathcal{H}}_0}(G^{(m,n)}(f_1), G^{(m,n)}(f_2)) < \gamma$$

C2 : for every $\delta > 0$, there exists an integer $N(\delta)$ such that, for $f \in \widehat{\mathcal{H}}_0(\delta)$ and for every $G \in \mathcal{G}$ and $(m,n) \in Q$ with $n - m \geq N(\delta)$,

$$\operatorname{mod} G^{(m,n)}(f) \geq \varepsilon_0$$

or, equivalently, $G^{(m,n)}(f) \in \widehat{\mathcal{H}}_0(\varepsilon_0)$ (ε_0 is the Beau bound for real maps).

6.4.1 Proof of an intermediate result

Suppose that $\mathcal{G}$ is a family of cocycles in $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ that satisfies **H1** but not **C1**. By **H1**, as for $f \in \widehat{\mathcal{H}}_0(\delta)$, $G^{(m,n)}(f)$ is in $\widehat{\mathcal{H}}_0(\varepsilon(\delta))$ for all $G \in \mathcal{G}$ and $(m,n) \in Q$, $\mathcal{G}$ is precompact and then $\mathcal{G}$ is uniformly almost periodic with values in the semigroup $\mathcal{S}_{\widehat{\mathcal{H}}_0}$. If **C1** is wrong, there exists $\delta > 0$ and $\gamma > 0$, a cocycle $G \in \mathcal{G}$ and two functions $f_1, f_2 \in \widehat{\mathcal{H}}_0(\delta)$ such that for all N, there exists $(m,n) \in Q$ with $n - m \geq N$ such that $\delta_{\widehat{\mathcal{H}}_0}(G^{(m,n)}(f_1), G^{(m,n)}(f_2)) > \gamma$. This means that $\mathcal{G}$ is not uniformly contracting on the compact set $\widehat{\mathcal{H}}_0(\delta)$ and consequently not uniformly sontracting on compact sets. Taking the converse of lemma 13, there exist a non constant idempotent in $\omega(\mathcal{G})$, ie there exists a sequence $(G_k)_{k\geq 1} \in \mathcal{G}^{\mathbb{N}}$ and two sequences of integers $(m_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ with $n_k - m_k \to \infty$ as $k \to \infty$ such that $(G_k^{(m_k,n_k)})_{k\geq 1}$ converges uniformly on compact sets to a non constant retraction $P \in \mathcal{S}_{\widehat{\mathcal{H}}_0}$. By lemma 14, $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ is closed so $P \in H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$. The following result sums up what has just been proved :

Theorem 28. *If $\mathcal{G}$ is a family of cocycles taking values in $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ and satisfying **H1** but not **C1**, then one can build sequences $(G_k)_{k\geq 1} \in \mathcal{G}^{\mathbb{N}}$, $(m_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ with $n_k - m_k \to \infty$ as $k \to \infty$ such that for every $f \in \widehat{\mathcal{H}}_0$,*

$$G_k^{(m_k,n_k)}(f) \to P(f)$$

where P is a non constant retraction in $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$.

6.4.2 Proof of C1 and C2

Suppose $\mathcal{G}$ is a cocycle amily taking values in the sub-semigroup $H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ and satisfies **H1** and **H2**. Now, suppose **C1** does not hold. thanks to the last paragraph (theorem 28), there exist sequences $(G_k)_{k\geq 1} \in \mathcal{G}^{\mathbb{N}}$, $(m_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ with $n_k - m_k \to \infty$ as $k \to \infty$ such that

$$G_k^{(m_k,n_k)}(f) \to P(f)$$

where P is a non constant retraction in $H(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$. But $G_k^{(m_k,n_k)} \in H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$, so taking the limit, this gives $P \in H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$. Now, **H2** gives that for every $\delta > 0$, $f_1 \in \widehat{\mathcal{H}}_0^{\mathbb{R}}(\delta)$ and $f_2 \in \widehat{\mathcal{H}}_0(\delta)$ there exist an integer $N(\delta)$ and $\eta(\delta) > 0$ such that, if $\delta_{\widehat{\mathcal{H}}_0}(f_1, f_2) \leq \eta(\delta)$, then

