Metric Spaces

MA222

David Preiss
d.preiss@warwick.ac.uk

Warwick University, Spring 2008/2009

Chapter 1. Metric Spaces

Definitions. A metric on a set M is a function $d : M \times M \to \mathbb{R}$ such that for all $x, y, z \in M$,

- $d(x, y) \geq 0$; and $d(x, y) = 0$ if and only if $x = y$ (*d is positive*)
- $d(x, y) = d(y, x)$ (*d is symmetric*)
- $d(x, z) \leq d(x, y) + d(y, z)$ (*d satisfies the triangle inequality*)

The pair (M, d) is called a metric space.

If there is no danger of confusion we speak about the metric space M and, if necessary, denote the distance by, for example, d_M.

The open ball centred at $a \in M$ with radius r is the set

$$B(a, r) = \{x \in M : d(x, a) < r\}$$

the closed ball centred at $a \in M$ with radius r is

$$\{x \in M : d(x, a) \leq r\}.$$

A subset S of a metric space M is bounded if there are $a \in M$ and $r \in (0, \infty)$ so that $S \subset B(a, r)$.

Normed linear spaces

Definition. A norm on a linear (vector) space V (over real or complex numbers) is a function $\|\cdot\| : V \to \mathbb{R}$ such that for all $x, y \in V$,

- $\|x\| \geq 0$; and $\|x\| = 0$ if and only if $x = 0$ (*positive*)
- $\|cx\| = |c|\|x\|$ for every $c \in \mathbb{R}$ (or $c \in \mathbb{C}$) (*homogeneous*)
- $\|x + y\| \leq \|x\| + \|y\|$ (*satisfies the triangle inequality*)

The pair $(V, \|\cdot\|)$ is called a normed linear (or vector) space.

Fact 1.1. *If $\|\cdot\|$ is a norm on V then $d(x, y) = \|x - y\|$ is a metric on V.*

Proof. Only the triangle inequality needs an argument:

$$\begin{aligned} d(x, z) &= \|x - z\| = \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z) \qquad \square \end{aligned}$$

Examples

Example (Euclidean n spaces). $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the norm

$$\|x\| = \sqrt{\sum_{i=1}^n |x_i|^2} \quad \text{so with metric} \quad d(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$$

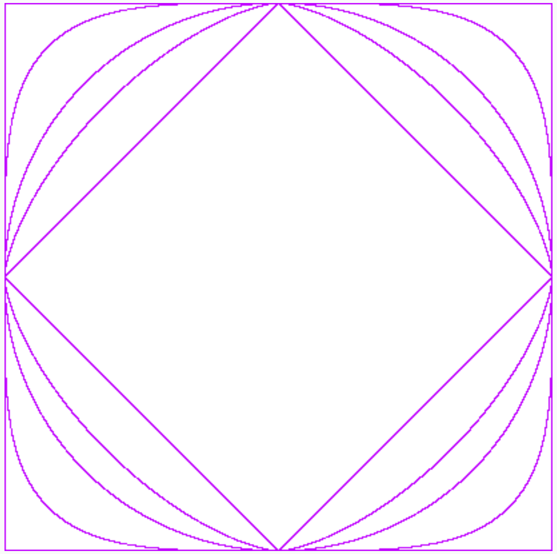
Example (n spaces with ℓ_p norm, $p \geq 1$). $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the norm

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \quad \text{so with metric} \quad d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{\frac{1}{p}}$$

Example (n spaces with max, sup or ℓ_∞ metric). $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the norm

$$\|x\|_\infty = \max{}_{i=1}^n |x_i| \quad \text{so with metric} \quad d_\infty(x, y) = \max{}_{i=1}^n |x_i - y_i|$$

Balls in ℓ_p norms



Balls in $\mathbb{R}^2$ with the ℓ_1, $\ell_{\frac{3}{2}}$, ℓ_2, ℓ_4 and ℓ_∞ norms.

Convexity of ℓ_p balls

We show that the unit ball (and so all balls) in ℓ_p norm are convex. (This is an important fact, although for us it is only a tool for proving the triangle inequality for the ℓ_p norms.) So we wish to prove:

If $\|x\|_p, \|y\|_p \leq 1$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ then $\|\alpha x + \beta y\|_p \leq 1$.

Proof for $p < \infty$; for $p = \infty$ it is left as an exercise.
Since the function $|t|^p$ is convex (here we use that $p \geq 1$!),

$$|\alpha x_i + \beta y_i|^p \leq \alpha |x_i|^p + \beta |y_i|^p.$$

Summing gives

$$\sum_{i=1}^{n} |\alpha x_i + \beta y_i|^p \leq \alpha \sum_{i=1}^{n} |x_i|^p + \beta \sum_{i=1}^{n} |y_i|^p \leq \alpha + \beta = 1.$$

So $\|\alpha x + \beta y\|_p \leq 1$, as required. □

Proof of triangle inequality for ℓ_p norms

Proof. (In this proof we write $\|\cdot\|$ instead of $\|\cdot\|_p$.)
The triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ is obvious if $x = 0$ or $y = 0$, so assume $x, y \neq 0$. Let

$$\hat{x} = \frac{x}{\|x\|},\ \hat{y} = \frac{y}{\|y\|},\ \lambda = \frac{1}{\|x\| + \|y\|},\ \alpha = \lambda\|x\| \text{ and } \beta = \lambda\|y\|.$$

Then

$$\|\hat{x}\| = 1,\ \|\hat{y}\| = 1,\ \alpha, \beta, \lambda > 0,\ \alpha + \beta = 1$$

and

$$\lambda(x + y) = \alpha\hat{x} + \beta\hat{y}.$$

Since $\|\alpha\hat{x} + \beta\hat{y}\| \leq 1$ by convexity of the unit ball,

$$\begin{aligned}\|x + y\| &= (\|x\| + \|y\|)\|\lambda(x + y)\| \\ &= (\|x\| + \|y\|)\|\alpha\hat{x} + \beta\hat{y}\| \leq \|x\| + \|y\|.\end{aligned}$$

□

Some exotic metric spaces

Example (Discrete spaces). Any set M with the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Example (Sunflower or French railways metric in $\mathbb{R}^2$).

$$d(x, y) = \begin{cases} \|x - y\| & \text{if } x, y \text{ lie on the same line passing through origin} \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

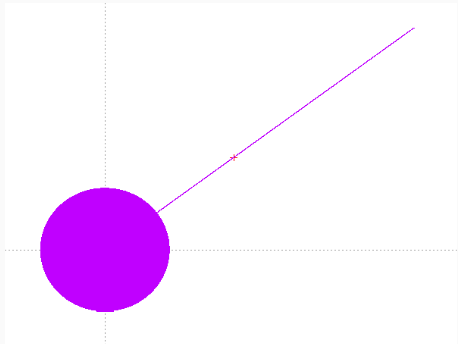
Example (Jungle river metric in $\mathbb{R}^2$).

$$d(x, y) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & \text{otherwise} \end{cases}$$

Balls in sunflower metric

$$d(x,y) = \begin{cases} \|x-y\| & x, y, 0 \text{ colinear} \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

centre $(4,3)$, radius 6



Subspaces, product spaces

Subspaces. If M is a metric space and $H \subset M$, we may consider H as a metric space in its own right by defining $d_H(x,y) = d_M(x,y)$ for $x, y \in H$. We call (H, d_H) a (metric) subspace of M.

Agreement. If we refer to $M \subset \mathbb{R}^n$ as a metric space, we have in mind the Euclidean metric, unless another metric is specified.

Warning. When subspaces are around, confusion easily arises. For example, in $\mathbb{R}$, the ball $B(0,1)$ is the interval $(-1,1)$ while in the metric space $[0,2]$, the ball $B(0,1)$ is the interval $[0,1)$.

Products. If M_i are metric spaces, the product $M_1 \times \cdots \times M_n$ becomes a metric space with any of the metrics

$$d(x,y) = \left(\sum_{i=1}^{n} (d_i(x_i,y_i))^p \right)^{\frac{1}{p}} \quad \text{or} \quad \max{}_{i=1}^{n} d_i(x_i,y_i)$$

where $1 \le p < \infty$.

Function & Sequence Spaces

$\mathcal{C}([a,b])$ with maximum norm. The set $\mathcal{C}([a,b])$ of continuous functions on $[a,b]$ with the norm

$$\|f\| = \sup_{x\in[a,b]} |f(x)| \left(= \max_{x\in[a,b]} |f(x)| \right)$$

$\mathcal{C}([a,b])$ with L_p norm. Very different norms on $\mathcal{C}([a,b])$ are defined for $p \ge 1$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

Spaces ℓ_p. For $p \ge 1$, the set of real (or complex) sequences such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ becomes a normed linear space with

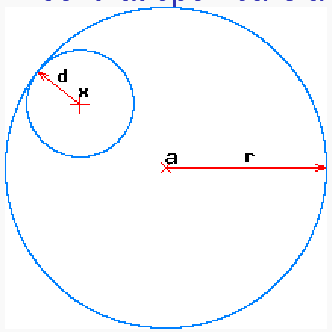
$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$$

Open and closed sets

Definition. A subset U of a metric space M is open (in M) if for every $x \in U$ there is $\delta > 0$ such that $B(x,\delta) \subset U$.
A subset F of a metric space M is closed (in M) if $M \setminus F$ is open.

Important examples. In $\mathbb{R}$, open intervals are open. In any metric space M: $\emptyset$ and M are open as well as closed; open balls are open and closed balls are closed. In $\mathbb{R}$, $[0,1)$ is neither open nor closed.

Proof that open balls are open.



If $x \in B(a,r)$, put $\delta = r - d(a,x) > 0$. Let $y \in B(x,\delta)$. Then

$$\begin{aligned} d(y,a) &\le d(y,x) + d(x,a) \\ &< \delta + d(a,x) = r. \end{aligned}$$

So $y \in B(a,r)$, and so

$$B(x,\delta) \subset B(a,r).$$

Properties of open sets

Recall. $U \subset M$ is open (in M) if for every $x \in U$ there is $\delta > 0$ such that $B(x, \delta) \subset U$.

Proposition 1.2. *Let $U_1, \ldots, U_k$ be open in M. Then $\bigcap_{i=1}^{k} U_i$ is open in M.*

Proof. Let $x \in \bigcap_{i=1}^{k} U_i$. Then $x \in U_i$ and U_i is open, so there are $\delta_i > 0$ so that $B(x, \delta_i) \subset U_i$. Let $\delta = \min(\delta_1, \ldots, \delta_k)$.Then $B(x, \delta) \subset B(x, \delta_i) \subset U_i$ for each i, hence $B(x, \delta) \subset \bigcap_{i=1}^{k} U_i$. □

Proposition 1.3. *The union of any collection of sets open in M is open in M.*

Proof. Let $U = \bigcup_{i \in I} U_i$ where U_i are open and I is any index set. Let $x \in U$. Then $x \in U_i$ for some i. Since U_i is open, there is $\delta > 0$ so that $B(x, \delta) \subset U_i$. So $B(x, \delta) \subset U_i \subset \bigcup_{i \in I} U_i = U$. □

Continuity

Definition. Let (M_1, d_1) and (M_2, d_2) be metric spaces and $f : M_1 \to M_2$.

- f is said to be continuous at $a \in M_1$ if

  $$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in M_1)\big(d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon\big)$$

- f is said to be continuous (on M_1) if it is continuous at every point of M_1
- f is said to be Lipschitz (continuous) if there is $C \in \mathbb{R}$ so that

  $$d_2(f(x), f(y)) \le C d_1(x, y) \text{ for all } x, y \in M.$$

  We also say that f is Lipschitz with constant C.

Fact 1.4. *A Lipschitz continuous map is continuous.*

Proof. Let $\delta = \varepsilon / C$. □

Example: Distance from a set

For a nonempty $A \subset M$ we define the distance of a point $x \in M$ from A by

$$d(x, A) = \inf_{z \in A} d(x, z)$$

Fact 1.5. *For any nonempty set $A \subset M$, the function $x \to d(x, A)$ is Lipschitz with constant one.*

Proof. Let $x, y \in M$. For every $z \in A$ we have

$$d(x, A) \le d(x, z) \le d(x, y) + d(y, z).$$

Hence, taking inf on the right side,

$$d(x, A) \le d(x, y) + d(y, A),$$

which is

$$d(x, A) - d(y, A) \le d(x, y).$$

Exchanging x, y and using symmetry gives the required inequality

$$|d(x, A) - d(y, A)| \le d(x, y). \quad □$$

Examples of continuous function

Example. By 1.4, the function $x \to d(x, A)$, where $\emptyset \ne A \subset M$ is continuous. In particular $x \to d(x, a)$ ($a \in M$) is continuous.

Example. Since $|x_k - y_k| \le \|x - y\|$ for $x, y \in \mathbb{R}^n$, the projections $x \in \mathbb{R}^n \to x_k$ are Lipschitz continuous.

Example. As for functions $\mathbb{R} \to \mathbb{R}$, we can prove that sums, products and ratios (provided the denominator is non-zero) of continuous functions $M \to \mathbb{R}$ are continuous.

Remark (and a warning). The above facts may be used to show that various functions $\mathbb{R}^n \to \mathbb{R}$ are continuous. For example,

- The function $(x, y) \to x/(1 + x^2 + y^2)$ is continuous on $\mathbb{R}^2$.
- The function $(x, y) \to x/(x^2 + y^2)$ is not continuous on $\mathbb{R}^2$, since it is not defined everywhere on $\mathbb{R}^2$!

Continuity and open sets

Recall that the preimage of U under f is $f^{-1}(U) = \{x : f(x) \in U\}$.
Notice that use of f^{-1} **does not** mean that f has an inverse.

Theorem 1.6 (Key Theorem). *A map $f : M_1 \to M_2$ is continuous iff for every open set $U \subset M_2$, the set $f^{-1}(U)$ is open (in M_1).*

Proof. ($\Rightarrow$) Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Hence, since U is open, $B(f(x), \varepsilon) \subset U$ for some $\varepsilon > 0$. Since f is continuous at x, there is $\delta > 0$ such that $d_1(y, x) < \delta$ implies $d_2(f(y), f(x)) < \varepsilon$.
We show that $B(x, \delta) \subset f^{-1}(U)$; this will mean that $f^{-1}(U)$ is open.
Let $y \in B(x, \delta)$. Then $d_1(x, y) < \delta$. Hence $d_2(f(y), f(x)) < \varepsilon$, which is $f(y) \in B(f(x), \varepsilon) \subset U$. So $y \in f^{-1}(U)$.
($\Leftarrow$) Let $x \in M_1$; we show that f is continuous at x.
Let $\varepsilon > 0$. Since $B(f(x), \varepsilon)$ is open, $f^{-1}(B(f(x), \varepsilon))$ is open.
Since $x \in f^{-1}(B(f(x), \varepsilon))$, there is $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$.
This means that $y \in B(x, \delta)$ implies $f(y) \in B(f(x), \varepsilon)$, which is the same as $d_1(y, x) < \delta$ implies $d_2(f(y), f(x)) < \varepsilon$. □

How show open/closed?

Question. *Show that $U = \{(x, y) \in \mathbb{R}^2 : y > 0, x/y > 7\}$ is open in $\mathbb{R}^2$.*

Answer. The functions $f(x, y) = y$ and $g(x, y) = x - 7y$ are continuous and so $U = f^{-1}(0, \infty) \cap g^{-1}(0, \infty)$ is open.

Question. *Show that open balls are open and closed balls are closed.*

Answer. The function $f(x) = d(x, a)$ is continuous and so $B(a, r) = f^{-1}(-\infty, r)$ is open. The complement of the closed ball $\{x : d(x, a) \le r\}$ is $f^{-1}(r, \infty)$, hence open.

Question. *Show that the preimages of closed sets under continuous maps are closed.*

Answer. Let $f : M \to N$ be continuous and $F \subset N$ be closed. Then

$$M \setminus f^{-1}(F) = f^{-1}(N \setminus F)$$

is open since $N \setminus F$ is open, and so $f^{-1}(F)$ is closed.

Proofs of “open” using continuity

Question. *Show that $\{(x, y) \in \mathbb{R}^2 : y/\sin(x^2y^3) > 2\}$ is open.*

Wrong Answer. The function $f(x, y) = y/\sin(x^2y^3)$ is continuous and the set is $f^{-1}(2, \infty)$. This is wrong since f is not continuous, since it is not defined everywhere on $\mathbb{R}^2$.

Answer. The set is

$$\begin{aligned} &\{(x,y) \in \mathbb{R}^2 : \sin(x^2y^3) > 0,\ y > 2\sin(x^2y^3)\} \\ &\qquad \cup \{(x, y) \in \mathbb{R}^2 : \sin(x^2y^3) < 0,\ y < 2\sin(x^2y^3)\} \\ &= \big(g^{-1}(0, \infty) \cap h^{-1}(0, \infty)\big) \cup \big(g^{-1}(-\infty, 0) \cap h^{-1}(-\infty, 0)\big) \end{aligned}$$

where $g(x, y) = \sin(x^2y^3)$ and $h(x, y) = y - 2\sin(x^2y^3)$ are continuous. This set is open since pre-images of open intervals under g and h are open, and finite intersections and unions preserve openness.

