MA2AA1 (ODE's): Lecture Notes

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Contents

0 Introduction

0.1 Practial Arrangement

- The lectures for this module will take place **Wednesday 9-11, Thursday 10-11** in Clore.
- Each week I will hand out a sheet with problems. It is very important you go through these thoroughly, as these will give the required training for the exam and class tests.
- Support classes: **Thursday 11-12, from January 22**.
- The **support classes** will be run rather differently from previous years. The objective is to make sure that you will get a lot out of these support classes.
- The main way to revise for the tests and the exam is by doing the exercises.
- There will be two **class tests**. These will take place on Tuesday 9th February and Tuesday 9th March. Each of these count for 5% .
- **Questions are most welcome**, during or after lectures and during office hour.
- My office hour is to be agreed with students reps. Office hour will in my office 6M36 Huxley Building.

0.2 Relevant material

- There are many books which can be used in conjunction to the module, but **none are required**.
- The **lecture notes** displayed during the lectures will be posted on my webpage: `http://www2.imperial.ac.uk/~svanstri/` Click on Teaching in the left column. The notes will be updated during the term.
- The lectures will also be recorded. See my webpage.
- There is no need to consult any book. However, recommended books are
  - Simmons + Krantz, *Differential Equations: Theory, Technique, and Practice*, about 40 pounds. This book covers a significant amount of the material we cover. Some students will love this text, others will find it a bit longwinded.
  - Agarwal + O'Regan, *An introduction to ordinary differential equations*.
  - Teschl, *Ordinary Differential Equations and Dynamical Systems*. These notes can be downloaded for free *from the authors webpage*.
  - Hirsch + Smale (or in more recent editions): Hirsch + Smale + Devaney, *Differential equations, dynamical systems, and an introduction to chaos*.
  - Arnold, *Ordinary differential equations*. This book is an absolute jewel and written by one of the masters of the subject. It is a bit more advanced than this course, but if you consider doing a PhD, then get this one. You will enjoy it.

  Quite a few additional exercises and lecture notes can be freely downloaded from the internet.

0.3 Notation and aim of this course

Notation: when we write $\dot{x}$ then we ALWAYS mean $\frac{dx}{dt}$. When we write y' then this usually means $\frac{dy}{dx}$ but also sometimes $\frac{dy}{dt}$; which one should always be clear from the context.

This course is about studying differential equations of the type

$$\dot{x} = f(x), \text{ resp. } \dot{y} = g(t, y)$$

which is short for finding a function $t \mapsto x(t)$ (resp. $t \mapsto y(t)$) so that

$$\frac{dx}{dt} = f(x(t)) \text{ resp. } \frac{dy}{dt} = g(y, y(t)).$$

In particular this means that (in this course) we will assume that $\frac{dx}{dt}$ is continuous and therefore $t \mapsto x(t)$ differentiable.

Aim of this course is to find out when or whether such an equation has a solution and determine its properties.

0.4 Examples of differential equations

- An example of a differential equation is the law of Newton: $m\ddot{x}(t) = F(x(t)) \quad \forall t$. Here F is the gravitational force. Using the gravitational force in the vicinity of the earth, we approximate this by

$$m\ddot{x}_1 = 0, m\ddot{x}_2 = 0, m\ddot{x}_3 = -g.$$

  This has solution

$$x(t) = x(0) + v(0)t - \frac{g}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t^2.$$

- According to Newton's law, the gravitational pull between two particles of mass m and M is $F(x) = \gamma m M x/|x|^3$. This gives

$$m\ddot{x}_i = -\frac{\gamma m M x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \text{ for } i = 1, 2, 3$$

  Now it is no longer possible to explicitly solve this equation. One needs some theory be sure that there are solutions and that they are unique.

- In **ODE's** *the independent variable is one-dimensional.* In a *Partial Differential Equation* **(PDE)** such as

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

  the unknown function u is differentiated w.r.t. several variables.

- The typical form for the ODE is the following **initial value problem**:

$$\frac{dx}{dt} = f(t, x) \text{ and } x(0) = x_0$$

  where $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The aim is to find some curve $t \mapsto x(t) \in \mathbb{R}^n$ so that the initial value problem holds. When does this have solutions? Are these solutions unique?

- An example of an ODE related to vibrations of bridges (or springs) is the following (see Appendix C, Subsection C.7):

$$Mx'' + cx' + kx = F_0 \cos(\omega t).$$

  One reason you should want to learn about ODE's is:

  - `http://www.ketchum.org/bridgecollapse.html`
  - `http://www.youtube.com/watch?v=3mclp9QmCGs`
  - `http://www.youtube.com/watch?v=gQK21572oSU`

0.5 Issues which will be addressed in the course include:

- do solutions of ODE's exist?
- are they unique?
- most differential equations, cannot be solved explicitly. One aim of this course is to develop methods which allow information on the behaviour of solutions anyway.

1 Existence and Uniqueness: Picard Theorem

In this chapter we will prove a theorem which gives sufficient conditions for a differential equation to have solutions. Before stating this theorem, we will cover the background needed for the proof of this theorem.

In this chapter X will denote a space of functions (so infinitely dimensional).

1.1 Banach spaces

- A **vector space** X is a space so that if $v_1, v_2 \in X$ then $c_1 v_1 + c_2 v_2 \in X$ for each $c_1, c_2 \in \mathbb{R}$ (or, more usually, for each $c_1, c_2 \in \mathbb{C}$).

- A **norm** on X is a map $|| \cdot ||\colon X \to [0, \infty)$ so that

  1. $||0|| = 0, ||x|| > 0 \; \forall x \in X \setminus \{0\}$.
  2. $||cx|| = |c|||x|| \; \forall c \in \mathbb{R}$ and $x \in X$
  3. $||x+y|| \leq ||x||+||y|| \; \forall x, y \in X$ (triangle inequality).