$$G_k^{(m_k,n_k)}(f_2) \in \widehat{\mathcal{H}}_0(\varepsilon_0)$$

for $n_k - m_k \geq N(\delta)$. Taking the limit, this gives $P(f_2) \in \widehat{\mathcal{H}}_0(\varepsilon_0)$. In that sense, this is clear that $P \in H^{\mathbb{R}}(\widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_0)$ satisfies the assumption of theorem 27 : taking $\mathcal{K} = \widehat{\mathcal{H}}_0(\varepsilon_0/2)$, for all sequence $(f_n)_{n\geq 1}$ converging to $f \in \widehat{\mathcal{H}}_0^{\mathbb{R}}$, $P(f_n) \in \mathcal{K}$ for n sufficiently large. Thus we can apply theorem 27 to P : P is constant, which gives the contradiction. Thus **C1** holds.

Now, for some $\delta > 0$, take $f_1 \in \widehat{\mathcal{H}}_0^{\mathbb{R}}(\delta)$ and $f_2 \in \widehat{\mathcal{H}}_0(\delta)$. Take some arbitrary $\gamma > 0$; by **C1**, there exists an integer N_1 such that, for every $G \in \mathcal{G}$ and $(n, m) \in Q$ with $n - m \geq N_1$,

$$\delta_{\widehat{\mathcal{H}}_0}(G^{(m,n)}(f_1), G^{(m,n)}(f_2)) < \gamma$$

As gamma is completely arbitrary, we can choose it so that **H2** holds for $G^{(m,n)}(f_1)$ and $G^{(m,n)}(f_2)$. Then there exists some integer N_2 such that, for all $G' \in \mathcal{G}$ and $(n', m') \in Q$ with $n' - m' \geq N_2$,

$$G'^{(m',n')}(G^{(m,n)}(f_2)) \in \widehat{\mathcal{H}}_0(\varepsilon_0)$$

Taking $G = G'$ and $m' = n$, we get $G^{(m,n')}(f_2) \in \widehat{\mathcal{H}}_0(\varepsilon_0)$ for $n' - m \geq N = N_1 + N_2$: **C2** holds.

6.4.3 General statement, without cocycle setting

The last thing to check is that the above result (adapted to the reduction of our main system to the cocycle setting) is indeed equivalent to the macroscopic contraction and beau bounds for real maps in the real hybrid class. In fact, this is obvious : we defined

$$\begin{array}{rcl} \Pi : \mathcal{C} & \to & \widehat{\mathcal{H}}_0 \\ f & \mapsto & i_0 \circ \pi(f) \end{array}$$

which restricts to a path holomorphic homeomorphism $\Pi_c : \widehat{\mathcal{H}}_c \to \widehat{\mathcal{H}}_0$ for every $c \in \mathcal{M}$. Moreover, we have seen that the family of cocycles

$$G_c : (m, n) \mapsto G_c^{(m,n)}(\Pi_c(f)) = \Pi_c(\mathcal{R}^{n-m}(\mathcal{R}^m f))$$

satisfies **H1** and **H2**, so **C1** and **C2** hold, considering a real-symmetric hybrid leaf. Thus this reduction is equivalent to the macroscopic contraction along one hybrid leaf ($\widehat{\mathcal{H}}_0$), but using Π, this is equivalent to the macroscopic contraction along every real-symmetric hybrid leaf. This enables to state a more general result :

Theorem 29. *There exists $\varepsilon_0 > 0$ such that, for any $\delta > 0$ and $\gamma > 0$, there exists an integer $N(\delta, \gamma)$ such that for every $f_1, f_2 \in \mathcal{C}(\delta)$ that belong to the same real-symmetric hybrid leaf $\widehat{\mathcal{H}}_c^{\mathbb{R}}$ and for all $n \geq N(\delta, \gamma)$,*

- *$\mathcal{R}^n f_1$ and $\mathcal{R}^n f_2$ belong to $\widehat{\mathcal{H}}_{c_n}(\varepsilon_0) \subset \mathcal{C}(\varepsilon_0)$, where $c_n = \chi(\mathcal{R}^n f_1) = \chi(\mathcal{R}^n f_2)$. (Beau bounds for complex maps)*
- *$\delta_{\widehat{\mathcal{H}}_{c_n}}(\mathcal{R}^n f_1, \mathcal{R}^n f_2) < \gamma$ (macroscopic contraction)*