Convergence of sequences

Definition. We say that a sequence $x_k \in M$ converges to $x \in M$ if $d(x_k, x) \to 0$.

Fact 1.7. *A sequence can have at most one limit.*

Proof. If $x_k \to x, y$ then $d(x, y) \le d(x, x_k) + d(x_k, y) \to 0$.
So $d(x, y) = 0$, implying that $x = y$. □

Fact 1.8. *$x_k \to x$ iff for every open set U containing x there is K such that $x_k \in U$ for all $k \ge K$.*

Proof. ($\Rightarrow$) Since U is open, there is $r > 0$ with $B(x, r) \subset U$.
Since $x_k \to x$, there is K so that $d(x_k, x) < r$ for $k \ge K$.
It follows that $x_k \in B(x, r)$ and so $x_k \in U$ for $k \ge K$.
($\Leftarrow$) Let $\varepsilon > 0$. Then $B(x, \varepsilon)$ is an open set, so there is K such that $x_k \in B(x, \varepsilon)$ for $k \ge K$. Hence $d(x_k, x) < \varepsilon$ for $k \ge K$.
This means that $d(x_k, x) \to 0$, in other words that $x_k \to x$. □
Note possible other forms of this Fact.

Sequences and closed sets

Theorem 1.9. *A subset F of a metric space M is closed iff for every sequence $x_k \in F$ that converges to some $x \in M$ we necessarily have that $x \in F$.*

Proof. ($\Rightarrow$) Let F be closed, $x_k \in F$ and $x_k \to x \in M$. If $x \notin F$ then $x \in M \setminus F$ which is open. So (by Fact 1.8) there is K such that $x_k \in M \setminus F$ for $k \geq K$. But this contradicts $x_k \in F$.
($\Leftarrow$) Let $x \in M \setminus F$. If $B(x, 1/k) \subset M \setminus F$ for some k, we are done. If not, then for every k there is $x_k \in B(x, 1/k) \cap F$. But then $d(x_k, x) < 1/k \to 0$, so $x_k \to x$ and $F \ni x_k \to x \notin F$, contradicting our assumption. □

Continuity and sequences are related as in $\mathbb{R}$, with the same proof.

Theorem 1.10. *A function $f : M_1 \to M_2$, where M_1, M_2 are metric spaces, is continuous at $x \in M_1$ iff $f(x_k) \to f(x)$ for every sequence $x_k \in M_1$ such that $x_k \to x$.*
Danger. We do **not** base our approach to continuity on this!!!

Continuity in different metrics

The fact that continuity is described using open sets only (by "preimages of open sets are open") has the following immediate corollaries that motivate the transition to topological spaces.

Observation. Let d_1, d_2 be two metrics on the same set M. The identity map ($f(x) = x$ for $x \in M$) is continuous from (M, d_1) to (M, d_2) iff every d_2 open set is d_1 open.

Theorem 1.11. *If d_1, d_2 are two metrics on the same set M, the following statements are equivalent.*

(1) *For every metric space (N, d), every $f : M \to N$ is d_1 continuous iff it is d_2 continuous.*
(2) *For every metric space (N, d), every $g : N \to M$ is d_1 continuous iff it is d_2 continuous.*
(3) *d_1 open and d_2 open sets coincide.*

Equivalence of metrics

Definition. Two metrics d_1, d_2 on the same set M are called topologically equivalent if d_1 open and d_2 open sets coincide.

Fact 1.12. *If there are $0 < c, C < \infty$ such that for all $x, y \in M$,*

$$c d_1(x, y) \leq d_2(x, y) \leq C d_1(x, y),$$

then d_1, d_2 are topologically equivalent.

Proof. Let U be d_2 open and $x \in U$. Find $\delta > 0$ such that $B_{d_2}(x, \delta) \subset U$. Then $B_{d_1}(x, \delta/C) \subset B_{d_2}(x, \delta) \subset U$, so U is d_1 open. The converse is similar. □

Example. The norms $\|\cdot\|_p$ in $\mathbb{R}^n$ are topologically equivalent.

Example. $\|\cdot\|_1$ and $\|\cdot\|_2$ in $\mathcal{C}[0, 1]$ are not topologically equivalent.

Example. On $\mathbb{R}$, the Euclidean metric and the metric $d(x, y) = \min(1, |x - y|)$ are topologically equivalent.

Isometries and homeomorphisms

If $f : M_1 \to M_2$ is a bijection such that $d_2(f(x), f(y)) = d_1(x, y)$, we would naturally consider that M_1 and M_2 are "the same" as metric spaces, since f just renames the points. In this situation we say that M_1 and M_2 are isometric and f is an isometry.

If $f : M_1 \to M_2$ is a bijection such that U is open in M_1 iff $f(U)$ is open in M_2, then M_1 and M_2 are "the same" not as metric spaces, but behave in the same way for questions concerning continuity. (This is just another way of saying the main point of the previous part that continuity can be fully described with the help of open sets.) In this situation we say that M_1 and M_2 are homeomorphic and f is a homeomorphism.

Notice also that the above condition of preservation of open sets is equivalent to saying that f and f^{-1} are both continuous.

Example. $(0, 1)$ and $\mathbb{R}$ are not isometric but are homeomorphic.

Topological properties

If P is some property which makes sense for every metric space, we say that it is a topological property of metric spaces (or topological invariant of metric spaces) if whenever M has property P so has every metric space homeomorphic to it.

Examples (of topological properties).

- M is open in M; M is closed in M.
- M is finite; countable; uncountable.
- M has an isolated point, ie, $\{x\}$ is open for some $x \in M$.
- M has no isolated points.
- Every subset of M is open.

Examples (of non-topological properties).

- M is bounded.
- M is totally bounded, ie, for every $r > 0$ there is a finite set F such that every ball with radius r contains a point of F.

Main points of Chapter 1

- Definition of metric spaces. Metrics induced by norms.
- Examples: Euclidean spaces, ℓ_p norms, proof of the triangle inequality.
- Examples: Discrete spaces, some exotic spaces, function spaces.
- Subspaces and product spaces
- Open and closed sets. Union and intersection. Openness of open balls.
- Continuity. Lipschitz continuity.
- Examples: Distance from a set, projections. Algebraic operations.
- A function is continuous if and only if preimages of open sets are open.
- Convergence of sequences. Description using open sets.
- Closed sets are those from which sequences cannot escape.
- Equivalence of metrics. Isometries. Homeomorphisms. Topological properties.

Chapter 2. Topological spaces

Definition. Topology on a set T is a collection $\mathcal{T}$ of subsets of T, whose members are called open sets, such that

(T1) $\emptyset$ and T are open;

(T2) the intersection of any finite collection of open sets is open;

(T3) the union of any collection of open sets is open.

The pair $(T, \mathcal{T})$ is called a topological space.

If there is no danger of confusion we speak about the topological space T and call the open sets T-open.

Example (Topology induced by a metric). If d is a metric on T, the collection of all d-open sets is a topology on T.

Example (Discrete topologies). All subsets of T are open.

Example (Indiscrete topologies). Only $\emptyset$ and T are open.

Defining a topology via closed sets

Definition. A set F in a topological space T is called closed if its complement is open.

By De Morgan Laws, the collection $\mathcal{F}$ of closed sets has the following properties.

(T1) $\emptyset$ and T are closed;

(T2) the union of any finite collection of closed sets is closed;

(T3) the intersection of any collection of closed sets is closed.

Conversely, again by De Morgan Laws, if a collection of subsets of T satisfies these conditions, there is unique topology on T for which it becomes the collection of closed sets.

Example (Zariski-type topology). Closed sets consist of finite subsets of T, together with T.

Warnings. Sets are not doors. They may be neither open nor closed. And they may be simultaneously open and closed.

Bases and sub-bases

Definition. A basis for a topology $\mathcal{T}$ on T is a collection $\mathcal{B} \subset \mathcal{T}$ such that every set from $\mathcal{T}$ is a union of sets from $\mathcal{B}$.

Example. In a metric space, open balls form a basis for the topology induced by the metric.

Proof. In a metric space, a set U is open iff for every $x \in U$ there is $\delta_x > 0$ such that $B(x, \delta_x) \subset U$. Hence $U = \bigcup_{x \in U} B(x, \delta_x)$. □

Definition. A sub-basis for a topology $\mathcal{T}$ on T is a collection $\mathcal{B} \subset \mathcal{T}$ such that every set from $\mathcal{T}$ is a union of finite intersections of sets from $\mathcal{B}$.

Example. The collection of intervals (a, ∞) and $(-\infty, b)$ is a sub-basis for the (Euclidean) topology of $\mathbb{R}$.

Funny agreements. In these definitions, we understand the union of an empty collection of sets as the empty set $\emptyset$, and the intersection of an empty collection of sets as the whole space T.

Defining a topology via bases

Immediately from the definition of basis we have

Proposition 2.1. *If $\mathcal{B}$ is any basis for the topology of T then*

(B1) *T is the union of sets from $\mathcal{B}$,*

(B2) *if $B, D \in \mathcal{B}$ then $B \cap D$ is a union of sets from $\mathcal{B}$.*

Proposition 2.2. *If $\mathcal{B}$ is any collection of subsets of a set T satisfying (B1) and (B2) then there is a unique topology on T with basis $\mathcal{B}$. Its open sets are exactly the unions of sets from $\mathcal{B}$.*

Proof. By definition of basis, if such a topology exists, it must be the collection $\mathcal{T}$ of the unions of sets from $\mathcal{B}$. So we just have to show that this $\mathcal{T}$ is a topology:

(T1) $\emptyset, T \in \mathcal{T}$ by Agreement and (B1), respectively.

(T2) If $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} D_j$ where $B_i, D_j \in \mathcal{B}$ then $U \cap V = \bigcup_{i,j} B_i \cap D_j$, which is a union of sets from $\mathcal{B}$ by (B2).

(T3) A union of unions of sets from $\mathcal{B}$ is a union of sets from $\mathcal{B}$. □

Defining a topology via sub-bases

Proposition 2.3. *If $\mathcal{B}$ is any collection of subsets of a set T then there is a unique topology on T with sub-basis $\mathcal{B}$. Its open sets are exactly the unions of finite intersections of sets from $\mathcal{B}$.*

Proof. By the definition of sub-basis, any topology with sub-basis $\mathcal{B}$ must have the collection $\mathcal{D}$ of finite intersections of sets from $\mathcal{B}$ as a basis. The only point to observe is that $\mathcal{D}$ satisfies (B1), (B2). Then Proposition 2.2 immediately implies that there is unique topology with basis $\mathcal{D}$. This topology is also the unique topology with sub-basis $\mathcal{B}$. □

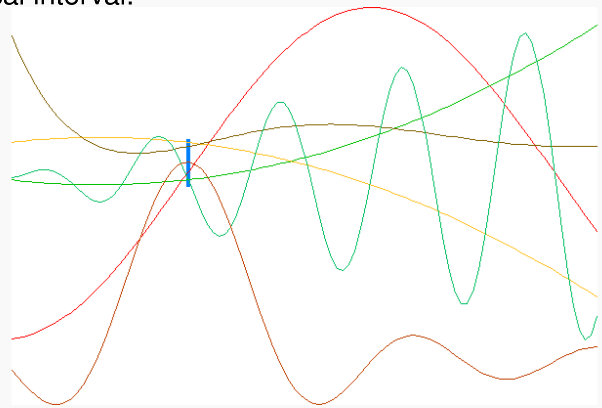
Definition. If $\mathcal{T}_0$ and $\mathcal{T}_1$ are two topologies on the same set, we say that $\mathcal{T}_0$ is coarser than $\mathcal{T}_1$, or that $\mathcal{T}_1$ is finer than $\mathcal{T}_0$, if $\mathcal{T}_0 \subset \mathcal{T}_1$.

Notice that the topology with the sub-basis $\mathcal{B}$ can be described as the coarsest topology containing $\mathcal{B}$.

The topology of pointwise convergence

The topology of pointwise convergence on the set $\mathcal{F}(X)$ of real functions on a set X is defined as the topology with a sub-basis formed by the sets $\{f \in \mathcal{F}(X) : a < f(x) < b\}$ $(x \in X,\ a, b \in \mathbb{R})$.

A set from the sub-basis consists of all functions that pass through one vertical interval.



Subspaces and product spaces

Subspaces. If $(T, \mathcal{T})$ is a topological space and $S \subset T$, we define the subspace topology on S as $\mathcal{T}_S = \{(U \cap S) : U \in \mathcal{T}\}$.
We call $(S, \mathcal{T}_S)$ a (topological) subspace of T.

Products. If $(T_1, \mathcal{T}_1)$, $(T_2, \mathcal{T}_2)$ are topological spaces, the product topology on $T_1 \times T_2$ is the topology $\mathcal{T}$ with basis

$$\mathcal{B} = \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}$$

We call $(T_1 \times T_2, \mathcal{T})$ the (topological) product of T_1, T_2.
The definition extends to any number of factors. In particular, $\mathbb{R}^n$ is the topological product of $\mathbb{R}, \dots, \mathbb{R}$.

Warning. Usually there are many more sets in the product than just those from $\mathcal{B}$.

Remark. For metric spaces, we can either take subspaces and products as defined above, or first take them in the sense of metric spaces and then use the induced topology. Fortunately, we get precisely the same topologies!

Neighbourhood, interior and closure.

Definitions. A neighbourhood of $x \in T$ is a set $H \subset T$ for which there is an open set U such that $x \in U \subset H$.
The closure $\overline{H}$ of a set $H \subset T$ is the set of points x such that every neighbourhood of x meets H.
The interior H^o is formed by points x of which H is a neighbourhood.

Fact 2.4. *$H^o = T \setminus \overline{(T \setminus H)}$ and $\overline{H} = T \setminus (T \setminus H)^o$.*

Proof. If $x \in H^o$, H is a neighbourhood of x not meeting $T \setminus H$, so $x \notin \overline{(T \setminus H)}$, so $x \in T \setminus \overline{(T \setminus H)}$.
If $x \in T \setminus \overline{(T \setminus H)}$, then $x \notin \overline{(T \setminus H)}$, which means that there is a neighbourhood of x not meeting $T \setminus H$. So this neighbourhood of x is contained in H, and so $x \in H^o$. □

Examples. In $\mathbb{R}$: $\overline{(a,b)} = [a,b]$, $[a,b]^o = (a,b)$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}^o = \emptyset$,
Note that in $[a,b]$, $[a,b]^o = [a,b]$ and in (a,b), $\overline{(a,b)} = (a,b)$.

Closure, interior, boundary

Using the definition of H^o and the formula $\overline{H} = T \setminus (T \setminus H)^o$, we get

- H^o is open and $\overline{H}$ is closed
- H^o is the largest open subset and $\overline{H}$ is the least closed superset of H
- H is open iff $H = H^o$ and H is closed iff $H = \overline{H}$
- $(H^o)^o = H^o$ and $\overline{\overline{H}} = \overline{H}$
- $H \subset K$ implies $H^o \subset K^o$ and $\overline{H} \subset \overline{K}$
- $(H \cap K)^o = H^o \cap K^o$ and $\overline{H \cup K} = \overline{H} \cup \overline{K}$

Definition. The boundary ∂H of H is the set of points x whose every neighbourhood meets both H and its complement.

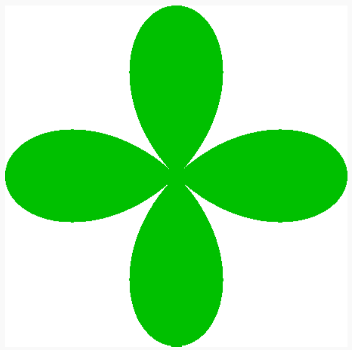
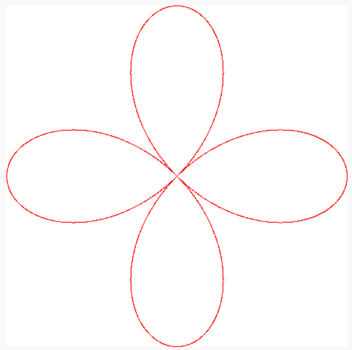
Fact 2.5. *$\partial H = \overline{H} \cap \overline{(T \setminus H)}$ is a closed set.*

Examples. In $\mathbb{R}$, $\partial(a,b) = \partial[a,b] = \{a,b\}$, $\partial\mathbb{Q} = \mathbb{R}$.

Boundaries on pictures

The (open) set

$$\{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^3 < (x^2-y^2)^2\}$$

and its boundary

$$\{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^3 = (x^2-y^2)^2\}$$

The Cantor set

Step 0. Start with the interval $[0, 1]$ and call it C_0.

Step 1. Remove the middle third; two closed intervals remain.

...

Step N. From each of the 2^{N-1} remaining intervals remove the (open) middle third; 2^N closed intervals remain.

- The set $C = \bigcap_{n=0}^{\infty} C_n$ is the (ternary) Cantor set.
- It is a closed set with empty interior, so $\partial C = C$.
- Notice that C has no isolated points.
- It has uncountably many points, so many more than just the end-points of the removed intervals. (A proof may be given in Chapter 5.)