- A **Cauchy sequence** in a vector space with a norm is a sequence $(x_n)_{n \geq 0} \in X$ so that for each $\epsilon > 0$ there exists N so that $||x_n - x_m|| \leq \epsilon$ whenever $n, m \geq N$.

- A vector space with a norm is **complete** if each Cauchy sequence $(x_n)_{n \geq 0}$ converges, i.e. there exists $x \in X$ so that $||x_n - x|| \to 0$ as $n \to \infty$.

- X is a **Banach space** if it is a vector space with a norm which is complete.

1.2 Metric spaces

- A **metric space** X is a space with together with a function $d\colon X \times X \to \mathbb{R}^+$ (called metric) so that

  1. $d(x,x) = 0$ and $d(x,y) = 0$ implies $x = y$.
  2. $d(x,y) = d(y,x)$
  3. $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

- A sequence $(x_n)_{n\geq 0} \in X$ is called **Cauchy** if for each $\epsilon > 0$ there exists N so that $d(x_n, x_m) \leq \epsilon$ whenever $n, m \geq N$.

- The metric space is **complete** if each Cauchy sequence $(x_n)_{n\geq 0}$ converges, i.e. there exists $x \in X$ so that $d(x_n, x) \to 0$ as $n \to \infty$.

1.3 Metric space versus Banach space

- Given a norm $||\cdot||$ on a vector space X one can also define the metric $d(x,y) = ||y - x||$ on X. So a Banach space is automatically a metric space. A metric space is **not** necessarily a Banach space.

1.4 Examples

Example 1. Consider $\mathbb{R}$ with the norm $|x|$. You have see in Analysis I that this space is complete.

In the next two examples we will consider $\mathbb{R}^n$ with two different norms. As is usual in year ≥ 2, we write $x \in \mathbb{R}^n$ rather than $\underline{x}$ for a vector.

Example 2. Consider the space $\mathbb{R}^n$ and define $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ where x is the vector $(x_1, \ldots, x_n)$. It is easy to check that $|x|$ is a norm (the main point to check is the triangle inequality). This norm is usually referred to as the Euclidean norm (as $d(x,y) = |x - y|$ is the Euclidean distance).

Typo corrected

Example 3. Consider the space $\mathbb{R}^n$ and the supremum norm $|x| = \max_{i=1}^n |x_i|$ (it is easy to check that this is a norm).

Regardless which of two two norms we put on $\mathbb{R}^n$, in both cases the space we obtain is complete (this follows from Example 1).

Without saying this explicitly everywhere, in this course, we will always endow $\mathbb{R}^n$ with the Euclidean metric. In other lectures, you will also come across other norms on $\mathbb{R}^n$ (for example the l^p norm $(\sum_{i=1}^n |x_i|^p)^{1/p}$, $p \geq 1$.

Example 4. One can define several norms on the space of $n \times n$ matrices. One, which is often used, is the **matrix norm** $||A|| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \dfrac{|Ax|}{|x|}$ when A is a $n \times n$ matrix. Here x, Ax are vectors and $|Ax|, |x|$ are the Euclidean norms of these vectors. By linearity of A we have $\sup_{x \in \mathbb{R}^n \setminus \{0\}} \dfrac{|Ax|}{|x|} = \sup_{x \in \mathbb{R}^n, |x|=1} |Ax|$ and so the latter also defines $||A||$. In particular $||A||$ is a finite real number.

Now we will consider a compact interval I and the vector space $C(I, \mathbb{R})$ of continuous functions from I to $\mathbb{R}$. In the next two examples we will put two different norms on $C(I, \mathbb{R})$. In one case, the resulting vector is complete and in the other it is not.

Example 5. The set $C(I, \mathbb{R})$ endowed with the **supremum norm** $||x||_\infty = \sup_{t \in I} |x(t)|$, is a Banach space. That $|| \cdot ||_\infty$ is a norm is easy to check, but the proof that $||x||_\infty$ is complete is more complicated and will not proved in this course (this result is shown in the metric spaces course).

Remark: in this course it will suffice that you know that $C(I, \mathbb{R}^n)$ with the supremum norm is complete - it is not necessary to know the proof of this fact.

Example 6. The space $C([0,1], \mathbb{R})$ endowed with the L^1 norm $||x||_1 = \int_0^1 |x(s)|\, ds$ is **not** complete.

(Hint: To prove this norm is not complete, use the sequence of functions $x_n(s) = \min(\sqrt{n}, 1/\sqrt{s})$ for $s > 0$ and $x_n(0) = \sqrt{n}$. That this sequence is Cauchy is easy to see: for $m > n$ then $\int_0^1 |x_n(s) - x_m(s)|\, ds = \int_0^{1/m} |\sqrt{m} - \sqrt{n}|\, ds + \int_{1/m}^{1/n} |1/\sqrt{s} - \sqrt{n}|\, ds \le 1/\sqrt{m} + 2/\sqrt{n} \le 3/\sqrt{n} \to 0$. Assume by contradiction that the sequence x_n converges: then there exists a continuous function $x \in C([0,1], \mathbb{R})$ so that $||x - x_n||_1$ converges to zero. Since x is continuous, there exists k so that $|x(s)| \le \sqrt{k}$ for all s. Then it is easy to show that $||x_n - x|| \ge 1/(2\sqrt{k}) > 0$ when n is large (check this!). So the Cauchy sequence x_n does not converge.

Typo, $n > m$ corrected into $m > n$.