Appendix A

Maximum modulus principle

A.1 Cauchy integral formula

We recall here the well-known Cauchy formula. Suppose we are given an open and simply connected set U in $\mathbb{C}$, a holomorphic function $f : U \to \mathbb{C}$ and a closed path γ in U. Now suppose $z \in U$ is inside the domain formed by γ. Then we have the following formula

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi \tag{A.1}$$

A.2 Statement of the principle

Suppose U is a connected and open subset in $\mathbb{C}$; we are given a holomorphic map $f : U \to \mathbb{C}$. If there exists any $a \in U$ such that, for all $z \in U$, $|f(a)| \geq |f(z)|$, then f is constant. Equivalently, the modulus $|f|$ of a non-constant holomorphic map $f : U \to \mathbb{C}$ cannot have a maximum in U.

A.3 Proof

Suppose that $|f(a)| \geq |f(z)|$ for all $z \in U$. We want to show that f is constant.

1. We choose some $\delta > 0$ such that $D(a, \delta) \subset U$, where $D(a, \delta)$ is the open disc centred in a and of radius δ. Then we can use the Cauchy integral formula for $\gamma = C(a, r)$ with $r < \delta$ where $C(a, r)$ is the circle centred in a and of radius r :
$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - a} d\xi \tag{A.2}$$

2. This is obvious to use the following parametrisation, for $\theta \in [0, 2\pi]$:

$$\xi = a + re^{i\theta} \tag{A.3}$$
$$d\xi = ire^{i\theta}d\theta \tag{A.4}$$

It gives

$$f(a) = \frac{1}{2\pi i}\int_{[0,2\pi]} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}d\theta = \frac{1}{2\pi}\int_{[0,2\pi]} f(a+re^{i\theta})d\theta \tag{A.5}$$

Taking the modulus and using that $|f(a)| \geq |f(z)|$ for all $z \in U$, we get :

$$|f(a)| \leq \frac{1}{2\pi}\int_{[0,2\pi]} |f(a+re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_{[0,2\pi]} |f(a)|d\theta = |f(a)| \tag{A.6}$$

Then we must have equality in the previous inequality. As $\theta \mapsto |f(a+re^{i\theta})|$ is a continuous and positive function, by a classical result in integration theory, we have that $|f(a+re^{i\theta})| = |f(a)|$ for $\theta \in [0, 2\pi]$. Note that we can do it for every $r < \delta$, meaning that $|f|$ is constant on $D(a, \delta)$.

3. We know that $|f|$ is constant on $D(a, \delta)$, so is $|f|^2$. But writing $f(x+iy) = u(x, y) + iv(x, y)$, this gives, for some constant $K \geq 0$:

$$u(x, y)^2 + v(x, y)^2 = K \tag{A.7}$$

We derive (A.7) with respect to x and y :

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \tag{A.8}$$
$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \tag{A.9}$$

If we susbtitute, using the Cauchy-Riemann equations (3.3) and (3.4), we get :

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \tag{A.10}$$
$$v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0 \tag{A.11}$$

The matrix form is

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{A.12}$$

The determinant of this system is $u^2 + v^2 = K$. If $K = 0$, then $u = 0$ and $v = 0$, ie $f(z) = 0$ in $D(a, \delta)$. If $K > 0$, using the system we clearly have $u_x = u_y = 0$. But

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial x} = u_x - iu_y = 0 \tag{A.13}$$

So f is constant on $D(a, \delta)$.

4. f coincides with a constant function on a non-empty open set $D(a, \delta)$, so by analytic continuation, f is constant on U itself.

Theorem 30 (Stronger version)**.** *The modulus of a non-constant analytic function defined on a connected open set U cannot have a local maximum in U.*

Proof : The deifference with the former result is that the maximum is **local** here : $|f(a)| \geq |f(z)|$ for all z in some $D(a, \delta) \subset U$ (not all U).By the previous result, f must be constant on $D(a, \delta)$ and by analytic continuation, it is constant on U itself.