Continuity

Motivated by description of continuity in metric spaces using open sets, we define:

Definition. A map $f : T_1 \to T_2$ between two topological spaces is said to be continuous if for every open set $U \subset T_2$, the set $f^{-1}(U)$ is open (in T_1).

Examples. Constant maps. Identity map. Continuous maps between metric spaces. Any map of a discrete space to any topological space.

Example. If T is an indiscrete topological space then $f : T \to \mathbb{R}$ is continuous iff it is constant.

Fact 2.6. *Suppose that T_1, T_2 are topological spaces and $\mathcal{B}$ is a sub-basis for T_2. Then $f : T_1 \to T_2$ is continuous iff $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.*

Proof. Preimages preserve unions and intersections. □

Continuity, composition and product spaces

Theorem 2.7. *If T_1, T_2, T_3 are topological spaces and $f : T_1 \to T_2$ and $g : T_2 \to T_3$ are continuous then $g \circ f : T_1 \to T_3$ is continuous.*

Proof. If U is open in T_3, $g^{-1}(U)$ is open in T_2 by continuity of g. So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in T_1 by continuity of f. □

Definition. The first projection of $T_1 \times T_2$ onto T_1 is defined by $\pi_1(x, y) = x$; the second projection π_2 is defined similarly.

Fact 2.8. *The projections π_1 and π_2 are continuous.*

Proof. If $U_1 \subset T_1$, then $\pi_1^{-1}(U_1) = U_1 \times T_2$. □

Proposition 2.9. *$f = (f_1, f_2) : T \to T_1 \times T_2$ is continuous iff f_1 and f_2 (ie $\pi_1 \circ f$ and $\pi_2 \circ f$) are continuous.*

Proof. ($\Rightarrow$) Since π_i are continuous, so are $f_i = \pi_i \circ f$.
($\Leftarrow$) If U_i are open in T_i, $f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$ is open in T. So preimages of sets from a basis are open. Now use 2.6. □

Continuity and algebraic operations.

Example. If $f, g : T \to \mathbb{R}$ are continuous then cf ($c \in \mathbb{R}$), $|f|$, $\max(f, g)$, $\min(f, g)$, $f + g$, fg and, if g is never zero, f/g are all continuous.

Proof for $f + g$. Notice $(f(x) + g(x) > a) \iff (f(x) > a - g(x))$
$\iff (\exists r)((f(x) > r) \;\&\; (r > a - g(x)))$
$\iff (\exists r)((f(x) > r) \;\&\; (g(x) > a - r))$.

Hence

$$\{x : f(x) + g(x) > a\} = \bigcup_{r \in \mathbb{R}} \Big(\{x : f(x) > r\} \cap \{x : g(x) > a - r\} \Big).$$

Similarly for $\{x : f(x) + g(x) < b\}$. Now use 2.6. □

Another proof for $f + g$. The function $h : (x, y) \in \mathbb{R}^2 \to x + y \in \mathbb{R}$ is continuous. (Since $\mathbb{R}^2$, $\mathbb{R}$ are metric spaces, this may be proved by observing that it is a Lipschitz function or by other methods of metric spaces.) The map $\Phi : x \in T \to (f(x), g(x))$ is continuous by Proposition 2.9. Now notice that $f + g = h \circ \Phi$. □

Examples of proofs of continuity

Question. *Show that* $(x, y) \in \mathbb{R}^2 \to (x + y, \sin(x^2y^3)) \in \mathbb{R}^2$ *is continuous.*

Answer. $(x, y) \to x$ and $(x, y) \to y$ are continuous (projections). So are $(x, y) \to x + y$ (sum), $(x, y) \to x^2y^3$ (products) and $(x, y) \to \sin(x^2y^3)$ (composition with $t \in \mathbb{R} \to \sin(t)$). Hence both components of our function are continuous, so it is continuous.

Question. *Show that the identity map from* $\mathcal{C}[0, 1]$ *with the topology* $\mathcal{T}_m$ *induced by the maximum norm to the topology of pointwise convergence* $\mathcal{T}_p$ *is continuous.*

Answer. By Fact 2.6, it suffices to show that for each set U from a sub-basis of $\mathcal{T}_p$, $U = \mathrm{id}^{-1}(U)$ is open in $\mathcal{T}_m$. Take U of the form $U = \{\varphi : a < \varphi(x) < b\}$ for some $x \in [0, 1]$ and $a, b \in \mathbb{R}$.
For $\varphi \in U$ denote $\delta_\varphi = \min\{\varphi(x) - a, b - \varphi(x)\} > 0$.
If $\|\psi - \varphi\| < \delta_\varphi$ then $a < \psi(x) < b$. Hence $B_m(\varphi, \delta_\varphi) \subset U$, implying that $U = \bigcup_{\varphi \in U} B_m(\varphi, \delta_\varphi)$ is open in $\mathcal{T}_m$.

Direct proofs of open/not open

Question. *Show that the set* $S = \{\varphi \in \mathcal{C}[0, 1] : \varphi > 0\}$ *is open in the topology* $\mathcal{T}_m$ *induced by the maximum norm but has empty interior in the topology of pointwise convergence* $\mathcal{T}_p$.

Answer. $S = \bigcup_{\varphi \in S} B_m(\varphi, \delta_\varphi) \in \mathcal{T}_m$ where $\delta_\varphi = \min_{x \in [0,1]} \varphi(x)$.
If U is in $\mathcal{T}_p$ and contains a function $\psi \in S$, there are $x_1, \dots x_n \in [0, 1]$ and $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$ such that

$$\psi \in \{\gamma : a_i < \gamma(x_i) < b_i \text{ for } i = 1, \dots n\} \subset U$$

But then any function which equals to ψ at $x_1, \dots x_n$ but attains a negative value is in U. So $U \not\subset S$, so ψ is not in the interior of U.

Question. *Show that the identity map from* $(\mathcal{C}[0, 1], \mathcal{T}_p)$ *to* $(\mathcal{C}[0, 1], \mathcal{T}_m)$ *is not continuous.*

Answer. We have to find a set in $\mathcal{T}_m$ which is not in $\mathcal{T}_p$. By the previous question, $\{\varphi \in \mathcal{C}[0, 1] : \varphi > 0\}$ has this property.

Defining topologies by continuity requirements

If T_j $(j \in J)$ are topological spaces, T is any set and $f_j : T \to T_j$ any mappings, the projective (or initial) topology on T is defined as the coarsest topology for which all f_j are continuous.
It can be equivalently described as the topology with the sub-basis $f_j^{-1}(U_j)$ where $j \in J$ and U_j is open in T_j.

Example. The topology of pointwise convergence on $\mathcal{F}(X)$ is the coarsest topology making all maps $f \in \mathcal{F}(X) \to f(x)$ continuous.

Definitions. Let T_j, $j \in J$ (where J is any set) be topological spaces. By their product we understand:

- the set $T = \prod_{j \in J} T_j$ of functions x on J such that $x(j) \in T_j$;
- the coarsest topology on T for which the coordinate projections, $\pi_j : T \to T_j$, $\pi_j(x) = x(j)$ are all continuous.

Example. If all $T_j = \mathbb{R}$, then $\prod_{j \in J} T_j$ (often denoted by $\mathbb{R}^J$) is exactly $\mathcal{F}(J)$ with the topology of pointwise convergence.

Quotient spaces

If T_j $(j \in J)$ are topological spaces, S is any set and $f_j : T_j \to S$ any mappings, the inductive (or final) topology on S is defined as the finest topology for which all f_j are continuous.
This means that $U \subset S$ is open iff $f_j^{-1}(U)$ is open in T_j for every j.
In the special case when T is a topological space, S is any set and $f : T \to S$ is surjective, this topology is called the quotient topology.
This situation often occurs when an equivalence relation on the space T is given ("gluing some points together") and f maps T onto the set of its equivalence classes.

Examples.

- Gluing together two opposite sides of a square gives a cylinder.
- Gluing together two opposite ends of a cylinder gives a torus.
- $\mathbb{R}$, with $x \sim y$ iff $y - x \in \mathbb{Z}$, becomes a circle.
- $\mathbb{R}$, with $x \sim y$ iff $y - x \in \mathbb{Q}$ becomes a large indiscrete space.

Metrizability

Agreement. If we refer to a metric space as a topological space, we have in mind the topology induced by the metric, unless another topology is specified.

Agreement. In particular, if we refer to a subset of $\mathbb{R}^n$ as a topological space, we have in mind the topology induced by Euclidean metric.

Definition. The topological space $(T, \mathcal{T})$ is called metrizable if there is a metric d on T such that $\mathcal{T}$ is exactly the collection of d-open sets.

Fact of Life. There are many non-metrizable topologies, and some of them are not only interesting but also important.

Idea. To prove that a given topology is non-metrizable, we find a property of topological spaces that holds for all metrizable topologies but fails for the given space.

Hausdorff property and non-metrizable topologies

Fact 2.10. *Any metrizable topology has the Hausdorff property that for any two distinct points x, y there are disjoint open sets U, V containing x, y respectively.*

Proof. Let $r = \frac{1}{2}d(x, y)$, $U = B(x, r)$ and $V = B(y, r)$. □

Example. If T has at least two points, the indiscrete topology on T is not metrizable.

Proof. Take any distinct $x, y \in T$. Then the only open set containing x is T, which contains y as well. □

Example. If T is an infinite set, the Zariski topology on T is not metrizable.

Proof. Suppose it has the Hausdorff property. Take any distinct $x, y \in T$ and disjoint Zariski open sets U, V containing x, y respectively. Then $T \setminus U$ is closed and does not contain x, so $T \setminus U$ is finite. Similarly, $T \setminus V$ is finite. But then $T \subset (T \setminus U) \cup (T \setminus V)$ is finite, a contradiction. □

Normal topological spaces.

Definition. T is called *normal* if for any disjoint closed $F_0, F_1 \subset T$ there are disjoint open $G_0, G_1 \subset T$ such that $F_0 \subset G_0$ and $F_1 \subset G_1$.

Fact 2.11. *Metrizable spaces are normal: take $G_0 = \{x : d(x, F_0) < d(x, F_1)\}$ and $G_1 = \{x : d(x, F_1) < d(x, F_0)\}$.*

Example. The space $\mathbb{R}$ with the topology $\mathcal{T}$ whose subbasis consists of open intervals together with the set $\mathbb{Q}$ is Hausdorff but not normal. In particular, it is not metrizable.

Proof. We have $\mathcal{T} = \{G \cup (H \cap \mathbb{Q}) : G, H \in \mathcal{E}\}$ where $\mathcal{E}$ the Euclidean topology of $\mathbb{R}$. The space $(\mathbb{R}, \mathcal{T})$ is Hausdorff since any two different points are contained in disjoint open intervals.
The sets $\{0\}$ and $\mathbb{R} \setminus \mathbb{Q}$ are disjoint and $\mathcal{T}$-closed. Suppose that $0 \in U_0 = G_0 \cup (H_0 \cap \mathbb{Q})$ and $\mathbb{R} \setminus \mathbb{Q} \subset U_1 = G_1 \cup (H_1 \cap \mathbb{Q})$. Then there is an open interval I such that $(I \cap \mathbb{Q}) \subset U_0$. But $G_1 \supset \mathbb{R} \setminus \mathbb{Q}$ is an $\mathcal{E}$-open set and $G_1 \cap I \neq \emptyset$, so $G_1 \cap (I \cap \mathbb{Q}) \neq \emptyset$, implying that $U_1 \cap U_0 \neq \emptyset$. Hence $(\mathbb{R}, \mathcal{T})$ is not normal. □

Continuous functions on normal spaces.

Theorem 2.12 (Urysohn's Lemma). *Suppose that F_0, F_1 are disjoint closed subsets of a normal topological space T. Then there is a continuous function $f : M \to \mathbb{R}$ such that*

- *$f(x) = 0$ for every $x \in F_0$,*
- *$f(x) = 1$ for every $x \in F_1$.*

We will not prove Urysohn's Lemma here, but only observe that if T is metrizable, such f may be defined by the formula

$$f(x) = \frac{d(x, F_0)}{d(x, F_0) + d(x, F_1)}$$

Theorem 2.13 (Tietze's Theorem). *Every real-valued function defined and continuous on a closed subset of a normal topological space T may be extended to a continuous function on the whole T.*

We prove Tietze's Theorem (from Urysohn's Lemma) in Chapter 5.

Closure in metrizable spaces

Recalling the description of convergence in metric spaces using open sets, we define that a sequence $x_k \in T$ converges to $x \in T$ if for every open set $U \ni x$ there is K such that $x_k \in U$ for $k > K$.

Example. In an indiscrete space, every sequence converges to every point.

Fact 2.14. *In a metrizable space T, the closure $\overline{S}$ of $S \subset T$ is the set of limits of convergent sequences whose terms are in S.*

Proof. Let d be a metric inducing the topology of T. Since $\overline{S}$ is closed, we know from Theorem 1.9 that every convergent sequence from $S \subset \overline{S}$ has its limit in $\overline{S}$.
Conversely, if $x \in \overline{S}$, then for every k the ball $B(x, 1/k)$, being open, meets S. So we may take $x_k \in B(x, 1/k) \cap S$ and get a sequence in S converging to x. □

Warning. In general topological spaces, this is false.

Some notions of descriptive set theory.

Definitions. A subset S of a topological space T is

- F_σ if it is a union of countably many closed sets
- G_δ if it is an intersection of countably many open sets
- $F_{\sigma\delta}$ if it is an intersection of countably many F_σ sets
- etc

Examples. Every closed set is F_σ, every open set is G_δ. The complement of an F_σ set is G_δ. The complement of a G_δ set is F_σ. In $\mathbb{R}$: $\mathbb{Q}$ is F_σ and $\mathbb{R} \setminus \mathbb{Q}$ is G_δ but $\mathbb{Q}$ is not G_δ and $\mathbb{R} \setminus \mathbb{Q}$ is not F_σ. (If you cannot prove the last two statements, try it again after you learn Baire's Theorem in Chapter 5.)

Theorem 2.15. *In a metrizable space, every closed set is G_δ and every open set is F_σ.*

Proof. If S is closed, then $S = \bigcap_{k=1}^\infty \{x : d(x, S) < \frac{1}{k}\}$. □

Application to non-metrizability

Fact 2.16. *The space $\mathcal{F}[0,1]$ of real-valued function on $[0,1]$ with the topology of pointwise convergence is not metrizable.*

Proof. If $\mathcal{F}[0,1]$ were metrizable, the set $\{0\}$ would be G_δ. Hence we would have open $G_k \subset \mathcal{F}[0,1]$ such that $\{0\} = \bigcap_{k=1}^\infty G_k$. By definition of the topology of pointwise convergence, for each k we can find a finite set S_k and $\varepsilon_k > 0$ such that

$$G_k \supset \{f : |f(x)| < \varepsilon_k \text{ for } x \in S_k\}$$

But then the function

$$g(x) = 0 \text{ for } x \in \bigcup_{k=1}^\infty S_k \text{ and } g(x) = 1 \text{ otherwise}$$

is not identically zero and belongs to $\bigcap_{k=1}^\infty G_k$. A contradiction. □

Idea of a different proof. Let S be the set of functions $f \in \mathcal{F}[0,1]$ such that $\{x : f(x) \neq 1\}$ is finite. Then $\overline{S} = \mathcal{F}[0,1]$ and there is no sequence $f_k \in S$ such that $f_k \to 0$. Now use Fact 2.14. □

Big and small sets in topological spaces

Definitions. A subset S of a topological space T is

- (everywhere) dense in T if $\overline{S} = T$
- nowhere dense in T if $T \setminus \overline{S}$ is dense in T
- meagre in T (or of the first category in T) if it is a union of a sequence of nowhere dense sets.

Examples. $\mathbb{Q}$ is dense in $\mathbb{R}$. In $\mathbb{R}$, one point sets are nowhere dense. So $\mathbb{Q}$ is meagre in $\mathbb{R}$. But $\mathbb{R} \setminus \overline{\mathbb{Q}} = \emptyset$, so $\mathbb{Q}$ is not nowhere dense in $\mathbb{R}$. Notice that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, so the closure in the definition of "nowhere dense" cannot be removed.

Fact 2.17. *A closed subset S of T is nowhere dense in T iff $T \setminus S$ is dense in T.*
Proof. If S is closed then $\overline{S} = S$. □

Fact 2.18. *A subset S of T is nowhere dense in T iff $\overline{S}^{\circ} = \emptyset$.*
Proof. $\overline{S}^{\circ} = T \setminus \overline{(T \setminus \overline{S})}$. □

Homeomorphisms and topological properties

Definitions. A bijection $f : T_1 \to T_2$ is a homeomorphism if U is open in T_1 iff $f(U)$ is open in T_2. If there is a homeomorphism $f : T_1 \to T_2$, we say that T_1 and T_2 are homeomorphic.

Simple facts. A bijection $f : T_1 \to T_2$ is a homeomorphism iff f and f^{-1} are both continuous. The inverse to a homeomorphism is a homeomorphism and the composition of two homeomorphisms is a homeomorphism.

Example. $x \to x/(1-|x|)$ is a homeomorphism of $(-1, 1)$ onto $\mathbb{R}$.