Indeed, for $n \ge k$ and $s \in [0, 1/k)$, we have $x_n(s) - x(s) \ge x_n(s) - \sqrt{k} > 0$. Hence $||x_n - x|| \ge \int_0^{1/k} |x_n(s) - x(s)| ds \ge \int_0^{1/k} x_n(s) - (1/k)\sqrt{k} \ge (1/n)\sqrt{n} + (2/\sqrt{k} - 2/\sqrt{n}) - (1/k)/\sqrt{k} \ge 1/(2\sqrt{k})$ when n is large.

Remark: The previous two examples show that the same set can be complete w.r.t. one metric and incomplete w.r.t. to another metric.

1.5 Banach Fixed Point Theorem

Theorem 1 (Banach Fixed Point Theorem)**.** Let X be a complete metric space and consider $F\colon X \to X$ so that there exists $\lambda \in (0,1)$ so that

$$d(F(x), F(y)) \leq \lambda d(x,y) \text{ for all } x, y \in X$$

Then F has a unique fixed point p:

$$F(p) = p.$$

Proof. (**Existence**) Take $x_0 \in X$ and define $(x_n)_{n\geq 0}$ by $x_{n+1} = F(x_n)$. This is a Cauchy sequence:

$$d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1})) \leq \lambda d(x_n, x_{n-1}).$$

Hence for each $n \geq 0$, $d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$. Therefore when $n \geq m$, $d(x_n, x_m) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \leq (\lambda^{n-1} + \cdots + \lambda^m) d(x_1, x_0) \leq \lambda^m/(1-\lambda) d(x_1, x_0)$. So $(x_n)_{n\geq 0}$ is a Cauchy sequence and has a limit p. As $x_n \to p$ one has $F(p) = p$.

(**Uniqueness**) If $F(p) = p$ and $F(q) = q$ then $d(p,q) = d(F(p), F(q)) \leq \lambda d(p,q)$. Since $\lambda \in (0,1)$, $p = q$. □

Remark: Since a Banach space is also a complete metric space, the previous theorem also holds for a Banach space.

Example 7. Let $g\colon [0,\infty) \to [0,\infty)$ be defined by $g(x) = (1/2)e^{-x}$. Then $g'(x) = (1/2)e^{-x} \leq 1/2$ for all $x \geq 0$ and so there exists a unique $p \in \mathbb{R}$ so that $g(p) = p$. (By the Mean Value Theorem $\dfrac{g(x) - g(y)}{x - y} = g'(\zeta)$ for some ζ between x, y. Since $|g'(\zeta)| \leq 1/2$ for each $\zeta \in [0,\infty)$ this implies that g is a contraction. Also note that $g(p) = p$ means that the graph of g intersects the line $y = x$ at (p,p).)

The proof of this theorem shows that whatever x_0 you choose the sequence x_n defined by $x_{n+1} = F(x_n)$ converges to a fixed point p (and this fixed point does **not** depend on the starting point x_0.

Here we use $F(x_n) = x_{n+1} \to p$ (since $x_n \to p$) and also $F(x_n) \to F(p)$ (since $d(F(x_n), F(p)) \leq \lambda d(x_n, p) \to 0$. Since a convergent sequence has only one limit, it follows that $F(p) = p$.

1.6 Lipschitz functions

Let X be a Banach space. Then we say that a function $f\colon X \to X$ is **Lipschitz** if there exists $K > 0$ so that

$$||f(x) - f(y)|| < K||x - y||.$$

Example 8. Let A be a $n \times n$ matrix. Then $\mathbb{R}^n \ni x \mapsto Ax \in \mathbb{R}^n$ is Lipschitz. Indeed, $|Ax - Ay| \leq K|x - y|$ where K is the **matrix norm** of A defined by $||A|| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \dfrac{|Ax|}{|x|}$. Remember that $||A||$ is also equal to $\max_{x\in\mathbb{R}^n;|x|=1} |Ax|$.

Example 9. The function $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ is not Lipschitz: there exists no constant K so that $|x^2 - y^2| \leq K|x - y|$ for all $x, y \in \mathbb{R}$.

Example 10. On the other hand, the function $[0, 1] \ni x \mapsto x^2 \in [0, 1]$ is Lipschitz.

Example 11. The function $[0, 1] \ni x \mapsto \sqrt{x} \in [0, 1]$ is not Lipschitz.

Example 12. Let U be an open set in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be continuously differentiable. Then $f\colon C \to \mathbb{R}$ is Lipschitz for any compact set $C \subset U$. When $n = 1$ this follows from the Mean Value Theorem, and for $n > 1$ this will be proved in Appendix A.

1.7 The Picard Theorem for ODE's (for functions which are globally Lipschitz)

Theorem 2. Picard Theorem (global version). Consider $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which satisfies the Lipschitz inequality $|f(s,u) - f(s,v)| \leq K|u - v|$ for all $s \in \mathbb{R}$, $u, v \in \mathbb{R}^n$. Let $h = \frac{1}{2K}$.

Then there exists a unique $x\colon (-h, h) \to \mathbb{R}^n$ satisfying the initial value problem

$$\frac{dx}{dt} = f(t, x) \text{ and } x(0) = x_0. \tag{1}$$

Proof. By integration it follows that (1) is equivalent to

$$x(t) - x(0) \overset{\bullet}{=} \int_0^t f(s, x(s))\, ds. \tag{2}$$

It follows that the initial value problem is equivalent to finding a fixed point of the operator $P\colon B \to B$ defined by

$$P(x)(t) := x_0 + \int_0^t f(s, x(s))\, ds$$

on the Banach space $B := C([-h, h], \mathbb{R}^n)$ with norm $||x|| = \max_{t \in [-h,h]} |x(t)|$.

