Period Three, Chaos and Fractals

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Imperial, May 2012

Three Different Projects

- (A) Real one-dimensional dynamics: builds on analysis from the first year.

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- (B) Complex dynamics: ask you to computer experiments and computer coding.
- (C) Complex dynamics: looks ahead at material from the 2nd year, but requires no material except what you already know from the first year (and knowledge about complex numbers - what you already know from secondary school).

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Let $f\colon [0,1] \to [0,1]$ be continuous and consider successive *iterates* $x_{n+1} = f(x_n)$ of a point $x_0 = x \in [0,1]$. So $x_n = f^n(x_0)$ where

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Theorem (Sarkovskii 1964)

If $p \prec q$ and f has a periodic point of (minimal) period p then it also has a periodic point of minimal period q.

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3. *If* $J_0, J_1 \subset [0,1]$ *are intervals and* $f(J_0) \supset J_1$ *then there exists an interval* $J \subset J_0$ *so that* $f(J) = J_1$*;*

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1. *Then $f(J)$ is also an interval;*
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3. *If $J_0, J_1 \subset [0,1]$ are intervals and $f(J_0) \supset J_1$ then there exists an interval $J \subset J_0$ so that $f(J) = J_1$;*
4. *Let $J_0, J_1, \dots, J_m$ be intervals in $[0,1]$ so that $f(J_i) \supset J_{i+1}$ for $i = 0, \dots, m-1$. Then there exists an interval $J \subset J_0$ so that $f^i(J) \subset J_i$ for $i = 0, 1, \dots, m-1$ and $f^m(J) = J_m$.*

(1) is the intermediate value theorem, (2) was proved last time. (4) follows from repeatedly applying (3). So why is (3) true: draw pictures!!! (Add some explanation during the lecture....)

Let x be a point of period three, so the set $x, f(x), f(x), \dots$ consists of three distinct points $a < b < c$.
Let

$$I_1 = [a, b] \text{ and } I_2 = [b, c].$$

Since f permutes points from the set $\{a, b, c\}$ (fixing none of these points), depending on whether the middle point goes to the left or to the right,

$$f(b) = a \text{ and } f(a) = c \text{ and } f(c) = b$$

or

$$f(b) = c \text{ and } f(a) = b \text{ and } f(b) = a.$$

Let us assume the former (the latter case goes the same up to relabelling). Then Statement (1) of the previous lemma implies

$$f(I_1) \supset I_1 \cup I_2 \text{ and } f(I_2) \supset I_1$$

A more abstract way to encapsulate the outcome of Step 1 is to say that we have the following graph.

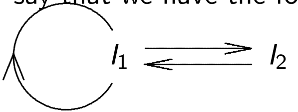

Figure: The Markov graph associated to a periodic point of period three.

The periodic orbits we construct correspond to paths in this graph.

$$I_1 \to I_1 \to \dots I_1 \to I_2 \to I_1.$$

Remember that

$$f(I_1) \supset I_1 \cup I_2 \text{ and } f(I_2) \supset I_1$$

Let m be an integer and let us take

$$J_0, \ldots, J_{m-2} \text{ all equal to } I_1,$$

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Since $f(J_i) \supset J_{i+1}$ for each $i = 0, 1, \ldots, m-1$, by Part (4) of the lemma, there exists an interval $J \subset J_0 = I_1$ so that $f^i(J) \subset J_i$ for $i = 0, 1, \ldots, m-1$ and

$$f^m(J) = J_m = I_1 \supset J.$$

The last inclusion and Part (2) of the lemma implies that there exists $x \in J$ so that $f^m(x) = x$.

Remember that

$$I_1 = [a,b]\ ,\ I_2 = [b,c] \text{ and } f(a) = c, f(b) = a, f(c) = b \tag{1}$$

$$x, f(x), \ldots, f^{m-2}(x) \in I_1 \text{ and } f^{m-1}(x) \in I_2. \tag{2}$$

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Suppose by contradiction that not all the points $x, f(x), \dots, f^{m-1}(x)$ are distinct.

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Suppose by contradiction that not all the points $x, f(x), \ldots, f^{m-1}(x)$ are distinct. Then $f^i(x) = f^j(x)$ for some $0 \le i < j < m$ and therefore $f^{i+k}(x) = f^{j+k}(x)$ for all $k \ge 0$. So

$$f^{i'}(x) = f^{m-1}(x) \text{ for some } 0 \le i' < m-1.$$

By (2), $f^{i'}(x) = f^{m-1}(x) = b$. Hence, using (1),

$$f^{i'+1}(x) = f^m(x) = f(b) = a \text{ and } x = f^m(x) = a. \tag{3}$$

But then

$$f(x) = f(a) = c \notin I_1$$

contradicting (2) unless $m = 2$. If $m = 2$ then $i' = 0$ and (3) gives $f(a) = a$, contradicting (1) and that a, b, c are all distinct.