Question. *Show that* $[0, 1]$ *and* $\mathbb{R}$ *are not homeomorphic.*

Answer. We know that every continuous function on $[0, 1]$ is bounded while this is clearly false for $\mathbb{R}$.

Moral. To show that two spaces are not homeomorphic we found a topological invariant, ie a property of topological spaces preserved by homeomorphisms, that one space has and the other doesn't.

Topological invariants

Definition. A property of topological spaces is a topological invariant (or topological property) if it is preserved by a homeomorphism.
They are (ultimately) defined in terms of set-theoretic properties (preserved by bijections) and in terms of open sets.

Examples (Topological invariants).

- T is finite.
- T is a Hausdorff space (has the Hausdorff property).
- T is metrizable.
- The closure $\overline{S}$ of $S \subset T$ is the set of limits of convergent sequences whose terms are in S.
- Every continuous function on T is bounded.

Example (A non-topological invariant).

- I am an element of T.

Main points of Chapter 2

- Topological spaces. Open and closed sets.
- Topology induced by a metric. Other examples of topological spaces.
- Bases and sub-bases. The topology of pointwise convergence.
- Subspaces and product spaces.
- Neighbourhoods, closure, interior, boundary.
- Continuity. Sufficiency of openness of preimages of elements of a sub-basis.
- Continuity of composition. Continuity of maps into product spaces.
- Preservation of continuity of real-valued functions by algebraic operations.
- Defining topologies by continuity requirements. Infinite products. Quotients.
- Metrizability. Non-metrizability via Hausdorff property.
- Normal topological spaces.
- Sequential closure.
- Some notions of descriptive set theory.
- Homeomorphisms and topological invariants.

Chapter 3. Compactness

Definition. A topological space T is compact if every open cover of T has a finite subcover.
Explanations. A cover of A is a collection $\mathcal{U}$ of sets whose union contains A. A subcover is a subcollection of $\mathcal{U}$ which still covers A. A cover is open if all its members are open.

Example. The family $(a, 1]$, where $0 < a < 1$, is an open cover of $(0, 1]$ which has no finite subcover.
Warning. Notice the word "every" in the definition of compactness. Some finite covers always exist! (Eg, T is covered by $\{T\}$.)
Notice that if T is a subspace of another topological space S, the definition speaks about sets open in T (so about a cover by subsets of T). However, it could be also interpreted as speaking about covers of T by sets open in S. Fortunately, the two interpretations lead to the same notion of compactness!

Compactness of $[a, b]$

Theorem 3.1 (Heine-Borel). *Any closed bounded interval $[a, b]$ in $\mathbb{R}$ is compact.*

Proof. It will be convenient to admit intervals $[x, x] = \{x\}$. Let $\mathcal{U}$ be a cover of $[a, b]$ by sets open in $\mathbb{R}$. Let A denote the set of $x \in [a, b]$ such that $[a, x]$ can be covered by a finite subfamily of $\mathcal{U}$. Then $a \in A$ (a point is easy to cover), so $A \neq \emptyset$ and bounded above by b.

Let $c = \sup A$. Then $a \leq c \leq b$, so $c \in U$ for some $U \in \mathcal{U}$. Since U is open, there is $\delta > 0$ such that $(c - \delta, c + \delta) \subset U$.

Since $c = \sup A$, there is $x \in A$, $x > c - \delta$. So $[a, c + \delta) \subset [a, x] \cup (c - \delta, c + \delta)$ can be covered by a finite subfamily of $\mathcal{U}$, since $[a, x]$ can be covered by such a subfamily and $(c - \delta, c + \delta)$ is covered by U.

It follows that $(c, c + \delta) \cap [a, b] = \emptyset$ (since points from this set would belong to A but be bigger than $c = \sup A$). Hence $c = b$ and so $[a, b] = [a, c] \subset [a, c + \delta)$ is covered by a finite subfamily of $\mathcal{U}$. □

Compact subspaces and closed subsets

Fact 3.2. *Any closed subset C of a compact space T is compact.*

Proof. Let $\mathcal{U}$ be a cover of C by sets open in T. Then $\mathcal{U} \cup \{T \setminus C\}$ is an open cover of T. So it has a finite subcover of T. Deleting from it $T \setminus C$ if necessary, we get a finite cover of C by sets from $\mathcal{U}$. □

Fact 3.3. *Any compact subspace C of a Hausdorff space T is closed in T.*

Proof. Let $a \in T \setminus C$. Use that T is Hausdorff to find for each $x \in C$ disjoint T-open $U_x \ni x$ and $V_x \ni a$. Then U_x form an open cover of C. Let $U_{x_1}, \dots, U_{x_n}$ be a finite subcover of C. Then $V = \bigcap_{i=1}^n V_{x_i}$ is an open set containing a and disjoint from C, since $C \subset \bigcup_{i=1}^n U_{x_i}$ and $V \cap \bigcup_{i=1}^n U_{x_i} = \emptyset$. Hence $a \in (T \setminus C)^{\circ}$, so $T \setminus C$ is open. □

Fact 3.4. *A compact subspace C of a metric space M is bounded.*

Proof. Fix $a \in M$. Then C is covered by open balls $B(a, r)$. So there are $r_1, \dots, r_n$ such that

$$C \subset \bigcup_{i=1}^n B(a, r_i) = B\big(a, \max(r_1, \dots, r_n)\big).$$ □

Intersections of compact sets

Theorem 3.5. *Let $\mathcal{F}$ be a collection of nonempty closed subsets of a compact space T such that every finite subcollection of $\mathcal{F}$ has a nonempty intersection. Then the intersection of all sets from $\mathcal{F}$ is non-empty.*

Proof. Suppose that the intersection of the sets from $\mathcal{F}$ is empty. Let $\mathcal{U}$ be the collection of their complements. By De Morgan's rules, $\mathcal{U}$ is a cover of T. Since the sets from $\mathcal{U}$ are open, it is an open cover, and so it has a finite subcover $U_1, \dots, U_n$. Then $F_i = T \setminus U_i$ belong to $\mathcal{F}$ and, by De Morgan's rules, their intersection is empty. This contradicts the assumptions of the Theorem. □

Corollary 3.6. *Let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be nonempty closed subsets of a compact space T. Then $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.*

Remark. Theorem 3.5 and its Corollary are often stated for Hausdorff spaces and compact sets. This follows since we know that then compact sets are closed.

Continuous image

Theorem 3.7. *A continuous image of a compact space is compact.*

Proof. Let $f : T \to S$ be continuous and T compact. We have to show that $f(T)$ is compact. Let $\mathcal{U}$ be an open cover of $f(T)$. Since f is continuous, the sets $f^{-1}(U)$, where $U \in \mathcal{U}$ are open. Moreover, they form a cover of T since for every $x \in T$, $f(x)$ belongs to some $U \in \mathcal{U}$. By compactness of T, there is a finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n)$ of T. But for every $y \in f(T)$ we have $y = f(x)$ where $x \in T$. Then $x \in f^{-1}(U_i)$ for some i and so $y = f(x) \in U_i$. Hence $U_1, \dots, U_n$ form the finite subcover we wanted. □

Theorem 3.8. *A continuous bijection of a compact space T onto a Hausdorff space S is a homeomorphism.*

Proof. If U is open in T, $T \setminus U$ is closed, hence compact by 3.2, so $f(T \setminus U)$ is compact by 3.7, so closed by 3.3, and so the preimage of U under f^{-1}, $(f^{-1})^{-1}(U) = f(U) = S \setminus f(T \setminus U)$ is open. □

Attainment of minima and maxima

Definition. A function $f : T \to \mathbb{R}$ is called lower semi-continuous if for every $c \in \mathbb{R}$, $\{x \in T : f(x) > c\}$ is open. It is called upper semi-continuous if for every $c \in \mathbb{R}$, $\{x \in T : f(x) < c\}$ is open.

Theorem 3.9. *A lower (upper) semi-continuous real-valued function f on a nonempty compact space T is bounded below (above) and attains its minimum (maximum).*

Proof. Let $c = \inf_{x \in T} f(x)$, where possibly $c = -\infty$. If f is lower semi-continuous and does not attain the value c (surely true if $c = -\infty$), then $f(x) > c$ for every x and so the open sets $\{x : f(x) > r\} = f^{-1}(r, \infty)$, where $r > c$, cover T. Find $r_1, \dots, r_n > c$ so that $T \subset \bigcup_{i=1}^n \{x : f(x) > r_i\}$. Then $f(x) > \min(r_1, \dots, r_n)$ for all x. Hence $c = \inf_{x \in T} f(x) \geq \min(r_1, \dots, r_n) > c$, a contradiction. □

Corollary 3.10. *A continuous real-valued function f on a compact space T (is bounded and) attains its maximum and minimum.*

Compactness of products

Theorem 3.11 (Special case of Tychonov's Theorem). *The product $T \times S$ of compact spaces T, S is compact.*

Let $\mathcal{U}$ be an open cover of $T \times S$. We first show how the statement follows from the following Lemma.

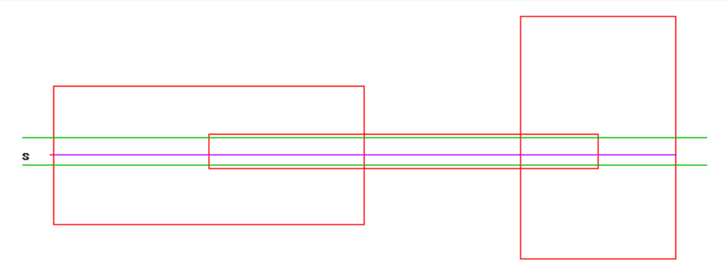
Lemma 3.12. *If $s \in S$, there is open $V \subset S$ containing s such that $T \times V$ can be covered by a finite subfamily of $\mathcal{U}$.*

Proof of Theorem. By Lemma, for each $s \in S$ there is open $V_s \subset S$ such that $s \in V_s$ and $T \times V_s$ can be covered by a finite subfamily of $\mathcal{U}$. Since S is compact and V_s ($s \in S$) form its open cover, there are $V_{s_1}, \dots, V_{s_m}$ covering S. Hence $T \times S = \bigcup_{j=1}^m T \times V_{s_j}$. Since the union is finite and each of the sets $T \times V_{s_j}$ can be covered by a finite subfamily of $\mathcal{U}$, $T \times S$ can be covered by a finite subfamily of $\mathcal{U}$. □

Cover of $T \times \{s\}$ where $s \in S$

Proof of Lemma: *It T is compact, $s \in S$ and $\mathcal{U}$ is an open cover of $T \times S$, there is open $V \subset S$ containing s such that $T \times V$ can be covered by a finite subfamily of $\mathcal{U}$.*

Proof. For each $x \in T$ find $W_x \in \mathcal{U}$ such that $(x, s) \in W_x$.



By definition of the product topology, there are open $U_x \subset T$ and $V_x \subset S$ such that $(x, s) \in U_x \times V_x \subset W_x$. Then the sets U_x ($x \in T$) form an open cover of T, so they contain a finite subcover $U_{x_1}, \dots, U_{x_n}$. Let $V = \bigcap_{i=1}^n V_{x_i}$. Then $V \subset S$ is open and $T \times V \subset \bigcup_{i=1}^n U_{x_i} \times V_{x_i} \subset \bigcup_{i=1}^n W_{x_i}$. □

Compact sets in $\mathbb{R}^n$

Theorem 3.13 (Heine-Borel). *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof. Metric spaces are Hausdorff. So we have already proved in 3.3 and 3.4 that any compact subset of $\mathbb{R}^n$ is closed and bounded. If $C \subset \mathbb{R}^n$ is closed and bounded, there is $[a, b]$ such that

$$C \subset [a, b] \times \cdots \times [a, b].$$

The space on the right is compact by Tychonov's Theorem. So C is a closed subset of a compact space, hence compact by 3.2. □

Warning. Although the implication $\Rightarrow$ is true, with the same proof, in all metric spaces, the implication $\Leftarrow$ is completely false in general metric spaces.

Example. $(0, 1)$ is a closed and bounded subset of the space $(0, 1)$. If you feel this is cheating, consider $\mathbb{R}$ with the metric $d_0(x, y) = \min(|x - y|, 1)$. Then $[0, \infty)$ is closed and bounded, yet non-compact.

Lebesgue number of a cover

Definition. Given a cover $\mathcal{U}$ of a metric space M, a number $\delta > 0$ is called a Lebesgue number of $\mathcal{U}$ if for any $x \in M$ there is $U \in \mathcal{U}$ such that $B(x, \delta) \subset U$.

Example. The sets $(x/2, x)$, where $x \in (0, 1)$, form an open cover of the open interval $(0, 1)$ which has no Lebesgue number.

Proposition 3.14. *Every open cover $\mathcal{U}$ of a compact metric space has a Lebesgue number.*

Proof. For $x \in M$ define $r(x)$ as the supremum of $0 < r \leq 1$ for which there is $U \in \mathcal{U}$ such that $B(x, r) \subset U$. We show that $r(x)$ is lower semi-continuous. Then it attains its minimum, say $r > 0$. So for every x, $r(x) \geq r$, and so $r/2$ is a Lebesgue number of $\mathcal{U}$.
Lower semi-continuity of $r(x)$: Let $W = \{x : r(x) > c\}$ and $x \in W$.
For $\varepsilon = \frac{1}{3}(r(x) - c)$ find $U \in \mathcal{U}$ such that $B(x, r(x) - \varepsilon) \subset U$.
If $d(x, y) < \varepsilon$, then $B(y, c + \varepsilon) \subset B(x, c + 2\varepsilon) = B(x, r(x) - \varepsilon) \subset U$.
Hence $r(y) \geq c + \varepsilon > c$. So $B(x, \varepsilon) \subset W$, and so W is open. □

Uniform continuity

Definition. A map f of a metric space M to a metric space N is called uniformly continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in M)\big(d_M(x, y) < \delta \implies d_N(f(x), f(y)) < \varepsilon\big)$$

Theorem 3.15. *A continuous map of a compact metric space M to a metric space N is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Then the sets

$$U_z = f^{-1}\big(B_N(f(z), \varepsilon/2)\big), \quad z \in M$$

form an open cover of M. Let δ be its Lebesgue number. If $x, y \in M$ and $d_M(x, y) < \delta$ then $y \in B(x, \delta)$. But $B(x, \delta)$ is contained in a single set U_z, hence

$$d_N(f(x), f(y)) \leq d_N(f(x), f(z)) + d_N(f(y), f(z)) < \varepsilon. \qquad \square$$

Sequential compactness in metric spaces

Definition. A metric space M is said to be sequentially compact if every sequence of its elements has a convergent subsequence.
Notice that in practice M is usually a subspace of another metric space N. It is important to notice that in this situation "convergent" means "convergent in the space M", that is to an element of M!

Theorem 3.16. *A metric space is compact iff it is sequentially compact.*

Proof of ($\Rightarrow$). Let x_j be a sequence of points of a compact metric space M. Let $F_j = \overline{\{x_j, x_{j+1}, \dots\}}$.
By Corollary 3.6 we have $\bigcap_{j=1}^{\infty} F_j \neq \emptyset$. Choose $x \in \bigcap_{j=1}^{\infty} F_j$.
Since $x \in \overline{\{x_1, x_2, \dots\}}$, there is $j(1)$ so that $d(x, x_{j(1)}) < 1$.
Continue recursively:
When $j(k)$ has been defined, use that $x \in \overline{\{x_{j(k)+1}, x_{j(k)+2}, \dots\}}$ to find $j(k+1) > j(k)$ so that $d(x, x_{j(k+1)}) < 1/k$.
Then $x_{j(k)}$ is a subsequence of x_j converging to x. □

Proof of Theorem 3.16($\Leftarrow$)

Let $\mathcal{U}$ be an open cover of M. For each $x \in M$ choose $0 < s(x) \leq 1$ such that $B(x, s(x)) \subset U$ for some $U \in \mathcal{U}$.
Greedy Algorithm. Denote $M_1 = M$, let $s_1 = \sup_{x \in M_1} s(x)$, find x_1 with $s(x_1) > \frac{1}{2}s_1$ and choose $U_1 \in \mathcal{U}$ so that $B(x_1, s(x_1)) \subset U_1$.
When $x_1, \dots, x_j$ have been defined, denote

$$M_{j+1} = M_j \setminus B(x_j, s(x_j)) = M \setminus \bigcup_{i=1}^{j} B(x_i, s(x_i)).$$

If $M_{j+1} = \emptyset$, $M \subset \bigcup_{i=1}^{j} B(x_i, s(x_i)) \subset \bigcup_{i=1}^{j} U_i$, and we are done.
If $M_{j+1} \neq \emptyset$, denote $s_{j+1} = \sup_{x \in M_{j+1}} s(x)$, find $x_{j+1} \in M_{j+1}$ with $s(x_{j+1}) > \frac{1}{2}s_{j+1}$ and pick $U_{j+1} \in \mathcal{U}$ so that $B(x_{j+1}, s(x_{j+1})) \subset U_{j+1}$.
Suppose the procedure ran for ever. The (infinite) sequence x_j has a subsequence $x_{j(k)}$ converging to some x. Since M_j are closed, x belongs to all of them and so $d(x, x_j) \geq \frac{1}{2}s_j \geq \frac{1}{2}s(x)$ for each j. So no subsequence of x_j can converge to x, a contradiction.
Hence $M_{j+1} = \emptyset$ for some j and $\{U_i : i \leq j\}$ is a finite subcover. □

Comments on sequential compactness

Recall that convergence of sequences is defined in an arbitrary topological space. We call a topological space T sequentially compact if every sequence in T has a convergent subsequence.
Theorem 3.16 says that a metric space is compact iff it is sequentially compact. However, both implications of this theorem are **false** in general topological spaces, and the restriction to metric spaces is absolutely essential!
Subsets M of N such that every $x_j \in M$ has a subsequence converging in N are called relatively sequentially compact in N.