Note that P assigns to function $x \in B$ another function which we denote by Px. To define the function $P(x)$, we need to evaluate its vector value at some $t \in [-h, h]$. This is what $P(x)(t)$ means. So a solution of $P(x) = x$ is equivalent to finding a solution of (2) and therefore of (1).

Let us show that

$$P(x)(t) := x_0 + \int_0^t f(s, x(s))\, ds$$

is a contraction. Take $x, y \in [-h, h] \to \mathbb{R}^n$. Then for all $t \in [-h, h]$ one has

$$|P(x)(t) - P(y)(t)| = |\int_0^t (f(s, x(s)) - f(s, y(s)))\, ds| \overset{*}{\leq}$$

$$\int_0^t |f(s, x(s)) - f(s, y(s))|\, ds \overset{**}{\leq} K \int_0^t |x(s) - y(s)|\, ds$$

$$\overset{***}{\leq} (hK)||x - y|| \leq (1/2)||x - y||.$$

So

$$||P(x) - P(y)|| = \sup_{t \in [-h,h]} |P(x)(t) - P(y)(t)|$$

$$\leq (1/2)||x - y||$$

and so P is a contraction on the Banach space B. By the previous theorem therefore P has a unique fixed point. $\square$

In inequality (*) we use that $| \int_0^t u(s)\, ds| \leq \int_0^t |u(s)|\, ds$ for any function $u \in B$

Inequality (**) follows from Lipschitz assumption.

Inequality (***) holds because $|x(s) - y(s)| \leq ||x - y||$ (because $||x - y|| = \sup_{s \in [-h,h]} |x(s) - y(s)|$). So $\int_0^t |x(s) - y(s)|\, ds \leq t \cdot || - y||$, and using $|t| \leq h$ inequality (***) follows.

1.8 Application to linear differential equations

Consider

$$x' = Ax \text{ with } x(0) = x_0 \tag{3}$$

where A is a $n \times n$ matrix and $x \in \mathbb{R}^n$. (When we say $x \in \mathbb{R}^n$ we mean here that $x(t) \in \mathbb{R}^n$.)

- Note that $|Ax - Ay| \leq K|x - y|$ where K is the **matrix norm** of A defined by $||A|| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \dfrac{|Ax|}{|x|}$. So the Picard Theorem implies that the initial value problem (3) has a unique solution $t \mapsto x(t)$ for $|t| < h$. It is important to remark that the Picard theorem states that there exists $h > 0$ (namely $h = 1/(2K)$) so that there exists a solution $x(t)$ for $|t| < h$. So **at this point** we cannot yet guarantee that there exists a solution all $t \in \mathbb{R}$.

- For each choice of $x_0 \in \mathbb{R}^n$ there exists a unique solution $x(t)$ (for $|t|$ small). For each $i = 1, \ldots, n$, let $u_i(t)$ be the (unique) solution so that $u_i(0) = e_i$. Since linear combinations of solutions of $x' = Ax$ are also solutions,

  $$c_1u_1(t) + \cdots + c_nu_n(t)$$

  is the general solution of $x' = Ax$.

  That each solution is of this form follows from the uniqueness part of Picard's theorem: Each $c = (c_1, \ldots, c_n) \in \mathbb{R}$ can be written in a **unique** way as a linear combination of the basis vectors e_i, namely $c = c_1e_1 + \cdots + c_ne_n$. So if we are looking for a solution u of $u' = Au$, $u(0) = c$ then by the uniqueness part of Picard's theorem necessarily $u(t)$ is equal to $c_1u_1(t) + \cdots + c_nu_n(t)$ for all $t \in [-h, h]$.

- What form do the solutions of (3) take?

  Remember that by the two previous theorems we can find a solution of $x' = Ax$, by taking $x_0 \colon [-h, h] \to \mathbb{R}^n$ to be any function (for example $x_0(t) := x_0$ and then defining a sequence of functions $x_0, x_1, \ldots$ by $x_{n+1} = P(x_n)$ (where the operator P is defined as in the proof of the Picard theorem). This sequence of functions will converge (in the supremum norm) to the (unique) fixed point of P and therefore solution of the differential equation. In this case $x_{n+1}(t) = P(x_n)(t) := x_0 + \int_0^t Ax_n(s)\,ds$.

  So apply Picard iteration, taking $x_0(t) :\equiv x_0$. Then $x_1(t) = x_0 + \int_0^t Ax_0(s)\,ds = x_0 + tAx_0$. $x_2(t) = x_0 + \int_0^t Ax_1(s)\,ds = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0$. By induction $x_n(t) = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \cdots + \frac{t^n}{n!}A^nx_0 = \sum_{k=0}^n \frac{t^kA^k}{k!}x_0$. So the solution of (3) is

  $$x(t) = e^{At}x_0 \text{ where we write } e^{At} = \sum_{k=0}^{\infty} \frac{t^kA^k}{k!}.$$

  The proof of the Picard Theorem shows that this infinite sum exists (i.e. converges) when $|t| < h$. Later on we shall show that it exists for all t.