Corollary 2. *Suppose U is a bounded connected and open subset in $\mathbb{C}$; we are given a holomorphic map $f : U \to \mathbb{C}$. Moreover we suppose that f is continuous on ∂U. Then the maximum of the modulus $|f|$ is attained on the boundary of U, ∂U :*

$$\sup_{z \in \overline{U}} |f(z)| = \sup_{z \in \partial U} |f(z)| \tag{A.14}$$

Proof : Since U is bounded, $\overline{U}$ is closed and bounded, and thus compact. $|f|$ is a continuous and positive function on a compact set $\overline{U}$, so $\sup_{z \in \overline{U}} |f(z)| < \infty$. If the maximum is attained on U, then by the maximum modulus principle, f is constant, and thus the maximum is also attained on ∂U. Otherwise the maximum is attained on ∂U and only on ∂U (by the maximum modulus principle) : (A.14) is true in the two cases.

Appendix B

Schwarz reflection principle

Here we state the Schwarz reflection principle, which enables to extend a holomorphic function defined on a domain of the upper half plane to the symmetric of this domain. Of course we need some hypothesis on the domain and the function.

Theorem 31 (Schwarz reflection principle)**.** *Let L be a segment on the real axis of the complex plane and $\Omega_+ \subset \mathbb{H}$ a region in the upper half plane. Suppose that for every $x \in L$, there exists some $r > 0$ such that the open disc $D(x,r)$ has its upper part into Ω_+, ie $D(x,r) \cap \mathbb{H} \subset \Omega_+$. We denote by Ω_- the symmetric of Ω_+ with respect to the real axis : $\Omega_- = \{z : \bar{z} \in \Omega_+\}$.*
Now let f be a holomorphic function in Ω_+ such that for every sequence $\{z_n\}_{n\geq 0}$ converging to a point of L, we have

$$\lim_{n\to\infty} \Im(f(z_n)) = 0 \tag{B.1}$$

Denote $\Omega = \Omega_+ \cup L \cup \Omega_-$. Then there exists an extension F of f, holomorphic in Ω such that $F = f$ in Ω_+ and for all $z \in \Omega$,

$$F(\bar{z}) = \overline{F(z)} \tag{B.2}$$

A proof is given by Rudin in Real and Complex Analysis. Here is another version of this principle, but the symmetry is done with respect to the unit circle.

Theorem 32. *Let L be an arc on $\partial\mathbb{D}$ and $\Omega_+ \subset \mathbb{C}\setminus\overline{\mathbb{D}}$ a region in the exterior of the unit circle. Suppose that for every $\rho \in L$, there exists some $r > 0$ such that the open disc $D(\rho,r)$ has its exterior part into Ω_+, ie $D(\rho,r) \cap (\mathbb{C}\setminus\overline{\mathbb{D}}) \subset \Omega_+$. We denote by Ω_- the symmetric of Ω_+ with respect to the unit circle : $\Omega_- = \{z : \bar{z}^{-1} \in \Omega_+\}$.*
Now let f be a holomorphic function in Ω_+ such that for every sequence $\{z_n\}_{n\geq 0}$ converging to a point of L, we have

$$\lim_{n\to\infty} |f(z_n)| = 1 \tag{B.3}$$

Denote $\Omega = \Omega_+ \cup L \cup \Omega_-$. *Then there exists an extension* F *of* f, *holomorphic in* Ω *such that* $F = f$ *in* Ω_+ *and for all* $z \in \Omega$,

$$F(\bar{z}^{-1}) = \overline{F(z)}^{-1} \tag{B.4}$$

Appendix C

A theorem for path holomorphic structures

Theorem 33. *Suppose that*

$$\begin{array}{rcl} \mathbb{D} & \rightarrow & \widehat{\mathcal{H}} \\ \lambda & \mapsto & f_\lambda \end{array}$$

is continuous. Let R be a radius $0 < R < 1/4$. The following statements are equivalent :

- *$(f_\lambda)_{\lambda\in\mathbb{D}}$ is a holomorphic path in $\widehat{\mathcal{H}}_0$.*
- *$(I_R(f_\lambda))_{\lambda\in\mathbb{D}}$ is a holomorphic path in $\mathcal{B}_{\mathbb{D}_R}$, where*

$$\begin{array}{rcl} I_R : \widehat{\mathcal{H}}_0 & \rightarrow & \mathcal{B}_{\mathbb{D}_R} \\ f & \mapsto & f_{|\mathbb{D}_R} \end{array}$$

  is the restriction operator.