Question. *Show: If N is a metric space, $M \subset N$ is relatively sequentially compact in N iff $\overline{M}$ is compact.*

Answer. If $x_j \in \overline{M}$, find $y_j \in M$ with $d(x_j, y_j) < 1/j$. Choose $y_{j(k)}$ converging to some $y \in N$. Then $y \in \overline{M}$ and $x_{j(k)}$ converge to y.
Hence $\overline{M}$ is sequentially compact, hence compact.
The converse is obvious. □

Maximal open covers without finite subcovers

We sketch a proof of the full version of Tychonov's Theorem that the (possibly infinite) product of compact spaces is compact.

Lemma 3.17. *If T is not compact, it has a maximal open cover $\mathcal{V}$ without a finite subcover. Moreover, for any continuous $\pi : T \to S$,*

(a) *the collection $\mathcal{U}$ of sets U open in S such that $\pi^{-1}(U) \in \mathcal{V}$ cannot contain a finite cover of S;*

(b) *if $U \subset S$ is open and not in $\mathcal{U}$ then $\pi^{-1}(S \setminus U)$ can be covered by finitely many sets from $\mathcal{V}$.*

Proof. One can argue that $\mathcal{V}$ can be defined by adding open sets to a starting cover as long as possible, perhaps infinitely many times, but keeping the property that the new cover has no finite subcover. A formal proof needs the Axiom of Choice.
(a) Preimages under π of a finite cover of S by sets from $\mathcal{U}$ would form a finite cover of T by sets from $\mathcal{V}$.
(b) $\mathcal{V}$ wouldn't be maximal since $\pi^{-1}(U)$ could be added to it. □

Tychonov's Theorem

Theorem 3.18. *The product of compact spaces is compact.*

Proof. By Lemma 3.17, assume that $\mathcal{V}$ is a maximal open cover of $C = \prod_{i \in I} C_i$ without a finite subcover. Let $\mathcal{U}_i$ be the collection of sets U_i open in C_i such that $\pi_i^{-1}(U_i) \in \mathcal{V}$. By Lemma 3.17 and compactness of C_i, $\mathcal{U}_i$ is not a cover of C_i. Choose $x(i) \in C_i$ not covered by $\mathcal{U}_i$. So Lemma 3.17 gives that for every open $U_i \ni x(i)$, $\pi_i^{-1}(C_i \setminus U_i)$ can be covered by finitely many sets from $\mathcal{V}$.
The point $x = (x(i))_{i \in I}$ belongs to some set from $\mathcal{V}$. Hence there is a finite set $J \subset I$ and sets $U_j \ni x(j)$ open in C_j such that $\bigcap_{j \in J} \pi_j^{-1}(U_j)$ is contained in a set from $\mathcal{V}$. But then

$$C = \Big(\bigcap_{j \in J} \pi_j^{-1}(U_j)\Big) \cup \bigcup_{j \in J} \pi_j^{-1}(C_j \setminus U_j)$$

is a finite union and each of the terms, and so all of C, can be covered by finitely many sets from $\mathcal{V}$; a contradiction. □

Main points of Chapter 3

- Covers. Definition of compactness.
- Compact subspaces and closed subsets.
- Intersections of compact sets.
- Continuous images of compact spaces.
- Attainment of minima and maxima.
- Compactness of products.
- Compact sets in $\mathbb{R}^n$.
- Lebesgue number of a cover.
- Uniform continuity.
- Equivalence of sequential compactness and compactness for metric spaces.
- Tychonov's Theorem.

Chapter 4. Connectedness

Definition. A topological space T is connected if the only decompositions of T into open sets are $T = \emptyset \cup T$ and $T = T \cup \emptyset$.

Explanation. A decomposition means $T = A \cup B$ and $A \cap B = \emptyset$.

Fact 4.1. *T is disconnected iff any one of the following holds.*

(a) *T has a decomposition into two nonempty open sets;*

(b) *T has a decomposition into two nonempty closed sets;*

(c) *T has a subset which is open, closed and is neither $\emptyset$ nor T;*

(d) *T admits a non-constant continuous function to a two-point discrete space $\{a, b\}$.*

Proof. The sets A, B in a decomposition are both open iff they are both closed. So (a), (b) and (c) are all equivalent to the definition. (d) follows by defining $f(x) = a$ on A and $f(x) = b$ on B. Finally, if $f : T \to \{a, b\}$ is as in (d), we let $A = f^{-1}(a)$ and $B = f^{-1}(b)$. □

Connected subsets

Warning. Even if T is a subspace of another space, "open" in the definition of connectedness means open in T!

Definition. A set $T \subset S$ is separated by subsets U, V of S if

$$T \subset U \cup V, \quad U \cap V \cap T = \emptyset, \quad U \cap T \neq \emptyset \quad \text{and} \quad V \cap T \neq \emptyset.$$

Proposition 4.2. *A subspace T of a topological space S is disconnected iff it is separated by some open subsets U, V of S.*

Proof. If T is disconnected, there are nonempty $A, B \subset T$ open in T such that $T = A \cup B$ and $A \cap B = \emptyset$. Since T is subspace of S, there are U, V open in S such that $A = U \cap T$ and $B = V \cap T$. Then U, V separate T.
If U, V separate T, $U \cap T$ is nonempty, open and closed in T. □

Remark. The second condition from the definition of "separated" would be nicer were it replaced by $U \cap V = \emptyset$. But this cannot be done. For example, consider the subspace $\{a, b\}$ of the three point space $\{a, b, c\}$ in which a set is open iff it is empty or contains c.

A slightly dangerous "equivalent" definition

Continuing the remark from the previous slide, we show that in metric spaces the definition of "separated" may be simplified as indicated.

Theorem 4.3. *A subset T of a metric space M is disconnected iff there are open subsets U, V of M such that*

$$T \subset U \cup V, \quad U \cap V = \emptyset, \quad U \cap T \neq \emptyset \quad \text{and} \quad V \cap T \neq \emptyset.$$

Proof. ($\Leftarrow$) is obvious.
($\Rightarrow$) Write $T = A \cup B$ where A, B are disjoint nonempty sets open in T. Define $U = \{x \in M : d(x, A) < d(x, B)\}$ and $V = \{x \in M : d(x, A) > d(x, B)\}$. Then U, V are disjoint and, since the functions $x \to d(x, A)$ and $x \to d(x, B)$ are continuous, they are open. We show $A \subset U$. Let $x \in A$. Since A is open in T, there is $\delta > 0$ such that $B(x, \delta) \cap T \subset A$. Since $B \subset T$ is disjoint from A, $B(x, \delta) \cap B = \emptyset$. So $d(x, B) \geq \delta > 0$. Since $d(x, A) = 0$, we have $x \in U$ as wanted. Similarly, $B \subset V$ and the statement follows. □

Warning: Metric Spaces Only!

Connectedness and Intervals

Theorem 4.4. *A subset of $\mathbb{R}$ is connected iff it is an interval.*

Before proving this, (on the next slide), we have to say what we mean by "an interval": It is one of the sets $\emptyset$; $\{a\}$;
$[a, b]$ (where $a < b$); $(a, b]$ (where $a < b$ and a may be $-\infty$);
$[a, b)$ (where $a < b$ and b can be ∞); and
(a, b) (where $a < b$ and any of the a, b may be infinite).

Fact 4.5. *A set $I \subset \mathbb{R}$ is an interval iff*
$(\forall x, y \in I)(\forall z \in \mathbb{R})(x < z < y \implies z \in I)$

Proof. Intervals clearly have this property. Conversely, suppose I has the described property and that I is non-empty and not a singleton. Let $a = \inf I$ and $b = \sup I$ (allowing $a, b = \pm\infty$).
We show that $(a, b) \subset I$: If $z \in (a, b)$ there are $x \in I$ with $x < z$ (because $z > \inf I$) and $y \in I$ with $y > z$ (because $z < \sup I$). So $z \in I$ by our condition. Hence $(a, b) \subset I \subset (a, b) \cup \{a, b\}$, showing that I is one of the sets listed above. □

Warning: $\mathbb{R}$ only!

Connected subsets of $\mathbb{R}$

Lemma 4.6. *If $I \subset \mathbb{R}$ is connected then it is an interval.*

Proof. Suppose I is not an interval. Then there are $x, y \in I$ and $z \in R$ such that $x < z < y$ and $z \notin I$. Let $A = (-\infty, z) \cap I$, $B = (z, \infty) \cap I$. Then A, B are disjoint, open (by definition of topology on I) and nonempty (since $x \in A$ and $y \in B$). Finally $A \cup B = I$ since $z \notin I$, and we see that I is not connected. □

Lemma 4.7. *Any interval $I \subset \mathbb{R}$ is connected.*

Proof #1. If not, there is a non-constant continuous f from I to discrete $\{0, 1\}$. Then $f : I \to \mathbb{R}$ is also continuous, which contradicts the intermediate value theorem. □

Proof #2. Suppose I is decomposed into nonempty open sets A, B. Choose $a \in A$, $b \in B$ and assume $a < b$. Then A, B form an open cover of $[a, b]$. Let δ be its Lebesgue number. Then $[a, a + \delta/2] \subset A$, $[a + \delta/2, a + 2\delta/2] \subset A$, ..., until we come to an interval containing b. So $b \in A$, and A, B are not disjoint. □

Unions of connected sets

Theorem 4.8. *If C and C_j ($j \in J$) are connected subspaces of a topological space T and if $C_j \cap \overline{C} \neq \emptyset$ for each j then $K = C \cup \bigcup_{j \in J} C_j$ is connected.*

Proof. Suppose that K is disconnected. Hence there are U, V open in T which separate K. So $K \subset U \cup V$, $U \cap V \cap K = \emptyset$, $U \cap K \neq \emptyset$ and $V \cap K \neq \emptyset$.
Now C, being connected, cannot be separated by U, V. So C does not meet one of the sets U, V. Suppose that it is V, so $C \cap V = \emptyset$. Since V is open, we even have $\overline{C} \cap V = \emptyset$ and so $K \cap \overline{C} \subset U$.
Since $C_j \cap \overline{C} \neq \emptyset$ it follows that $C_j \cap U \neq \emptyset$. Since C_j is connected, the same argument as for C shows that either $C_j \subset U$ or $C_j \subset V$.
But $C_j \subset V$ is impossible since $C_j \cap U \neq \emptyset$. Hence $C_j \subset U$.
So $K = C \cup \bigcup_{j \in J} C_j \subset U$, which contradicts $V \cap K \neq \emptyset$. □

Theorem 4.9. *If $C \subset T$ is connected, so is any $C \subset K \subset \overline{C}$.*

Proof. $K = C \cup \bigcup_{x \in K} \{x\}$ and $\{x\} \cap \overline{C} \neq \emptyset$ for each $x \in K$. □

Continuous Images

Theorem 4.10. *The continuous image of a connected space is connected.*

Proof. Let $f : T \to S$ be continuous with T connected. If $f(T)$ is disconnected, there are open sets $U, V \subset S$ separating $f(T)$, which means that $f(T) \subset U \cup V$, $U \cap V \cap f(T) = \emptyset$, $U \cap f(T) \neq \emptyset$ and $V \cap f(T) \neq \emptyset$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty, open in T (since U, V are open and f is continuous), disjoint (since $f^{-1}(U) = f^{-1}(U \cap f(T))$ and $f^{-1}(V) = f^{-1}(V \cap f(T))$ are preimages of disjoint sets), and cover T (since U, V cover $f(T)$).
This contradicts connectedness of T. □

Notice that together with the description of connected subsets of $\mathbb{R}$, this Proposition implies the Intermediate Value Theorem. (But to use this, we should better prove connectedness of intervals without the use of the Intermediate Value Theorem.)

Examples. Curves in $\mathbb{R}^2$ are connected, graphs of functions continuous on intervals are connected (as subsets of $\mathbb{R}^2$).

Proofs of connectedness

Theorem 4.11. *The product of connected spaces is connected.*

Proof. Let T, S be connected and $s_0 \in S$. Denote $C = T \times \{s_0\}$ (fixed horizontal line) and $C_t = \{t\} \times S$ (vertical lines above $t \in T$). Then C and C_t, being homeomorphic to T and S, respectively, are connected. Since $C_t \cap C \neq \emptyset$ (it contains (t, s_0)) and $T \times S = C \cup \bigcup_{t \in T} C_t$, the statement follows from 4.8. □

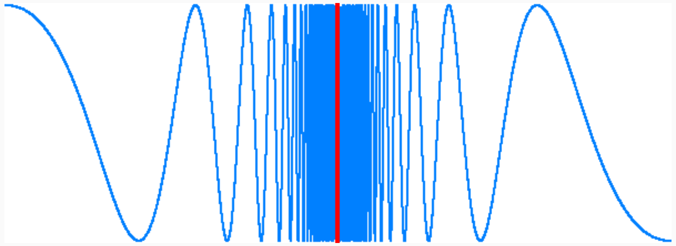
Proofs of connectedness: We combine connectedness of continuous images of connected spaces 4.10 (often of intervals) with connectedness of sufficiently intersecting unions 4.8.

Example.

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is connected by 4.11.
- Circles are connected (continuous images of intervals).
- $\mathbb{R}^2 \setminus \{0\}$ is connected since it is the union of circles about $(0, 0)$ each of which meets a fixed half-line, eg the positive x-axis.

Example: The "topologist's sin curve"

Example. The "topologist's sin curve," which is the graph of $\sin(1/t)$ together with the vertical segment I from $(0,-1)$ to $(0,1)$ is connected.



Proof. Let $C = \{(t, \sin(1/t)), t > 0\}$ and $D = \{(t, \sin(1/t)), t < 0\}$. Then C, D and I are connected (continuous images of intervals). The point $(0,0) \in I$ belongs to $\overline{C}$ since $(t_k, \sin(1/t_k)) \to (0,0)$ when $t_k = 1/k\pi$. Hence $I \cap \overline{C} \neq \emptyset$ and $I \cup C$ is connected by 4.8. Similarly $I \cup D$ is connected. Moreover, $(I \cup C) \cap (I \cup D) \neq \emptyset$, so their union is connected by 4.8. □

Connected components

Connectedness gives a natural equivalence relation on T:
$x \sim y$ if x and y belong to a common connected subspace of T.
Reflexivity and symmetry are easy. Transitivity follows from 4.8.

Definition. The equivalence classes of $\sim$ are called the (connected) components of T.

Fact 4.12. *Directly from the definition we get:*

- *A component containing x is the union of all connected subsets of T containing x.*
- *Components are connected (by 4.8).*
- *Components are closed (by 4.9).*
- *Components are maximal connected subsets of T.*

Examples. The components of $(0,1) \cup (1,2)$ are $(0,1)$ and $(1,2)$. The components of $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ and the Cantor set are just points.

Topological invariants based on connectedness.

From the definition of connectedness (or from 4.10) we see

Fact 4.13. *Connectedness is a topological invariant (it is preserved by homeomorphisms). Hence the number of components is also a topological invariant.*

Example. $(0,1)$, $(0,1) \cup (1,2)$ and $(0,1) \cup (1,2) \cup (2,3)$ are mutually non-homeomorphic.

Example. The property "$T \setminus \{x\}$ is connected for every $x \in T$" is also a topological invariant. This shows that

- $[0,1]$ is not homeomorphic to a circle;
- $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$;
- $[0,1]$ is not homeomorphic to a square.

Path-connected spaces

Definitions. Let $u, v \in T$. A path from u to v in T is a continuous map $\varphi : [0,1] \to T$ such that $\varphi(0) = u$ and $\varphi(1) = v$.
T is called path-connected if any two points in T can be joined by a path in T.

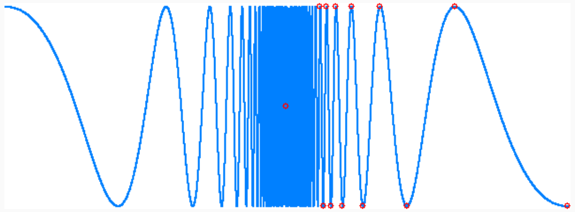
Proposition 4.14. *A path-connected space T is connected.*

Proof. Let $u \in T$. For every $v \in T$, the image C_v of a path from u to v is connected, and all the C_v contain u. So $T = \{u\} \cup \bigcup_{v \in T} C_v$ is connected by 4.8. □
Our main reason for introducing path-connectedness is that it is easier to imagine than connectedness. (But it is not equivalent to it.) We will give only one result (Theorem 4.16) in the proof of which the following simple observation will be useful.