1.9 The Picard Theorem for functions which are locally Lipschitz

Theorem 3. Picard Theorem (local version).
Let U be an open subset of $\mathbb{R} \times \mathbb{R}^n$ containing $(0, x_0)$ and assume that

- $f: U \to \mathbb{R}^n$ is continuous,
- $|f| \leq M$
- $|f(t,u) - f(t,v)| \leq K|x - y|$ for all $(t,u),(t,v) \in U$
- $h \in (0, \frac{1}{2K})$ is chosen so that $[-h,h] \times \{y; |y - x_0| \leq hM\} \subset U$ (such a choice for h is possible since U open).

$$\frac{dx}{dt} = f(t,x) \text{ and } x(0) = x_0. \tag{4}$$

Proof. Fix $h > 0$ as in the theorem, write $I = [-h,h]$, and let $B := \{y \in \mathbb{R}^n; |y - x_0| \leq hM\}$. Next define $C(I,B)$ as the space of continuous functions $x: I \to B \subset \mathbb{R}^n$ and

$$P: C(I,B) \to C(I,B) \text{ by } P(x)(t) = x_0 + \int_0^t f(s, x(s))\, ds$$

Then the initial value problem (4) is equivalent to the fixed point problem

$$x = P(x).$$

We need to show that P **is well-defined**, i.e. that the expression $P(x)(t) = x_0 + \int_0^t f(s,x(s))\, ds$ makes sense, and that when $x \in C(I,B)$ then $P(x) \in C(I,B)$. To see this first note that $h > 0$ is chosen so that when $B := \{y; |y - x_0| \leq hM\}$ then $[-h,h] \times B \subset U$. So

- when $x \in C(I,B)$ then $f(t,x(t))$ is well-defined for all $t \in [-h,h]$;

The **autonomous version** of this theorem goes as follows: Let $V \subset \mathbb{R}^n$ be open and $g: V \to \mathbb{R}^n$ continuous, $|g| \leq M$, $|g(u) - g(v)| \leq K|u - v|$, $0 < h < 1/(2K)$ and $\{y; |y - x_0)| \geq hM\} \subset V$. Then there is a unique solution $x \in (-h,h) \to \mathbb{R}^n$ of $x' = g(x), x(0) = x_0$. This follows from Theorem 3, taking $U = \mathbb{R} \times V$ and $f(t,x) = g(x)$ on U.

This property we call **Locally Lipschitz**

- hence $x_0 + \int_0^t f(s, x(s))\, ds$ is well-defined;
- $|f| \le M$ implies $[-h, h] \ni t \mapsto x_0 + \int_0^t f(s, x(s))\, ds$ is continuous;
- hence $t \mapsto P(x)(t)$ is a continuous map;
- finally, $|P(x)(t) - x_0| \le \int_0^h |f(s, x(s))|\, ds \le hM$. So $P(x)(t) \in B$ for all $t \in [-h, h]$ and therefore $P(x) \in C(I, B)$.

Let us next show that

$$P\colon C(I,B) \to C(I,B) \ \text{ by } \ P(x)(t) = x_0 + \int_0^t f(s, x(s))\, ds$$

is a **contraction**: for each $t \in [-h, h]$,

$$\begin{aligned}
|P(x)(t) - P(y)(t)| &= \left|\int_0^t (f(s, x(s)) - f(t, y(s)))\, ds\right| \\
&\le \int_0^t |f(s, x(s)) - f(s, y(s))|\, ds \\
&\le K \int_0^t |x(s) - y(s)|\, ds \quad \text{(Lipschitz)} \\
&\le Kt \max_{|s| \le t} |x(s) - y(s)| \\
&\le Kh||x - y|| \le ||x - y||/2 \qquad \text{(since } h \in (0, \tfrac{1}{2K})\text{)}
\end{aligned}$$

Since this holds for all $t \in [-h, h]$ we get $||P(x) - P(y)|| \le ||x - y||/2$. So P has a unique fixed point, and hence the integral equation, and therefore the ODE, has a unique solution. $\square$

1.10 Some comments on the assumptions in Picard's Theorem

- To obtain existence in Theorem 3 it is enough to find *some* open set $U \ni (0, x_0)$.

- Often one can apply Theorem 3, but not Theorem 2. Take for example $x' = (1 + x^2)$. Then the r.h.s. is **not Lipschitz** on all of $\mathbb{R}$. It is locally Lipschitz though.

- It is not necessary to take the initial time to be $t = 0$. The Picard Theorem also gives that there exists $h > 0$ so that the initial value problem

$$x' = f(t, x), x(t_0) = x_0$$

  has a solution $(t_0 - h, t_0 + h) \ni t \mapsto x(t) \in \mathbb{R}^n$.

- Let $V \subset \mathbb{R} \times \mathbb{R}^n$ and assume that the Jacobian matrix $\frac{\partial f}{\partial x}(t, x)$ exists for $(t, x) \in V$ and $(t, x) \ni V \mapsto \frac{\partial f}{\partial x}(t, x)$ is continuous. Then for each convex, compact subset $C \subset V$ there exists $K \in \mathbb{R}$ so that

$$|f(t, x) - f(t, y)| \le K|x - y|.$$

  This follows from the Mean Value Theorem in $\mathbb{R}^n$, see Appendix A. (So one can apply the previous theorem for each open set $U \subset C$.)

This remark implies that the local Picard Theorem Theorem 3 implies a much punchy statement in the most usual setting that the **right hand side of the ODE is continuously differentiable**: Let $f\colon V \to \mathbb{R}^n$ be continuously differentiable. Then for each $(0, x_0) \in U$ there exists $h > 0$ and a unique solution $x\colon (-h, h) \to \mathbb{R}^n$ of $\dot{x} = f(t, x), x(0) = x_0$.

- If $(t, x) \mapsto f(t, x)$ has additional smoothness, the solutions will be more smooth. For example, suppose that $f(t, x)$ is real analytic (i.e. $f(t, x)$ can be written as a convergent power series), then the solution $t \mapsto x(t)$ is also real analytic.

1.11 Some implications of uniqueness in Picard's Theorem

- If the assumptions of the previous theorem hold and

$$x_1\colon I_1 \to \mathbb{R}^n, x_2\colon I_2 \to \mathbb{R}^n$$

are both solutions of the initial value problem. Then

$$x_1(t) = x_2(t) \text{ for all } t \in I_1 \cap I_2.$$

(See exercises.)