Topic A. Some more examples

What if you consider the map $f(x) = 4x(1-x)$ and $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Then

$$f(I_1) \supset I_1 \cup I_2 \text{ and } f(I_2) \supset I_1 \supset I_2.$$

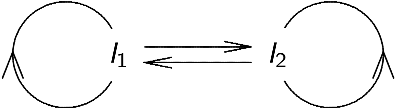

Figure: The Markov graph associated to the map $f(x) = 4x(1-x)$.

How many periodic points are there? Yes, there is a periodic point associated to

$$I_1 \to I_2 \to I_1 \to I_2 \ldots I_1.$$

What if you consider the map $f(x) = 2x(1-x)$ and $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Then

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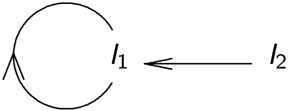

Figure: The Markov graph associated to the map $f(x) = 2x(1-x)$.

How many periodic points are there? Only the one associated to

$$I_1 \to I_1 \to I_1 \to I_1 \to \cdots \to I_1.$$

Task for the next session: take a periodic point of period 5 and let $I_1, \dots, I_4$ be the four intervals associated to these five points. Can you associate a graph to this? Consider several possibilities, on how the five points are permuted. Can you make a general conclusion?

Topics B/C: Julia sets

As agreed, these two topics are now merged, but you can choose to emphasise in your presentation the numerical or the mathematical part.

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We showed that $J(f) = [-2, 2]$ when $f(z) = z^2 - 2$. Please include a proof of this in your presentation.

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We also showed that f has infinitely many periodic points. Also include a proof of this in your presentation.

Any questions on this?

Last time we discussed whether the Julia set was 'connected'.
Let's make precise what we mean by this,

- a set $A \subset \mathbb{C}$ is called **open** if around each $x \in A$ there exists a ball $B(x,r)$ so that $B(x,r) \subset A$.

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  - $X \cap U_1, X \cap U_2 \neq \emptyset$ and
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Example 2: The set $\mathbb{C} \setminus \mathbb{R}$ is not connected. This follows from the definition: let U_i be the upper and lower half planes.

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Example 3: Assume that $\alpha\colon [0,1] \to \mathbb{C}$ is a closed path without self-intersections (i.e. $\alpha(0) = \alpha(1)$ and $\alpha(s) \neq \alpha(t)$ for all $0 < s < t < 1$). Then $\mathbb{C} \setminus \alpha[0,1]$ is not connected.

The proof of this is challenging, and I do **not** expect you to include a proof. This result is closely related to the Jordan Theorem, which is a deep theorem.

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More generally, if X is 'disk-like' then the same holds.

What happens if X is an annulus?

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What happens if X is an annulus?

Hint (if you prefer to do something more computational): Use Matlab or Maple to draw $f^{-1}(B)$ for various choice of balls B and annuli. Observe whether the resulting set if connected.

Let's apply this to the **Julia set**

$$J(f) = \partial B(\infty) = \partial\{z; |f^n(z)| \to \infty \text{ as } n \to \infty\}$$

and assume that c so that **each** point outside the ball $B(0, 10)$ centred at 0 and with radius goes off to infinity.

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Then

$$J(f) = \partial\{z; |f^n(z)| \geq 10 \text{ for some } n \geq 0\}.$$

So

$$J(f) \subset \partial \cap_{n\geq 0} f^{-n}(B(0; 10)).$$

Proposition

If $|f(0)| \geq 10$ then $J(f)$ is **not** *connected.*

Proposition

If $|f^n(0)| \geq 10$ for some $n \geq 0$ then $J(f)$ is **not** *path connected.*

Project B/C: A dichotomy

Theorem

The Julia set of $f_c(z) = z^2 + c$ is connected if and only if the sequence $|f_c^n(0)|$ is bounded.

The **Mandelbrot set** is the set of c so that $|f_c^n(0)|_{n\geq 0}$ is bounded.

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Hint: for presentation. Draw pictures, either by hand or by computer of the set $f^{-2}(B(0, 10))$ when $f(0) \notin B(0, 10)$.

Project B/C: What to do by the next session

Your presentation could include the following:

- Draw pictures of regions D in $\mathbb{C}$ and determine $f^{-1}(D)$. Here you do not need to be formal, but just to show understanding.
- Try to show that a line segment in $\mathbb{C}$ is connected.
- Try to show that the union of several segments in $\mathbb{C}$ which all go through $0 \in \mathbb{C}$ is connected.
- Show pictures of the Julia set of $f(z) = z^2 + c$ for various choices of c.