Fact 4.15. *A path φ_1 from u to v and a path φ_2 from v to w may be joined to a path φ from u to w by defining $\varphi(t) = \varphi_1(2t)$ for $t \in [0,1/2]$ and $\varphi(t) = \varphi_2(2t-1)$ for $t \in [1/2,1]$.*

A connected not path-connected space

Example. The "topologist's sin curve," S is not path-connected.



Proof. Let $t_k = 2/((2k+1)\pi)$, so $x_k = (t_k, (-1)^k) \in S$. Suppose that φ is a path from $(0,0)$ to x_1 in S. We find a decreasing sequence $s_k \in [0,1]$ such that $\varphi(s_k) = x_k$. Then s_k converge, so $\varphi(s_k)$ have to converge, but they obviously don't, a contradiction.
Finding s_k: Let $s_1 = 1$. If s_k has been defined, observe that there is a point $s_{k+1} \in (0, s_k)$ such that $\varphi(s_{k+1}) = x_{k+1}$: if not, the connected image $\varphi[0, s_k]$ would be separated by the sets $\{(x,y) : x < t_{k+1}\}$ and $\{(x,y) : x > t_{k+1}\}$. □

Open sets in $\mathbb{R}^n$

Theorem 4.16. *Connected open sets in $\mathbb{R}^n$ are path-connected.*

Proof. Let $u \in U$ and denote by V the set of points $x \in U$ that can be joined to u by a path in U. We show that for every $z \in U \cap \overline{V}$ there is $\delta > 0$ such that $B(z,\delta) \subset V$. Then V is both open and closed in U and contains u, so $V = U$, and the statement follows. So suppose $z \in U \cap \overline{V}$. Find $\delta > 0$ such that $B(z,\delta) \subset U$. Since $z \in \overline{V}$, there is $y \in V \cap B(z,\delta)$. Then every $x \in B(z,\delta)$ is in V since we may join a path in U from u to y (which exists since $y \in V$) with the linear path from y to x (which is in U since it is in $B(z,\delta)$). □

Theorem 4.17. *Open subsets of $\mathbb{R}^n$ have open components.*

Proof. Let C be a component of open $U \subset \mathbb{R}^n$ and $x \in C$. Find $\delta > 0$ with $B(x,\delta) \subset U$. Since $B(x,\delta)$ is connected and C is the union of all connected subsets of U that contain x, we have $B(x,\delta) \subset C$. Hence C is open. □

Warning: $\mathbb{R}^n$ Only! Open Sets Only!

Open sets in $\mathbb{R}$

Theorem 4.18. *A subset U of $\mathbb{R}$ is open iff it is a disjoint union of countably many open intervals.*

Explanation. So $U = \bigcup_i (a_i, b_i)$ where (a_i, b_i) are mutually disjoint and i runs through a finite set or through $\mathbb{N}$.

Observation. *Each of the (a_i, b_i) must be a component of U: it is connected and $U \subset (a_i, b_i) \cup (\mathbb{R} \setminus \overline{(a_i, b_i)})$. So these intervals are determined uniquely; and this tells us how to prove the Theorem.*

Proof. $(\Leftarrow)$ Any union of open intervals is open.
$(\Rightarrow)$ Let $U \subset \mathbb{R}$ be open and let C_j $(j \in J)$ be its components. Since C_j are connected and open, they are open intervals. As components, they are mutually disjoint. So we only have to show that J can be ordered into a sequence. For each j there is a rational number $r_j \in C_j$. By disjointness of the C_j's, for different j we get different rational numbers r_j. So $j \to r_j$ is an injection into $\mathbb{Q}$, showing that J can be ordered into a sequence. □

Warning: $\mathbb{R}$ Only! Open Sets Only!

Main points of Chapter 4

- Definition(s) of (dis)connectedness.
- Connected subsets of $\mathbb{R}$ are exactly intervals.
- Connectedness of sufficiently intersecting unions.
- Connectedness of continuous images and products.
- Connected components.
- Path-connectedness.
- A connected space need not be path-connected.
- Connected open sets in $\mathbb{R}^n$ are path-connected.
- Components of open sets in $\mathbb{R}^n$ are open.
- Open sets in $\mathbb{R}$ are countable disjoint unions of open intervals.

Chapter 5. Completeness

Recall that a sequence $x_n \in M$ converges to $x \in M$ if $d(x_n, x) \to 0$. As in $\mathbb{R}$ we will show that convergent sequences have the following property.

Definition. A sequence x_n in a metric space M is Cauchy if

$$(\forall \varepsilon > 0)(\exists k \in \mathbb{N})(\forall m, n \geq k)\, d(x_n, x_m) < \varepsilon$$

Definition. A metric space M is complete if every Cauchy sequence in M converges.

Warning. Although every Cauchy sequence in $(0,1)$ converges (it is a bounded sequence of real numbers), the space $(0,1)$ is incomplete: Some of these sequences do not converge in the space $(0,1)$ (only in $\mathbb{R}$). Not to make an error, we may finish the definition of completeness by "…converges to a point of M."

Warning. Completeness is not a topological invariant: $\mathbb{R}$ is complete and $(0,1)$ is homeomorphic to it, yet incomplete.

From convergence to Cauchy

Proposition 5.1. *A convergent sequence is Cauchy.*

Proof. For every $\varepsilon > 0$ there is n_ε such that $d(x_n, x) < \varepsilon/2$ for $n \geq n_\varepsilon$. If $m, n \geq n_\varepsilon$ then $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \varepsilon$. □

Recall that a subset S of a metric space M is closed iff for every sequence $x_n \in S$ converging to a point $x \in M$ we have that $x \in S$.

Proposition 5.2.

(a) *A complete subspace S of any metric space M is closed.*

(b) *A closed subset S of a complete metric space M is complete.*

Proof. (a) Suppose $S \ni x_n \to x \in M$. Then x_n is Cauchy in M, so in S (the definition of Cauchy sequence depends only on points of the sequence). So $x_n \to y \in S$. But S is a subspace of M, so $x_n \to y$ also in M. By uniqueness of limits, $x = y$, and so $x \in S$.
(b) Let (x_n) be Cauchy in S. Then it is Cauchy in M so it converges to $x \in M$. Since S is closed, $x \in S$. Hence $x_n \to x$ in S. □

Proving "Cauchy"

Fact 5.3. *A sequence x_n in a metric space M is Cauchy iff there is a sequence $\varepsilon_n \geq 0$ such that $\varepsilon_n \to 0$ and $d(x_m, x_n) \leq \varepsilon_n$ for $m > n$.*

Proof. Suppose that x_n is Cauchy. Define $\varepsilon_n = \sup_{m>n} d(x_m, x_n)$. Then $\varepsilon_n \to 0$: Given $\varepsilon > 0$, there is k such that $d(x_m, x_n) < \varepsilon$ for $m, n \geq k$; hence $\varepsilon_n \leq \varepsilon$ for $n \geq k$.
Suppose $d(x_m, x_n) \leq \varepsilon_n$ for $m > n$ and $\varepsilon_n \to 0$. Given $\varepsilon > 0$, find k so that $\varepsilon_n < \varepsilon$ for $n \geq k$. Then $d(x_m, x_n) \leq \varepsilon_n < \varepsilon$ for $m > n \geq k$; exchanging m, n and noting that $d(x_m, x_n) = 0 < \varepsilon$ if $m = n$ we see that $d(x_m, x_n) < \varepsilon$ for $m, n \geq k$. So x_n is Cauchy. □

Fact 5.4. *Let x_n be a sequence in a metric space M for which there are $\tau_n \geq 0$ such that $\sum_{n=1}^{\infty} \tau_n < \infty$ and $d(x_n, x_{n+1}) \leq \tau_n$ for each n. Then the sequence x_n is Cauchy.*

Proof. Use the previous Fact with $\varepsilon_n = \sum_{k=n}^{\infty} \tau_k$. Since $\sum_{n=1}^{\infty} \tau_n < \infty$, we have $\varepsilon_n \to 0$ and by the triangle inequality,

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \tau_k \leq \varepsilon_n. \quad \square$$

Completeness of $\mathbb{R}^n$

Theorem 5.5. *$\mathbb{R}^n$ is complete.*

Proof. Let $x(k) = (x_1(k), \ldots, x_n(k)) \in \mathbb{R}^n$ be a Cauchy sequence of elements of $\mathbb{R}^n$.
For every $\varepsilon > 0$ there is k_ε such that $\|x(k) - x(l)\| < \varepsilon$ for $k, l \geq k_\varepsilon$.
For each $i = 1, \ldots, n$ and $k, l \geq k_\varepsilon$,

$$|x_i(k) - x_i(l)| \leq \|x(k) - x(l)\| < \varepsilon$$

Hence the sequence $\big(x_i(k)\big)_{k=1}^{\infty}$ of real numbers is Cauchy, and so it converges to some $x_i \in \mathbb{R}$.

Let $x = (x_1, \ldots, x_n)$. Then

$$\lim_{k\to\infty} \|x(k) - x\| = \lim_{k\to\infty} \sqrt{\sum_{i=1}^{n} |x_i(k) - x_i|^2} = 0.$$

So $x(k) \to x$ as required. □

Spaces of bounded functions

Proposition 5.6. *For any set S, the space $\mathcal{B}(S)$ of bounded real-valued functions on S with the norm $\|f\| = \sup_{x\in S}|f(x)|$ is complete.*

Proof. Let (f_n) be a Cauchy sequence. Given any $\varepsilon > 0$, there is n_ε such that $\sup_{x\in S}|f_m(x) - f_n(x)| < \varepsilon$ whenever $m, n \geq n_\varepsilon$. Hence for each fixed $x \in S$ the sequence $(f_n(x))$ is Cauchy in $\mathbb{R}$. So it converges to some $f(x) \in \mathbb{R}$.
Note that, if $n \geq n_\varepsilon$, then $|f_m(x) - f_n(x)| < \varepsilon$ for each $m \geq n_\varepsilon$. Limit as $m \to \infty$ gives $|f(x) - f_n(x)| \leq \varepsilon$ for $x \in S$ and $n \geq n_\varepsilon$.
This implies both that f is bounded (say, since $|f(x) - f_{n_1}(x)| \leq 1$ and f_{n_1} is bounded), which we need in order to know that $f \in \mathcal{B}(S)$ and that the sequence (f_n) converges to f in our norm (since $\|f - f_n\| \leq \varepsilon$ for $n \geq n_\varepsilon$). □

Note that the convergence in the norm of $\mathcal{B}(S)$ is the uniform convergence.

Continuous functions form a closed subset of $\mathcal{B}(T)$

Lemma 5.7. *If T is any topological space then the set $\mathcal{C}_b(T)$ of bounded continuous functions on T is a closed subset of $\mathcal{B}(T)$.*

Proof. Let $f \in \overline{\mathcal{C}_b(T)}$. Then for every $\varepsilon > 0$ there is $g_\varepsilon \in C_b(T)$ with

$$\sup_{x\in T}|f(x) - g_\varepsilon(x)| < \varepsilon.$$

We show that for every $a \in \mathbb{R}$,

$$\{x : f(x) > a\} = \bigcup_{\varepsilon>0}\{x : g_\varepsilon(x) > a + \varepsilon\}$$

If $f(x) > a$, we let $\varepsilon = (f(x) - a)/2$ and get

$$g_\varepsilon(x) = f(x) - (f(x) - g_\varepsilon(x)) > f(x) - \varepsilon = a + \varepsilon.$$

If $g_\varepsilon(x) > a + \varepsilon$ then

$$f(x) = g_\varepsilon(x) - (g_\varepsilon(x) - f(x)) > (a + \varepsilon) - \varepsilon = a.$$

So $\{x : f(x) > a\}$ and similarly $\{x : f(x) < a\}$ are open. Continuity of f follows from 2.6. □

Spaces of continuous functions

Corollary 5.8. *If T is any topological space then the space $\mathcal{C}_b(T)$ with the norm $\|f\| = \sup_{x\in T}|f(x)|$ is complete.*

Proof. Immediate from Lemma 5.7, Proposition 5.6 and Proposition 5.2. □

Corollary 5.9. *If K is a compact topological space then the space $\mathcal{C}(K)$ of continuous functions on K equipped with the norm $\|f\| = \max_{x\in K}|f(x)|$ is complete.*

Proof. Since every continuous function on K is bounded and attains its maximum, $\mathcal{C}(K) = \mathcal{C}_b(K)$ and

$$\|f\| = \sup_{x\in T}|f(x)| = \max_{x\in K}|f(x)|.$$

Hence this is a special case of 5.8. □

The space $\mathcal{C}[0,1]$ with the L^1 norm

Example. The space $\mathcal{C}[0,1]$ with the norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$ is incomplete.

Proof. Let $f_n(x) = \min(\sqrt{n}, 1/\sqrt{x})$ for $x > 0$ and $f_n(0) = \sqrt{n}$.
So $f_n \in \mathcal{C}[0,1]$. First we show that (f_n) is Cauchy. If $m > n$ then

$$\int_0^1 |f_m(x) - f_n(x)|\,dx = \int_0^{1/m} (\sqrt{m} - \sqrt{n})\,dx + \int_{1/m}^{1/n} (1/\sqrt{x} - \sqrt{n})\,dx$$
$$\leq 1/\sqrt{m} + 2/\sqrt{n} \leq 3/\sqrt{n} \to 0.$$

Now let $f \in \mathcal{C}[0,1]$. Find $k \in \mathbb{N}$ such that $|f| \leq \sqrt{k}$. Then for $n > k$,

$$\int_0^1 |f_n(x) - f(x)|\,dx \geq \int_{1/n}^{1/k} (1/\sqrt{x} - f(x))\,dx$$
$$\geq 2(1/\sqrt{k} - 1/\sqrt{n}) - 1/\sqrt{k}$$
$$= 1/\sqrt{k} - 2/\sqrt{n} \to 1/\sqrt{k} > 0.$$

□

Proof of Tietze's Theorem — a lemma

Recall that Urysohn's Lemma says that in a normal topological space T, for any two disjoint closed sets A, B there is a continuous $g : T \to [0,1]$ such that $g(x) = 0$ on A and $g(x) = 1$ on B.

Lemma 5.10. *Let $f \in C_b(S)$. Then there is $h \in C_b(T)$ such that $\|f - h\|_{C_b(S)} \le \frac{2}{3}\|f\|_{C_b(S)}$ and $\|h\|_{C_b(T)} \le \frac{1}{3}\|f\|_{C_b(S)}$.*

Proof. If $f = 0$, let $h = 0$. So we may assume $\|f\|_{C_b(S)} = 1$.
The sets $A = \{x \in S : f(x) \le -\frac{1}{3}\}$ and $B = \{x \in S : f(x) \ge \frac{1}{3}\}$ are disjoint and closed in S, and so in M since S is closed in M.

Hence there is continuous $g : M \to [0,1]$ such that $g(x) = 0$ on A and $g(x) = 1$ on B. Let $h(x) = \frac{2}{3}(g(x) - \frac{1}{2})$. Clearly $\|h\|_{C_b(T)} \le \frac{1}{3}$.

If $x \in A$, $-1 \le f(x) \le -\frac{1}{3}$ and $h(x) = -\frac{1}{3}$, hence $|f(x) - h(x)| \le \frac{2}{3}$.
If $x \in B$, $\frac{1}{3} \le f(x) \le 1$ and $h(x) = \frac{1}{3}$, hence $|f(x) - h(x)| \le \frac{2}{3}$.
If $x \in S \setminus (A \cup B)$, $-\frac{1}{3} \le f(x), h(x) \le \frac{1}{3}$ hence $|f(x) - h(x)| \le \frac{2}{3}$.

Consequently, $\|h\|_{C_b(T)} \le \frac{1}{3}$. □

Proof of Tietze's Theorem for bounded functions

Proposition 5.11. *Suppose that S is a closed subspace of a normal topological space T. Then every function from $\mathcal{C}_b(S)$ can be extended to a function from $\mathcal{C}_b(T)$.*

Proof. Let $f_0 \in \mathcal{C}_b(S)$. By Lemma 5.10 find $h_0 \in \mathcal{C}_b(T)$ such that

$$\|f_0 - h_0\|_{C_b(S)} \le \tfrac{2}{3}\|f_0\|_{C_b(S)} \quad \text{and} \quad \|h_0\|_{C_b(T)} \le \tfrac{1}{3}\|f_0\|_{C_b(S)}.$$

We let $f_1 = f_0 - h_0$ and continue recursively: Whenever f_k has been defined, we find $h_k \in \mathcal{C}_b(T)$ such that

$$\|f_k - h_k\|_{C_b(S)} \le \tfrac{2}{3}\|f_k\|_{C_b(S)} \quad \text{and} \quad \|h_k\|_{C_b(T)} \le \tfrac{1}{3}\|f_k\|_{C_b(S)}$$

and let $f_{k+1} = f_k - h_k = f_0 - \sum_{j=0}^{k} h_j$. It follows that

$$\|f_k\|_{C_b(S)} \le (\tfrac{2}{3})^k\|f_0\|_{C_b(S)} \quad \text{and} \quad \|h_k\|_{C_b(T)} \le \tfrac{1}{3}(\tfrac{2}{3})^k\|f_0\|_{C_b(S)}.$$

By completeness of $\mathcal{C}_b(T)$, $h = \sum_{k=0}^{\infty} h_k$ is a function from $\mathcal{C}_b(T)$.
On S we have $f_0 = \sum_{k=0}^{\infty} h_k = h$ and we are done. □

Proof of Tietze's Theorem

Proposition 5.12. *Suppose that S is a closed subset of a normal topological space T. Then every continuous function on S can be extended to a continuous function to T.*

Proof. Let ψ be a homeomorphism of $\mathbb{R}$ onto $(-1,1)$. If f is a continuous function on S, $\psi \circ f$ is a bounded continuous function on S, hence it can be extended to a continuous function h on T.
We change h so that it has values only in $(-1,1)$:

The sets $A = \{x \in T : |h(x)| \ge 1\}$ and S are disjoint closed subsets of T. Hence by Urysohn's Lemma there is a continuous function $g : M \to [0,1]$ such that $g(x) = 0$ on A and $g(x) = 1$ on S.