- $f\colon U \to \mathbb{R}^n$ does not depend on t (in this case we could take $U = \mathbb{R} \times V$ but in any case $f(t,x) = f(0,x)$ for all t and all x). This case is called **autonomous** (or **time-independent**), and so we can write $x' = f(x), x(0) = x_0$. In this setting solutions **cannot cross**:

More precisely, if x_1, x_2 are solutions with $x_1(t_1) = x_2(t_2) = p \in V$ then

$$x_3(t) = x_1(t + t_1) \text{ and } x_4(t) = x_2(t + t_2)$$

are both solutions to $x' = f(x)$ with $x(0) = p$. So

$$x_3 \equiv x_4.$$

- The following three important implications for **autonomous** systems from local existence uniqueness are explored in Assignment 2 (this material is **examinable**):
  - the existence of a **maximal interval** $(t_-, t_+) \ni 0$ **of existence**;
  - when $t_+ < \infty$ then $|x(t)| \to \infty$ as $t \to t_+$;
  - the **flow property**.

1.12 Higher order differential equations

Consider a higher order differential equation of the form

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = b(t) \tag{5}$$

where $y^{(i)}$ stands for the i-th derivative of y w.r.t. t.

- One can rewrite (5) as a first order ODE, by defining

  $$z_1 = y, z_2 = y^{(1)}, \ldots, z_n = y^{(n-1)}.$$

  The higher order differential equation (5) is equivalent to

  $$\frac{d}{dt}\begin{pmatrix} z_1 \\ \cdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ \cdots \\ z_n \\ b(t) - [a_{n-1}(t)z_n + \cdots + a_0(t)z_1] \end{pmatrix}$$

- Picard's theorem implies $\exists!$ solution of this ODE which satisfies $(z_1(0), \ldots, z_n(0)) = (y(0), \ldots, y^{(n-1)}(0))$.

- One can rewrite the vectorial equation as

  $$\frac{d}{dt}\begin{pmatrix} z_1 \\ \cdots \\ z_{n-1} \\ z_n \end{pmatrix} = A(t)\begin{pmatrix} z_1 \\ \cdots \\ z_{n-1} \\ z_n \end{pmatrix}$$

  where $A(t)$ is matrix with coefficients depending on t. Therefore, as in subsection 1.8, the general solution of the non-homogeneous ODE is of the form $c_1 y_1 + \cdots + c_n y_n + p$ where p is a particular solution. There are at most n degrees of freedom.

1.13 Continuous dependence on initial conditions

Theorem 4. Continuous dependence on initial conditions Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be open, $f, g \colon U \to R^n$ be continuous and assume that

$$K = \sup_{(t,u),(t,v) \in U} \frac{|f(t,u) - f(t,v)|}{|u - v|}, \; M = \sup_{(t,u) \in U} |f(t,u) - g(t,u)|$$

are finite. If $x(t)$ and $y(t)$ are respective solutions of the IVP's

$$\begin{cases} x' & = f(t,x) \\ x(0) & = x_0 \end{cases} \quad \text{and} \begin{cases} y' & = g(t,y) \\ y(0) & = y_0 \end{cases}$$

Then

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{K|t|} + \frac{M}{K}(e^{K|t|} - 1).$$

1.14 Gronwall Inequality

Proof:

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t |f(s, x(s)) - g(s, y(s))| \, ds.$$

Moreover,

$$\begin{aligned} &|f(s, x(s)) - g(s, y(s))| \leq \\ &\leq |f(s, x(s)) - f(s, y(s))| + |f(s, y(s)) - g(s, y(s))| \leq \\ &\leq K|x(s) - y(s)| + M. \end{aligned}$$

Hence, writing $u(t) := |x(t) - y(t)|$ we have

$$u(t) \leq |x_0 - y_0| + \int_0^t (K|u(s)| + M)$$

and therefore the required inequality follows from the following lemma.

Lemma 1. Gronwall Inequality

$$u(t) \le C_0 + \int_0^t (Ku(s) + M)\, ds \text{ for all } t \in [0,h] \implies$$

$$u(t) \le C_0 e^{Kt} + \frac{M}{K}\left(e^{Kt} - 1\right) \text{ for all } t \in [0,h].$$

Proof. Let's only prove this only when $M = 0$. Define

$$U(t) = C_0 + \int_0^t (Ku(s))\, ds.$$

Then $u(t) \le U(t)$. Differentiating, we obtain

$$U'(t) = Ku(t).$$

Hence

$$U'(t)/U(t) = Ku(t)/U(t) \le K$$

and therefore

$$\frac{d}{dt}\log(U(t)) \le K.$$

Since $U(0) = C_0$ this gives

$$u(t) \le U(t) \le C_0 e^{Kt}.$$

$\square$

1.15 Consequences of Gronwall inequality

- Let us interpret the previous result for $f = g$. Then $M = 0$ and

  $$\left\{ \begin{array}{ll} x' & = f(t,x) \\ x(0) & = x_0 \end{array} \right. \quad \text{and} \left\{ \begin{array}{ll} y' & = f(t,y) \\ y(0) & = y_0 \end{array} \right. \quad implies$$

  $$|x(t) - y(t)| \le |x_0 - y_0| e^{K|t|}.$$

  In particular, uniqueness follows.

- The previous inequality states:

$$|x(t) - y(t)| \leq |x_0 - y_0|e^{K|t|} + 0.$$

So orbits can separate exponentially fast.

1.16 The butterfly effect

If solutions indeed separate exponentially fast, the the differential equation is said to have *sensitive dependence on initial conditions*. (The flapping of a butterfly in the Amazon can cause a hurricane over the Atlantic.)

This sensitive dependence occurs in very simple differential equations, for example in the famous Lorenz differential equation

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{6}$$

with $\sigma = 10, r = 28, b = 8/3$.