Proof. It has already been shown that I_R is path holomorphic in the proof of the Carathéodory hyperbolicity of the hybrid leaves. Thus the only remaining thing to prove is that if $(I_R(f_\lambda))_{\lambda\in\mathbb{D}}$ is a holomorphic path in $\mathcal{B}_{\mathbb{D}_R}$, then $(f_\lambda)_{\lambda\in\mathbb{D}}$ is a holomorphic path in $\widehat{\mathcal{H}}_0$. Remember that a holomorphic path in $\widehat{\mathcal{H}}_0$ is a continuous family $(f_\lambda : U_\lambda \rightarrow V_\lambda) \in \widehat{\mathcal{H}}_0$ for $\lambda \in \mathbb{D}$ such that there exists a holomorphic motion $(h_\lambda) : \mathbb{C} \rightarrow \mathbb{C}$ of $\mathbb{C}$ over $\mathbb{D}$ with basepoint 0 such that $h_\lambda(K(f_0)) = K(f_\lambda)$, $\bar{\partial} h_\lambda = 0$ a.e. on $K(f_0)$ and $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ on $K(f_0)$. Our goal here is to construct such a holomorphic motion.

First it should be constructed from $intK(f_0)$ to $intK(f_\lambda)$ but by the extended λ-lemma, (theorem 2), we can assume that $h_\lambda : \mathbb{C} \rightarrow \mathbb{C}$. By the Böttcher theorem (see theorem 17), one can find a holomorphic one-to-one map (which is also onto) $\phi_\lambda : intK(f_\lambda) \to \mathbb{D}$ such that $\phi_\lambda \circ f_\lambda = P_0 \circ \phi_\lambda$ where $P_0 : z \mapsto z^d$. Now, define $h_\lambda = \phi_\lambda^{-1} \circ \phi_0$. Then, on $K(f_0)$:

$$h_\lambda \circ f_0 = \phi_\lambda^{-1} \circ \phi_0 \circ f_0 = \phi_\lambda^{-1} \circ P_0 \circ \phi_0$$

and

$$f_\lambda \circ h_\lambda = f_\lambda \circ \phi_\lambda^{-1} \circ \phi_0 = \phi_\lambda^{-1} \circ P_0 \circ \phi_0$$

proving that h_λ conjugates f_λ and f_0. Moreover, by analyticity and bijectivity of ϕ_λ, h_λ is holomorphic and injective.

The remaining thing to prove is that

$$\begin{array}{ccc} \mathbb{D} \times \mathbb{D} & \to & intK(f) \\ (z, \lambda) & \mapsto & \phi_\lambda^{-1}(z) \end{array}$$

is holomorphic. By assumption, $I_R(f_\lambda)$ is holomorphic in $(\lambda, z) \in \mathbb{D} \times \mathbb{D}_R$. Consequently, $(\lambda, z) \mapsto f_\lambda(z)$ is holomorphic in $\mathbb{D} \times \mathbb{D}_R$. We would like to construct a sequence of holomorphic (both in λ and z) maps $(\phi_{\lambda,n})_{n\geq 1}$ such that $\phi_{\lambda,n} \to \phi_\lambda$. To achieve this, we need to adapt the Köebe-1/4 theorem : let f satisfy the assumption of the Köebe-1/4 theorem. Then for $0 < R < 1/4$ and all $n \geq 1$:

$$\mathbb{D}_{R^{n+1}} \subset f(\mathbb{D}_{R^n})$$

In our proof of the Carathéodory hyperbolocity of the hybrid leaves, we have already seen that the inverse of the Böttcher coordinate satisfies the assumptions of the Köebe-1/4 theorem when $f \in \widehat{\mathcal{H}}_0$. Applying the above form of the Köebe-1/4 theorem to ϕ_λ^{-1} and $n = 2$ gives

$$\mathbb{D}_{R^3} \subset \phi_\lambda^{-1}(\mathbb{D}_{R^2})$$

Or, equivalently,

$$\phi_\lambda(\mathbb{D}_{R^3}) \subset \mathbb{D}_{R^2}$$

As the degree d is more than 2, we then have that for $z \in \mathbb{D}_{R^3}$, $(\phi_\lambda(z))^d \in \mathbb{D}_{R^4}$. But as we now see ϕ_λ as a dunction from $\mathbb{D}_{R^3}$ to $\mathbb{D}_{R^2}$, we can also apply the Köebe-1/4 theorem to it :