Then the product gh still extends f and has values in $(-1,1)$.

It follows that $\psi^{-1} \circ (gh)$ is a continuous extension of f to T. □

Fixed point theorems

Fixed point theorems speak about existence and/or approximation of the solution of equations of the form

$$x = f(x)$$

Many equations can be brought to this form.

Definition. Let $f : S \to S$ (where S is any set). A point $x \in S$ such that $f(x) = x$ is called a fixed point of f.

We will only consider the case when S is a metric space and the Lipschitz constant of f is < 1; such maps are called contractions.

Definition. A map f of a metric space M into itself is a contraction if there is a constant $\kappa < 1$, called contraction ratio, such that $d(f(x), f(y)) \le \kappa d(x,y)$ for every $x, y \in M$.

Fact 5.13. *Contractions are continuous.*

Proof. Take $\delta = \varepsilon$ or recall that Lipschitz maps are continuous. □

Contraction Mapping Theorem

Theorem 5.14 (Banach). *If f is a contraction on a complete metric space M then f has a unique fixed point.*

Proof of uniqueness. If $f(x) = x$ and $f(y) = y$ then $d(x, y) = d(f(x), f(y)) \leq \kappa d(x, y)$. Since $\kappa < 1$, this gives $d(x, y) = 0$, so $x = y$. □

Proof of existence. Choose $x_0 \in M$ and set $x_{n+1} = f(x_n)$. Then

$$d(x_j, x_{j+1}) = d(f(x_{j-1}), f(x_j)) \leq \kappa d(x_{j-1}, x_j) \leq \ldots \leq \kappa^j d(x_0, x_1).$$

Since $\sum_{j=1}^{\infty} \kappa^j d(x_0, x_1) < \infty$, the sequence (x_n) is Cauchy by 5.4. Since M is complete, x_n converge to some $x \in M$, so $f(x_n) \to f(x)$. But $f(x_n) = x_{n+1} \to x$, so $f(x) = x$ as required. □

Notice also various inequalities that may be deduced, for example $d(x, x_{n+1}) \leq \kappa d(x_n, x_{n+1})/(1 - \kappa)$. They are useful numerically.

Numerical approximation

Question. *Solve $x^7 - x^3 - 21x + 5 = 0$, $x \in [0, 1]$ with error $< 10^{-6}$.*

Answer. Rewrite the equation as $x = (x^7 - x^3 + x + 5)/22$. Use contraction mapping theorem with $M = [0, 1]$ and $f(x) = (x^7 - x^3 + x + 5)/22$.
Since clearly $0 \leq f(x) \leq 1$, we have $f : [0, 1] \to [0, 1]$. Since $|f'(x)| = |(7x^6 - 3x^2 + 1)/22| \leq 1/2$, we see from the Mean Value Theorem that f is a contraction with contraction ratio $\kappa = 1/2$.
Starting with $x_0 = 0$ gives:
$x_1 = 0.2272727$, $x_2 = 0.2370711$, $x_3 = 0.2374449$, $x_4 = 0.2374591$, $x_5 = 0.2374596$, $x_6 = 0.2374596, \ldots$
We know that $d(x, x_6) \leq \kappa d(x_5, x_6)/(1 - \kappa)$, where $\kappa = 1/2$ and $d(x_5, x_6) < 10^{-7}$.
Hence $x_6 = 0.2374596$ approximates the solution with error $< 10^{-6}$.

Example of application

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which is Lipschitz in the second variable; this means that there is C such that

$$|f(x, y) - f(x, z)| \leq C|y - z|.$$

We show that for any given $x_0, y_0 \in \mathbb{R}$, the differential equation

$$y' = f(x, y) \qquad y(x_0) = y_0$$

has a unique solution on $(a, b) = (x_0 - \delta, x_0 + \delta)$ where $\delta = 1/2C$.
Rewrite the equation in the form

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt = F(y)(x)$$

Define $F : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ by this. We want $y \in \mathcal{C}[a, b]$ such that $y = F(y)$, ie a fixed point of F!
Is F a contraction?

$$|F(y)(x) - F(z)(x)| = \left|\int_{x_0}^{x} (f(t, y(t)) - f(t, z(t)))\, dt\right|$$
$$\leq \int_{x_0}^{x} |f(t, y(t)) - f(t, z(t))|\, dt$$
$$\leq C \int_{x_0}^{x} |y(t) - z(t)|\, dt \leq \|y - z\|/2 \qquad \text{YES!}$$

Total boundedness

Definition. A metric space M is totally bounded if for every $\varepsilon > 0$ there is a finite set $F \subset M$ such that every point of M has distance $< \varepsilon$ of a point of F.
Explanation. The expression "every point of M has distance $< \varepsilon$ of a point of F" says that $M \subset \bigcup_{x \in F} B(x, \varepsilon)$.

Lemma 5.15. *A subspace M of a metric space N is totally bounded iff for every $\varepsilon > 0$ there is a finite set $H \subset N$ such that $M \subset \overline{\bigcup_{z \in H} B(z, \varepsilon)}$.*

Proof. $(\Rightarrow)$ is obvious, take $H = F$.
$(\Leftarrow)$ Given $\varepsilon > 0$, let $H \subset N$ be a finite set such that $M \subset \overline{\bigcup_{z \in H} B(z, \frac{\varepsilon}{3})} = \bigcup_{z \in H} \overline{B(z, \frac{\varepsilon}{3})}$. From each nonempty $M \cap \overline{B(z, \frac{\varepsilon}{3})}$ choose one point. Let F be the set of these points. So F is a finite subset of M. If $y \in M$ then y is in one of the $\overline{B(z, \frac{\varepsilon}{3})}$. So $M \cap \overline{B(z, \frac{\varepsilon}{3})} \neq \emptyset$ and so there is $x \in F \cap \overline{B(z, \frac{\varepsilon}{3})}$. Hence $y \in B(x, \varepsilon)$ and we see that $M \subset \bigcup_{x \in F} B(x, \varepsilon)$. □

Warning: Metric Spaces Only!

Total boundedness of subspaces, closures etc

Corollary 5.16. *A subspace of a totally bounded metric space is totally bounded.*

Proof. If $M \subset N$ and N is totally bounded, for every $\varepsilon > 0$ there is a finite set $H \subset N$ such that $M \subset N \subset \bigcup_{x \in H} B(x, \varepsilon)$. So the condition of Lemma 5.15 is satisfied. □

Corollary 5.17. *If a subspace M of a metric space N is totally bounded then so is $\overline{M}$.*

Proof. For every $\varepsilon > 0$ we have a finite set $F \subset M$ such that $M \subset \bigcup_{x \in F} B(x, \varepsilon)$. Hence $\overline{M} \subset \overline{\bigcup_{x \in F} B(x, \varepsilon)}$, and we may use Lemma 5.15. □

Remark. A totally bounded space is bounded: There are finitely many points $x_1, \ldots, x_n$ such that every x is in distance < 1 from one of the x_i. Hence $d(x, x_1) < r$ where $r = 1 + \max_i d(x_i, x_1)$. The converse is false: Take $\mathbb{R}$ with $\mathrm{dist}(x, y) = \min(1, |x - y|)$.

Total boundedness and Cauchy sequences.

Theorem 5.18. *A metric space M is totally bounded iff every sequence in M has a Cauchy subsequence.*

Proof. ($\Rightarrow$) Let $x_n \in M$. Let $N_0 = \mathbb{N}$. Suppose inductively that we have defined an infinite set $N_{k-1} \subset \mathbb{N}$. Since M is covered by finitely many balls of radius $1/2k$, there is one such ball B_k such that $N_k = \{n \in N_{k-1} : x_n \in B_k\}$ is infinite.
Let $n(1)$ be the least element of N_1 and choose inductively the least $n(k) \in N_k$ such that $n(k) > n(k-1)$. Then $(x_{n(k)})$ is a subsequence of (x_n) such that $x_{n(i)} \in B_k$ for $i \geq k$. Hence $d(x_{n(i)}, x_{n(j)}) < 1/k$ for $i, j \geq k$, and we see that $(x_{n(k)})$ is Cauchy.
($\Leftarrow$) Suppose M is not totally bounded. So for some $\varepsilon > 0$ there is no finite set with all points of M within ε of it. Choose $x_1 \in M$ and continue by induction: when $x_1, \ldots, x_{k-1}$ have been defined our assumption implies that there is x_k such that $d(x_k, x_i) \geq \varepsilon$ for all $i < k$. So we get an infinite sequence (x_k) such that $d(x_i, x_j) \geq \varepsilon$ for every $i \neq j$. So no subsequence of (x_k) is Cauchy. □

Completeness and compactness

Theorem 5.19. *A subspace C of a complete metric space M is compact iff it is closed and totally bounded.*

Proof. ($\Rightarrow$) C is closed by 3.3 and totally bounded since for each $\varepsilon > 0$ its open cover $B(x, \varepsilon)$, $(x \in C)$ has a finite subcover.
($\Leftarrow$) Every sequence in C has a Cauchy subsequence by 5.18, which converges to a point of M since M is complete. But C is closed, so the limit belongs to C. So C is sequentially compact, hence compact by 3.16. □

Theorem 5.20. *A subspace S of a complete metric space M is totally bounded iff its closure is compact.*

Proof. ($\Rightarrow$) $\overline{S}$ is totally bounded by 5.17 and so compact by 5.19.
($\Leftarrow$) $\overline{S}$ is totally bounded by 5.19 and so is its subset S by 5.16. □

Arzelà-Ascoli Theorem

Definitions. A subset $\mathcal{S}$ of $\mathcal{C}(M)$ is called

- equicontinuous at x if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $f \in \mathcal{S}$ and $y \in B(x, \delta)$.
- equicontinuous if it is equicontinuous at every $x \in M$
- uniformly equicontinuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $f \in \mathcal{S}$ and $d(y, x) < \delta$.

Fact 5.21. *If M is compact, $\mathcal{S} \subset \mathcal{C}(M)$ is equicontinuous iff it is uniformly equicontinuous.*

Theorem 5.22. *Let M be a compact metric space. A subset of $\mathcal{C}(M)$ is totally bounded iff it is bounded and equicontinuous.*

Theorem 5.23. *Let M be a compact metric space. A subset of $\mathcal{C}(M)$ is compact iff it is closed, bounded and equicontinuous.*

We first indicate an application.

Application to differential equations

Theorem 5.24 (Peano). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $(x_0, y_0) \in \mathbb{R}^2$. There is $\delta > 0$ such that the problem*

$$u' = f(x, u) \qquad u(x_0) = y_0$$

has a solution on $(x_0 - \delta, x_0 + \delta)$.

Idea of the proof. The time delayed equations

$$u_k'(x) = f(x, u_k(x - 1/k)) \text{ for } x > x_0, \quad u_k(x) = y_0 \text{ for } x \le x_0$$

are easy to solve. Use the Arzelà-Ascoli Theorem (this will be explained on the next slide) to find a subsequence u_{k_i} uniformly convergent on $[x_0, x_0 + \delta)$ to some u. Since x and $u_{k_i}(x)$ lie in a bounded set on which f is uniformly continuous, $f(t, u_{k_i}(t - 1/k_i)) \to f(t, u(t))$ uniformly on $[x_0, x_0 + \delta)$. Hence we may take the limit as $i \to \infty$ in the equation $u_{k_i}'(x) = f(x, u_{k_i}(x - 1/k_i))$ to get $u' = f(x, u)$ on $[x_0, x_0 + \delta)$. The interval $(x_0 - \delta, x_0]$ may be treated similarly. □

Use of the Arzelà-Ascoli Theorem

Assume $x_0 = y_0 = 0$. Since f is continuous, there is $C \in [1, \infty)$ such that $|f(x, y)| \le C$ whenever $\max(|x|, |y|) \le 1$. Let $\delta = 1/C$. Let u_k be the solutions of the "time delayed equations:"

$$u_k'(x) = f(x, u_k(x - 1/k)) \text{ for } x > 0, \quad u_k(x) = 0 \text{ for } x \le 0$$

These u_k may be defined inductively, first on $[0, 1/k)$, then on $[1/k, 2/k)$ etc. By induction along the construction we prove that $|u_k(x)| \le 1$ for $x \le \delta$:

$$|u_k(x)| \le \int_0^x \big|f(t, u_k(t - 1/k))\big|\, dt \le Cx \le 1$$

So u_k are bounded by 1 on $[0, \delta]$. They are also equicontinuous:

$$|u_k(x_1) - u_k(x_2)| \le \int_{x_1}^{x_2} \big|f(t, u_k(t - 1/k))\big|\, dt \le C|x_1 - x_2|.$$

Hence by Arzelà-Ascoli Theorem, a subsequence of u_k converges to some $u \in \mathcal{C}[0, \delta]$. □

Proof of Theorem 5.22($\Rightarrow$)

Statement. *Let $S \subset \mathcal{C}(M)$ be totally bounded. Then it is bounded and uniformly equicontinuous. So it is also bounded and equicontinuous.*

Proof. A totally bounded subset is bounded by 5.16.

Let $\varepsilon > 0$. Use that S is totally bounded to find $f_1, \dots, f_n \in S$ so that for every f there is i with $\|f - f_i\| < \varepsilon/3$. Since f_i are uniformly continuous, there is $\delta > 0$ such that for each $i = 1, \dots, n$,

$$|f_i(y) - f_i(x)| < \varepsilon/3 \quad \text{whenever} \quad d(x, y) < \delta.$$

For every $f \in S$ choose i with $\|f - f_i\| < \varepsilon/3$. Then for $d(x, y) < \delta$,

$$\begin{aligned} |f(y) - f(x)| &\le |f(y) - f_i(y)| + |f_i(y) - f_i(x)| + |f_i(x) - f(x)| \\ &\le \|f - f_i\| + |f_i(y) - f_i(x)| + \|f - f_i\| < \varepsilon. \end{aligned}$$

Hence S is uniformly equicontinuous.

Of course, uniform equicontinuity implies equicontinuity. □

Proof of Theorem 5.22($\Leftarrow$)

Statement. *Let $S \subset \mathcal{C}(M)$ be bounded and equicontinuous. Then it is totally bounded.*

Proof. Let $\varepsilon > 0$. For every $x \in M$ find $\delta(x) > 0$ such that $|f(y) - f(x)| < \varepsilon/3$ whenever $f \in S$ and $y \in B(x, \delta(x))$. By compactness, there are $x_1, \dots, x_n \in M$ such that $M \subset \bigcup_{i=1}^n B(x_i, \delta(x_i))$.

For any $q_1, \dots, q_n \in \mathbb{Z}$ for which there is a function $g \in S$ with $g(x_i) \in [q_i\varepsilon/3, (q_i + 1)\varepsilon/3]$ choose one such g. Since all $g \in S$ are bounded by the same constant, there are only finitely many such $q_1, \dots, q_n$. Hence the set F of the chosen functions g is finite.

Let $f \in S$. There are $q_i \in \mathbb{Z}$ so that $f(x_i) \in [q_i\varepsilon/3, (q_i + 1)\varepsilon/3]$. So there is a function $g \in F$ with $g(x_i) \in [q_i\varepsilon/3, (q_i + 1)\varepsilon/3]$. For every $x \in M$ we find i so that $x \in B(x_i, \delta(x_i))$. Then

$$|f(x) - g(x)| \le |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)| < \varepsilon.$$

Hence $\|f - g\| < \varepsilon$. So S is totally bounded. □

Cantor's Theorem

Definition. The diameter of a nonempty subset S of a metric space is defined by

$$\operatorname{diam}(S) = \sup_{x,y\in S} d(x,y)$$

Note. S is bounded iff $\operatorname{diam}(S) < \infty$.

Theorem 5.25. *Let F_n be a decreasing sequence of nonempty closed subsets of a complete metric spaces such that $\operatorname{diam}(F_n) \to 0$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

Explanation. "F_n decreasing" means $F_1 \supset F_2 \supset F_3 \supset \cdots$

Proof. Choose $x_n \in F_n$. Then for $i \geq n$, $x_i \in F_i \subset F_n$. Hence for $i, j \geq n$, $x_i, x_j \in F_n$, giving $d(x_i, x_j) \leq \operatorname{diam}(F_n)$. Since $\operatorname{diam}(F_n) \to 0$, the sequence (x_n) is Cauchy. By completeness of M, x_n converge to some x.