This equation has solutions which are chaotic and have sensitive dependence.

`http://www.youtube.com/watch?v=ByH8_nKD-ZM`

1.17 Double pendulum

There are many physical system where sensitive dependence of initial conditions occurs. For example the **double pendulum**, see for example `https://www.youtube.com/watch?v=U39RMUzCjiU` or `https://www.youtube.com/watch?v=fPbExSYcQgY`.

2 Linear systems in $\mathbb{R}^n$

In this section we consider

$$x' = Ax \text{ with } x(0) = x_0 \tag{7}$$

where A is a $n \times n$ matrix and $\mathbb{R} \ni t \mapsto x(t) \in \mathbb{R}^n$.

In Example 1.8 we saw that

$$e^{tA} = \sum_{k\geq 0} \frac{1}{k!}(At)^k$$

is defined for $|t|$ small and that $x(t) = e^{tA}x_0$ is a solution of (7) for $|t|$ small. In this section we will show that e^{tA} is well-defined for all $t \in \mathbb{R}$ and show how to compute this matrix.

Example 13. Let $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Then one has inductively $(tA)^k = \begin{pmatrix} (t\lambda)^k & 0 \\ 0 & (t\mu)^k \end{pmatrix}$. So $e^{tA} = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\mu} \end{pmatrix}$.

Example 14. Let $A = \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$. Then one has inductively $(tA)^k = \begin{pmatrix} (t\lambda)^k & \epsilon k t^k \lambda^{k-1} \\ 0 & (t\lambda)^k \end{pmatrix}$. By calculating the infinite sum of each entry we obtain $e^{tA} = \begin{pmatrix} e^{t\lambda} & \epsilon t e^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix}$.

Lemma 2. e^A is well-defined for any matrix $A = (a_{ij})$.

Proof. let $a_{ij}(k)$ be the matrix coefficients of A^k and define $a := ||A||_\infty := \max |a_{ij}|$. Then

$$\begin{aligned}
|a_{ij}(2)| &= \textstyle\sum_{k=1}^n |a_{ik}a_{kj}| \leq na^2 \leq (na)^2 \\
|a_{ij}(3)| &= \textstyle\sum_{k,l}^n |a_{ik}a_{kl}a_{lj}| \leq n^2a^3 \leq (na)^3 \\
&\vdots \\
|a_{ij}(k)| &= \textstyle\sum_{k_1,k_2,\ldots,k_n=1}^n |a_{k_1k_2}a_{k_2k_3}\cdots a_{k_{n-1}k_n}| \leq n^{k-1}a^k \leq (na)^k
\end{aligned}$$

So $\sum_{k=0}^{\infty} \frac{|a_{ij}(k)|}{k!} \leq \sum_{k=0}^{\infty} \frac{(na)^k}{k!} = \exp(na)$ which means that the series $\sum_{k=0}^{\infty} \frac{a_{ij}(k)}{k!}$ converges absolutely by the comparison test. So e^A is well-defined. $\square$

2.1 Some properties of $\exp(A)$

Lemma 3. Let A, B, T be $n \times n$ matrices and T invertible. Then

1. If $B = T^{-1}AT$ then $\exp(B) = T^{-1}\exp(A)T$;
2. If $AB = BA$ then $\exp(A+B) = \exp(A)\exp(B)$
3. $\exp(-A) = (\exp(A))^{-1}$

Proof. (1) $T^{-1}(A+B)T = T^{-1}AT + T^{-1}BT$ and $(T^{-1}AT)^k = T^{-1}A^kT$. Therefore

$$T^{-1}(\sum_{k=0}^{n} \frac{A^k}{k!})T = \sum_{k=0}^{n} \frac{(T^{-1}AT)^k}{k!}.$$

(2) follows from the next lemma and (3) follows from (2) taking $B = -A$. $\square$

For general matrices $\exp(A+B) \neq \exp(A)\exp(B)$. Note that if $AB = BA$ then $(A+B)^n = n! \sum_{j+k=n} \frac{A^j}{j!} \frac{B^k}{k!}$. So (2) in the previous lemma follows from:

Lemma 4.

$$\sum_{n=0}^{\infty} \sum_{j+k=n} \frac{A^j}{j!} \frac{B^k}{k!} = \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{k=0}^{\infty} \frac{B^k}{k!}.$$

Proof: A computation shows

$$\sum_{n=0}^{2m}\sum_{j+k=n}\frac{A^j}{j!}\frac{B^k}{k!}-\sum_{j=0}^{m}\frac{A^j}{j!}\sum_{k=0}^{m}\frac{B^k}{k!}=\sum{}'\frac{A^j}{j!}\frac{B^k}{k!}+\sum{}''\frac{A^j}{j!}\frac{B^k}{k!}$$

where $\sum'$ respectively $\sum''$ denote the sum over terms

$$j+k\le 2m, 0\le j\le m, m+1\le k\le 2m,$$

$$j+k\le 2m, m+1\le j\le 2m, 0\le k\le m.$$

So the absolutely values of the coefficients in $\sum' \dfrac{A^j}{j!}\dfrac{B^k}{k!}$ are bounded by $\sum_{j=0}^{m}\dfrac{||A^j||_\infty}{j!}\sum_{k=m+1}^{2m}\dfrac{||B^k||_\infty}{k!}$. As in the proof Lemma 2 the latter term goes to zero as $m\to\infty$.

Similarly $\sum'' \dfrac{A^j}{j!}\dfrac{B^k}{k!}$ goes to zero as $\to\infty$. This completes the proof of Lemma 4.