$$\mathbb{D}_{R^4} \subset \phi_\lambda(\mathbb{D}_{R^3})$$

This gives, for all $z \in \mathbb{D}_{R^3}$, $\phi_\lambda^{-1}((\phi_\lambda(z))^d) = f_\lambda(z) \in \mathbb{D}_{R^3}$. By iteration, we easily get that for all $n \geq 1$ and $z \in \mathbb{D}_{R^3}$, $f_\lambda^n(z) \in \mathbb{D}_{R^3}$. As $(\lambda, z) \mapsto f_\lambda(z)$ is holomorphic in $\mathbb{D} \times \mathbb{D}_R$, we conclude that $(\lambda, z) \mapsto f_\lambda^n(z)$ is holomorphic in $\mathbb{D} \times \mathbb{D}_{R^3}$.

Now, defining $\phi_{\lambda,n} : intK(f_\lambda) \to \mathbb{C}$ such that $\phi_{\lambda,n}^{d^n} = f_\lambda^n$ with a derivative at 0 equal to 1, we see that $\phi_{\lambda,n}$ uniformly converges to ϕ_λ on compacts sets of $intK(f_\lambda)$. But as $(\lambda, z) \mapsto f_\lambda^n(z)$ is holomorphic in $\mathbb{D} \times \mathbb{D}_{R^3}$, $(\lambda, z) \mapsto \phi_{\lambda,n}(z)$ is holomorphic and by uniform convergence, $(\lambda, z) \mapsto \phi_\lambda(z)$ is also holomorphic.

Now, use Köebe-1/4 theorem to $\phi_\lambda : \mathbb{D}_{R^3} \to \mathbb{C}$. This gives

$$\mathbb{D}_{R^4} \subset \phi_\lambda(\mathbb{D}_{R^3})$$

Equivalently,

$$\phi_\lambda^{-1}(\mathbb{D}_{R^4}) \subset \mathbb{D}_{R^3}$$

it follows that $(\lambda, z) \mapsto \phi_\lambda^{-1}(z)$ is holomorphic in $\mathbb{D} \times \mathbb{D}_{R^4}$. To conclude, we need to state Hartog's theorem :

Theorem 34. *If $f : U \to \mathbb{C}$ where U is an open set of $\mathbb{C}^n$ such that f is holomorphic in everyone of the n variables while the others are fixed, then f is a holomorphic function of all the n variables.*

But for each $\lambda \in \mathbb{D}$, $\phi_\lambda^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic and one-to-one. By holomorphicity in $\mathbb{D} \times \mathbb{D}_{R^4}$ and in $\mathbb{D}$ when λ is fixed, applying the Hartog's theorem, we obtain the holomorphicity of $(\lambda, z) \mapsto \phi_\lambda^{-1}(z)$ in $\mathbb{D} \times \mathbb{D}$. □

Appendix D

Convegence of cocycles, almost periodic cocycles

Here is a background that Avila and Lyubich put in their proof. The almost periodic cocycles play a key role in the proof, but this notion and its developments are topological. Then we choose to present it, just the way Avila and Lyubich did but without proving anything.

Definition 40 (convergence, ω-limit set). *Suppose we are given a semigroup $\mathcal{S}$ and an $\mathcal{S}$-cocycle G.*

- *if for all $m \in \mathbb{N}$, the limit $\lim_{n\to\infty} G^{(m,n)}$ exists (we will denote this limit $G^{(m,\infty)}$ for more simplicity), we say that the cocycle G **converges**.*
- *if $\lim_{m\to\infty} G^{(m,\infty)}$ exists, we say that the cocycle G **double converges** and we denote by $G^{(\infty,\infty)}$ the corresponding limit.*
- *the ω-limit set of a cocycle G, $\omega(G)$, consists of the set of existing limits $\lim_{m\to\infty, n-m\to\infty} G^{(m,n)}$.*

Note. *The property of cocycles still hold for $G^{(m,\infty)}$ and $G^{(\infty,\infty)}$ (when these limits exist) : for all $(m,n) \in Q$, $G^{(m,\infty)} = G^{(n,\infty)} \circ G^{(m,n)}$; for all $m \in \mathbb{N}$, $G^{(m,\infty)} = G^{(\infty,\infty)} \circ G^{(m,\infty)}$.*
*In particular putting $m = \infty$ gives that $(G^{(\infty,\infty)})^2 = G^{(\infty,\infty)}$: it is an **idempotent**.*