For each n we use once more that $x_i \in F_n$ for $i \geq n$: Since F_n is closed, it implies that $x \in F_n$. So $x \in \bigcap_{n=1}^{\infty} F_n$ and we are done. □

Warning: Metric Spaces Only!

Baire's (Category) Theorem

Theorem 5.26. *A nonempty complete metric space is not meagre in itself. Moreover, a set S meagre in M has a dense complement.*

Proof. Let $S \subset M$ be meagre. Write $S = \bigcup_{k=1}^{\infty} S_k$ where S_k are nowhere dense in M. Let $G_k = M \setminus \overline{S_k}$. Since S_k are nowhere dense, G_k are dense in M. Since $\overline{S_k}$ are closed, G_k are open.

Let U be a nonempty open set. Then $U \cap G_1 \neq \emptyset$. Choose $x_1 \in U \cap G_1$ and find $\delta_1 > 0$ so that $B(x_1, \delta_1) \subset U \cap G_1$.
We continue inductively: when x_{k-1} and δ_{k-1} have been defined, we use that G_k is dense to find $x_k \in G_k \cap B(x_{k-1}, \delta_{k-1}/2)$ and then use that G_k is open to find $0 < \delta_k < \delta_{k-1}/2$ so that $B(x_k, \delta_k) \subset G_k$.

Then $\delta_k \to 0$ and $\overline{B(x_k, \delta_k)} \subset B(x_{k-1}, \delta_{k-1})$. By Cantor's Theorem,

$$\emptyset \neq \bigcap_{k=1}^{\infty} \overline{B(x_{k+1}, \delta_{k+1})} \subset \bigcap_{k=1}^{\infty} B(x_k, \delta_k) \subset U \cap \bigcap_{k=1}^{\infty} G_k$$
$$= U \setminus \bigcup_{k=1}^{\infty} \overline{S_k} \subset U \setminus \bigcup_{k=1}^{\infty} S_k = U \setminus S = U \cap (M \setminus S).$$

Hence $M \setminus S$ is dense in M. □

Warning: Metric Spaces Only!

Exercises on Baire's Theorem

Question. *Show that the Cantor set C is uncountable.*

Answer. We observe that for every $x \in C$ there are points $y \in C$, $y \neq x$ arbitrarily close to x. In other words, $C \setminus \{x\}$ is dense in C, which, since $\{x\}$ is closed, shows that $\{x\}$ is nowhere dense.
If C were countable, we would have $C = \bigcup_{j=1}^{\infty}\{x_j\}$, showing that C is meagre in itself and contradicting the Baire Theorem.

Question. *Let $f : [1,\infty) \to \mathbb{R}$ be a continuous function such that for some $a \in R$ there are arbitrarily large x with $f(x) < a$. Show that for each $k \in \mathbb{N}$, the set $S = \bigcap_{n=k}^{\infty}\{x \in [1,\infty) : f(nx) \geq a\}$ is nowhere dense.*

Answer. Since f is continuous, S is closed. Let $1 \leq \alpha < \beta < \infty$. We have to show $(\alpha, \beta) \setminus S \neq \emptyset$. For large n, $(n+1)/n < \beta/\alpha$, so $(n+1)\alpha < n\beta$. It follows that the set $\bigcup_{n=k}^{\infty}(n\alpha, n\beta)$ contains some interval (r, ∞) and so a point y such that $f(y) < a$. Find n such that $y \in (n\alpha, n\beta)$. Then $x = y/n \in (\alpha, \beta)$ and $f(nx) < a$, so $x \notin S$.

Exercises on Baire's Theorem

Question. *Let $f : [1,\infty) \to \mathbb{R}$ be a continuous function such that for every $x \geq 1$ the limit $\lim_{n\to\infty} f(nx)$ exists. Show that $\lim_{x\to\infty} f(x)$ exists.*

Answer. If $\lim_{x\to\infty} f(x)$ does not exist there are $a < b$ such that there are arbitrarily large x with $f(x) < a$ and arbitrarily large y with $f(y) > b$. By the previous question, the set

$$\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\{x \in [1,\infty) : f(nx) \geq a\} \cup \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\{x \in [1,\infty) : f(nx) \leq b\}$$

is meagre. By Baire's Theorem there is x not belonging to this set. The fact that x does not belong to the first union means that for every k there is $n \geq k$ such that $f(nx) < a$. In other words, there are arbitrarily large n such that $f(nx) < a$. Similarly we see that there are arbitrarily large n such that $f(nx) > b$. Hence the sequence $f(nx)$ does not converge.

Application: nowhere differentiable functions

Theorem 5.27. *There is a continuous $f : [0,1] \to \mathbb{R}$ that is non-differentiable at any point of* $[0,1]$.

Brief sketch of proof. We will work in the complete space $\mathcal{C}[0,1]$ equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Define

$$S_n = \{f \in \mathcal{C}[0,1] : (\exists x \in [0,1])(\forall y \in [0,1])\, |f(y) - f(x)| \le n|y-x|\}$$

and show

(a) S_n is closed,

(b) the complement of S_n is dense,

(c) if $f'(x)$ exists for some x, then $f \in S_n$ for some n.

The first two points imply that S_n are nowhere dense. Hence by Baire Category Theorem,

$$C[0,1] \setminus \bigcup_{n=1}^{\infty} S_n \ne \emptyset$$

Any function belonging to this set is nowhere differentiable thanks to (c). □

S_n is closed

$$S_n = \{f \in \mathcal{C}[0,1] : (\exists x \in [0,1])(\forall y \in [0,1])\, |f(y) - f(x)| \le n|y-x|\}$$

Proof. Let $f_k \in S_n$, $f_k \to f$. Find $x_k \in [0,1]$ so that

$$(\forall y \in [0,1])\, |f_k(y) - f_k(x_k)| \le n|y - x_k|.$$

Since x_k has a convergent subsequence, we may assume that $x_k \to x$. Clearly, $x \in [0,1]$.

For any $y \in [0,1]$ we get

$$\begin{aligned} |f(y) - f(x)| &\le |f(y) - f(x_k)| + |f(x_k) - f(x)| \\ &\le |f_k(y) - f_k(x_k)| + 2\|f - f_k\| + |f(x_k) - f(x)| \\ &\le n|y - x_k| + 2\|f - f_k\| + |f(x_k) - f(x)| \\ &\quad \to n|y - x|. \end{aligned}$$

Hence $|f(y) - f(x)| \le n|y - x|$ and so $f \in S_n$. □

Complement of S_n is dense

$$S_n = \{f \in \mathcal{C}[0,1] : (\exists x \in [0,1])(\forall y \in [0,1])\, |f(y) - f(x)| \le n|y-x|\}$$

Proof. Let $n \in \mathbb{N}$, $g \in \mathcal{C}[0,1]$ and $0 < \varepsilon < 1$. We have to show that

$$B(g, \varepsilon) \setminus S_n \ne \emptyset.$$

Use that g is uniformly continuous to find $\delta > 0$ such that $|g(x) - g(y)| < \frac{1}{4}\varepsilon$ for $|x - y| \le \delta$.

Let $k \in \mathbb{N}$, $k > \max(1/\delta, 4n/\varepsilon)$. Consider

$$f(x) = g(x) + \tfrac{1}{2}\varepsilon \sin(2k\pi x)$$

Given any $x \in [0,1]$ there is $y \in [0,1]$ such that $|y - x| \le 1/k$ and $|\sin(2k\pi y) - \sin(2k\pi x)| = 1$. Then

$$\begin{aligned} |f(y) - f(x)| &\ge \tfrac{1}{2}\varepsilon - |g(y) - g(x)| \ge \tfrac{1}{4}\varepsilon \\ &> n/k \ge n|y - x|. \end{aligned}$$

Hence $f \in B(g, \varepsilon) \setminus S_n$, as required. □

Proof of: $f'(x)$ exists $\Rightarrow f \in S_n$ for some n

$$S_n = \{f \in \mathcal{C}[0,1] : (\exists x \in [0,1])(\forall y \in [0,1])\, |f(y) - f(x)| \le n|y-x|\}$$

Proof. If $f'(x)$ exists, find $\delta > 0$ so that $\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < 1$ whenever $0 < |y - x| < \delta$.

Let $n \in \mathbb{N}$, $n \ge \max(1 + |f'(x)|, \frac{2\|f\|}{\delta})$. We show that for every $y \in [0,1]$,

$$|f(y) - f(x)| \le n|y - x|.$$

This is obvious if $y = x$. If $|y - x| < \delta$, we have

$$\begin{aligned} |f(y) - f(x)| &\le |y - x|\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| + |y - x||f'(x)| \\ &\le (1 + |f'(x)|)|y - x| \le n|y - x|. \end{aligned}$$

If $|y - x| \ge \delta$, then

$$|f(y) - f(x)| \le 2\|f\| = \frac{2\|f\|}{\delta}\delta \le \frac{2\|f\|}{\delta}|y - x| \le n|y - x|.$$

Hence $f \in S_n$. □

Topological completeness

Notice that validity of Baire's Theorem is a topological property while completeness is not. In particular, Baire's Theorem holds in a metric space provided that there is a topologically equivalent metric in which it is complete.

Spaces for which there is a topologically equivalent metric in which they are complete a are called *topologically complete*.
We will answer the question which spaces have this property. We give our answer in two independent steps. First we consider the case that our space is a subspace of a complete metric space and show

Theorem 5.28. *A subspace of a complete metric space is topologically complete iff it is G_δ.*

Later we complete this description by showing that every metric space is a subspace of a complete space.

Topologically complete subspaces are G_δ

Proof of ($\Rightarrow$). Assume that S is a subspace of a complete metric space (M, d) and d_0 is a topologically equivalent complete metric on S. Since $\overline{S}$ is a G_δ subset of M, we may replace M by $\overline{S}$ and so assume that S is dense in M.

For every $x \in S$ and $n \in \mathbb{N}$, $B_{d_0}(x, 2^{-n})$ is an open subset of S. So there is an open subset $U_{x,n}$ of M such that $U_{x,n} \cap S = B_{d_0}(x, 2^{-n})$.

Let $G_n = \bigcup_{x \in S} U_{x,n}$. Clearly $S \subset \bigcap_{n=1}^{\infty} G_n$.

Let $x \in \bigcap_{n=1}^{\infty} G_n$. For every n find $x_n \in S$ such that $x \in U_{x_n,n}$. Hence for each n, m, $U_{x_n,n} \cap U_{x_m,m}$ is a nonempty open set. Since S is dense in M, there is $x_{n,m} \in S \cap U_{x_n,n} \cap U_{x_m,m}$. Hence

$$d_0(x_n, x_m) \le d_0(x_n, x_{n,m}) + d_0(x_{n,m}, x_m) \le 2^{-n} + 2^{-m}.$$

It follows that x_n is a d_0 Cauchy sequence , so x_n converges to some point $y \in S$. Hence in the space M, x_n converges to x as well as to y, implying that $x = y \in S$. □

G_δ subspaces are topologically complete

Proof of ($\Leftarrow$). Assume (M, d) is complete and $S = \bigcap_{n=1}^{\infty} G_n$ where $G_n \subset M$ are open. Define $g_n : S \to \mathbb{R}$ by $g_n(x) = 1/d(x, M \setminus G_n)$ and let

$$d_0(x, y) = d(x, y) + \sum_{n=1}^{\infty} \min(2^{-n}, |g_n(x) - g_n(y)|).$$

If $x_k \to_d x$ then for each m,

$$d_0(x_k, x) \le d(x_k, x) + \sum_{n=1}^{m} |g_n(x_k) - g_n(x)| + \sum_{n=m+1}^{\infty} 2^{-n} \to_{k\to\infty} 2^{-m}.$$

Hence $x_k \to_{d_0} x$. Since clearly $x_k \to_{d_0} x$ implies $x_k \to_d x$, we see that d_0 is topologically equivalent to d.

Let now x_k be a d_0 Cauchy sequence. Then x_k is d-Cauchy. Hence there is $x \in M$ such that $x_k \to_d x$. We have to show that $x \in S$.

Assuming this is not the case, fix n such that $x \notin G_n$. Then $d(x_k, M \setminus G_n) \le d(x_k, x) \to 0$, hence $g_n(x_k) \to \infty$. Then for every k there is $j > k$ with $|g_n(x_j) - g_n(x_k)| > 2^{-n}$, so $d_0(x_j, x_k) \ge 2^{-n}$. Hence the sequence x_k is not Cauchy, a contradiction. □

Completion – discussion

In the proof that the $\mathcal{C}[0,1]$ is incomplete in the L^1-norm, we found a sequence f_n that should have converged to $1/\sqrt{x}$, but this function was 'missing' from our space. It is possible to add those 'missing functions' to $\mathcal{C}[0,1]$ by building the powerful theory of the Lebesgue integral; Lebesgue integrable functions form a complete space when normed by $\|f\|_1 = \int_0^1 |f(x)|\,dx$.
We will consider the problem of making a space complete abstractly: Given an (incomplete) metric space, can we add to it points so that it becomes complete?
For example, to $(-1, 1)$ we may add $-1, 1$ to make it complete, but we may also add all $x \in \mathbb{R}$ with $|x| \ge 1$. Observing that in the first case $(-1, 1)$ is dense in the bigger space while in the latter it is not, we will also require that the original space be dense in its completion.
Recall that a set $S \subset M$ is dense in M if $\overline{S} = M$.

Definitions of completion

Definition. A completion of a metric space M is a complete metric space N such that M is a dense subspace of N.

Definition (Modern). A completion of M is a complete metric space N together with an isometry i of M onto a subset of N such that $i(M)$ is dense in N.

Example. In the old-fashioned definition, $\mathbb{R}$ is a completion of $\mathbb{Q}$ because it is complete and $\mathbb{Q}$ is its dense subset. The same fact is said in the modern definition that $\mathbb{R}$ together with the identity map from $\mathbb{Q}$ to $\mathbb{R}$ is a completion of $\mathbb{Q}$. However notice that $\mathbb{R}$ together with the map $i(x) = -x$ from $\mathbb{Q}$ to $\mathbb{R}$ is also a completion of $\mathbb{Q}$.

Remark. It can be proved that completions are unique in the following sense: if N, i and N', i' are two completions of M, there is an isometry $j : N \to N'$ so that $j(i(x)) = i'(x)$. (We will not prove it in this lecture.)

Existence of completion

Theorem 5.29. *Any metric space can be isometrically embedded into a complete metric space.*

Proof. We find an isometry of M onto a subset of $\mathcal{B}(M)$, which is complete by 5.6. Fix a point $a \in M$ and define $F : M \to \mathcal{B}(M)$ by

$$F(x)(z) = d(z, x) - d(z, a).$$

The inequality $|d(z, x) - d(z, a)| \le d(x, a)$ says that $F \in \mathcal{B}(M)$. We have $|F(x)(z) - F(y)(z)| = |d(z, x) - d(z, y)| \le d(x, y)$ and the inequality becomes equality when $z = y$ (or $z = x$). So $\|F(x) - F(y)\| = d(x, y)$ and so F is an isometry (onto some subset of $\mathcal{B}(M)$). □

Corollary 5.30. *Any metric space has a completion.*

Proof. Embed M into a complete metric space N. Then $\overline{M}$ (the closure taken in N) is complete by 5.2 and M is dense in $\overline{M}$. So $\overline{M}$ is a completion of M. □

Compactness and the Cantor Set

Recall the Cantor set. Notice that the remaining intervals give rise to pieces of the Cantor set that are both open and closed.

Theorem 5.31. *Every compact metric space M is a continuous image of the Cantor set C.*

Sketch of Proof. Let $A_k \subset M$ be finite sets such that every point of M is within 2^{-k} of some point of A_k. By induction one constructs a sequence $f_k : C \to M$ of continuous functions such that $f_k(C) = A_k$ and $d(f_k(x), f_{k+1}(x)) \le 2^k$ for $x \in C$. Then f_k form a Cauchy sequence in the space of continuous functions $C \to M$, so they converge to a continuous $f : C \to M$. Moreover, $f(C)$ is dense in M. It is also compact, hence closed, hence $f(C) = M$. □

Corollary 5.32. *There is a continuous surjective map $f : [0, 1] \to [0, 1]^2$. (Such maps are called Peano curves.)*

Proof. Extend a surjective continuous map $f : C \to [0, 1]^2$ linearly to each interval removed during the construction of C. □

Main points of Chapter 5

- Cauchy Sequences. Complete Metric Spaces.
- Complete subspaces and closed subsets.
- Completeness of $\mathbb{R}^n$, spaces of bounded and continuous functions.
- Incompleteness of $\mathcal{C}[0, 1]$ with the L^1 norm.
- Proof of Tietze's Theorem.
- Contraction Mapping Theorem and applications.
- Total boundedness, Cauchy sequences, completeness and compactness.
- Arzelà-Ascoli Theorem.
- Cantor's Theorem. Baire's (Category) Theorem. Applications.
- Topological completeness.
- Completion: definition and existence.
- Compact metric spaces as images of the Cantor Set.