Definition 41 (almost periodic cocycle). *If the family $\left\{G^{(m,n)}\right\}_{(m,n)\in Q}$ is precompact in the semigroup $\mathcal{S}$, then the cocycle G is said to be **almost periodic**.*

In particular, the ω-limit set of an almost periodic cocycle is compact as a closed subset of a precompact set. We naturally define a **subcocycle** as the restriction of a cocycle to a subsequence $(k_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$: $(G^{(k_m,k_n)})_{(n,m)\in Q}$.

Proposition 12. *Given an almost periodic cocycle G, there exists a subsequence $(k_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$ such that the subcocycle $(G^{(k_m,k_n)})_{(n,m)\in Q}$ is double converging.*

It has the following consequence

Corollary 3. *Given an almost periodic cocycle G, its ω-limit set $\omega(G)$ contains an idempotent element.*

Definition 42 (convergence in the space of cocycles)**.** *A sequence of cocycles $(G_n)_{n\geq 1}$ is said to be* ***convergent*** *to a cocycle G ($G_k \to G$ as $k \to \infty$) if, for all $(m,n) \in Q$*

$$\lim_{k\to\infty} G_k^{(m,n)} = G^{(m,n)}$$

Definition 43. *Suppose $\rho : \mathcal{S} \to \mathbb{R}_+$ is a positive function. A cocycle G is said to be* ***uniformly ρ-contracting*** *if for any $\gamma > 0$, there exists an integer $N(\gamma)$ such that for all $(m,n) \in Q$ with $n - m \geq N(\gamma)$,*

$$\rho(G^{(m,n)}) < \gamma$$

Definition 44 (Lyapunov pair)**.** *Given two positive functions $\rho_1 : \mathcal{S} \to \mathbb{R}_+$ and $\rho_2 : \mathcal{S} \to \mathbb{R}_+$ such that $\rho_2 \geq \rho_1$ and a cocycle G, we say that (ρ_1, ρ_2) is a* ***Lyapunov pair*** *adapted to the cocycle G if for any integers $l < m < n$,*

$$\rho_1(G^{(l,n)}) \leq \rho_1(G^{(l,m)}) \tag{D.1}$$
$$\rho_1(G^{(l,n)}) \leq \rho_2(G^{(m,n)}) \tag{D.2}$$

Now, we will consider a family $\mathcal{G} = (G_s)_{\sigma\in\Sigma}$ of cocycles.

Definition 45 (uniform almost periodicity)**.** *Given a family $\mathcal{G}$ as above, if the family $(G_\sigma^{(m,n)})_{(m,n)\in Q,\sigma\in\Sigma}$ is precompact in $\mathcal{S}$, then $\mathcal{G}$ is said to be* ***uniformly almost converging.***

- *The ω-limit set of a family $\mathcal{G}$, denoted $\omega(\mathcal{G})$, consists of the limits of the converging sequences $(G_{\sigma_k}^{(m_k,n_k)})_{k\geq 1} \in \mathcal{S}^{\mathbb{N}}$ where $(m_k, n_k, \sigma_k) \in (\mathbb{N}^2 \times \Sigma)^{\mathbb{N}}$ is such that $n_k - m_k \to \infty$ as $k \to \infty$.*

- *$\mathcal{G}$ is* ***uniformly ρ-contracting*** *for a positive $\rho : \mathcal{S} \to \mathbb{R}_+$ if, for any $\gamma > 0$, there exists an integer $N(\gamma)$ such that, for all $s \in \Sigma$ and $(n,m) \in Q$ with $n - m \geq N(\gamma)$,*
$$\rho(G_s^{(m,n)}) < \gamma$$

Lemma 15. *Suppose we are given a uniformly almost periodic family of cocycles $\mathcal{G}$ and a Lyapunov pair (ρ_1, ρ_2) adapted to every cocycle in $\mathcal{G}$. If for all idempotent $E \in \omega(\mathcal{G})$, $\rho_2(E) = 0$, then the family $\mathcal{G}$ is uniformly ρ_1-contracting.*