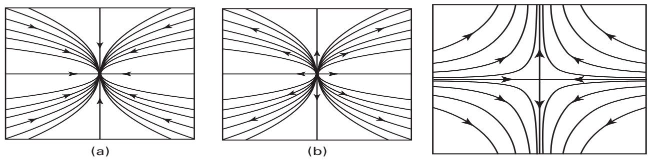
Example 15.

$$\exp\begin{pmatrix} a & b \\ -b & a\end{pmatrix}=\begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt)\end{pmatrix}.$$

This is proved in the first assignment of week 3 and also in Section 2.4.

Each coefficient of e^{tA} depends on t. So define $\dfrac{d}{dt}e^{tA}$ to be the matrix obtained by differentiating each coefficient.

Lemma 5. $\dfrac{d}{dt}\exp(tA)=A\exp(tA)=\exp(tA)A.$



Proof.

$$\frac{d}{dt}\exp(tA) = \lim_{h\to 0}\frac{\exp((t+h)A) - \exp(tA)}{h} =$$

$$= \lim_{h\to 0}\frac{\exp(tA)\exp(hA) - \exp(tA)}{h} =$$

$$= \exp(tA)\lim_{h\to 0}\frac{\exp(hA) - I}{h} = \exp(tA)A.$$

Here the last equality follows from the definition of $\exp(hA) = I + hA + \frac{h^2}{2!}A^2 + \dots$. □

2.2 Solutions of 2×2 systems

$x(t) = e^{tA}x_0$ is the solution of $\dot{x} = Ax$, $x(0) = x_0$ because $\dot{x} = Ae^{tA}x_0 = Ax(t)$ and $x(0) = e^{0A}x_0 = x_0$.

Example 16. Take $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. So $x(t) = e^{tA} = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\mu} \end{pmatrix} x_0$ is a solution of the differential equation.
(Case a) $\lambda, \mu < 0$ (**sink**). Then $x(t) \to 0$ as $t \to \infty$.
(Case b) If $\lambda, \mu > 0$ (**source**). Then $x(t) = e^{tA}x_0 \to \infty$ as $t \to \infty$ for any $x_0 \neq 0$.
(Case c) $\lambda < 0 < \mu$ (**saddle**). Then $x(t) = e^{tA}x_0 \to \infty$ as $t \to \infty$ if the 2nd component of x_0 is non-zero, and $x(t) \to 0$ otherwise.

Example 17. Take $A = \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$ and let us compute e^{tA} again. $tA = t\Lambda + tN$ where $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $N = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}$. Note that $\Lambda N = \lambda N = N\Lambda$ and that $N^2 = 0$. So

$$e^{tN} = I + tN = \begin{pmatrix} 1 & t\epsilon \\ 0 & 1 \end{pmatrix}, e^{t\Lambda} = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\lambda} \end{pmatrix}$$

and

$$e^{tA} = e^{t\Lambda}e^{tN} = \begin{pmatrix} e^{t\lambda} & \epsilon t e^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix}.$$

In general it is not so easy to compute e^{tA} directly from the definition. For this reason we will discuss

- eigenvalues and eigenvectors;
- using eigenvectors to put a matrix in a new form;
- using eigenvectors and eigenvalues to obtain solutions directly.

2.3 n linearly independent eigenvectors

Given a concrete $n \times n$ matrix A, one usually solves the solutions of $\dot{x} = Ax$ using eigenvalues and eigenvectors.

Reminder: A vector $v \neq 0$ is an **eigenvector** if $Av = \rho v$ for some $\rho \in \mathbb{C}$ where ρ is called the corresponding **eigenvalue**. So, $(A - \rho I)v = 0$ and $\det(A - \rho I) = 0$. The equation $\det(A - \rho I) = 0$ is a polynomial of degree in ρ.

Example 18. Take $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}$. Consider

$$\det \begin{pmatrix} 1-\rho & 2 & -1 \\ 0 & 3-\rho & -2 \\ 0 & 2 & -2-\rho \end{pmatrix} = -(-1+\rho)(-2-\rho+\rho^2).$$

So A has eigenvalues $2, 1, -1$. Eigenvector w.r.t. 2:

$$\begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{pmatrix} v = 0$$

which gives $v = (3, 2, 1)$ (or multiples). A has eigenvalues $2, 1, -1$ with eigenvectors $(3, 2, 1), (1, 0, 0), (0, 1, 2)$.

- Case 1: n linearly independent eigenvectors. Suppose that $v_1, \dots, v_n$ are eigenvectors of A with eigenvectors $\rho_1, \dots, \rho_n$ and assume that these eigenvectors are **linearly independent**.

  Lemma from Linear Algebra: if all ρ_i are distinct then the eigenvectors $v_1, \dots, v_n$ are lin. independent and span $\mathbb{R}^n$.

  Then $x_i(t) = e^{\rho_i t} v_i$ is a solution because

  $$\dot{x}_i = \rho_i e^{\rho_i t} v_i = e^{\rho_i t} A v_i = A x_i(t_i)$$

  Hence
  $$x(t) = c_1 e^{\rho_1 t} v_1 + \dots + c_n e^{\rho_n t} v_n$$

  is the general solution of the differential equation.

  To determine the solution with $x(0) = x_0$ one needs to solve c_i so that $c_1 v_1 + \dots + c_n v_n = x_0$ (which can be done since $v_1, \dots, v_n$ are linearly independent and span $\mathbb{R}^n$).

2.4 Complex eigenvectors

- Case 2: Complex eigenvectors. If v_1 is **non-real** (which implies since A is real that ρ_1 is also non-real), then there exists another eigenvector, say v_2 with $\bar{v}_2 = v_1$, $\bar{\rho}_2 = \rho_1$.)

  So write $v_1 = \zeta_1 + i\zeta_2$, $v_2 = \zeta_1 - i\zeta_2$, $\rho_1 = a + ib$ and $\rho_2 = a - ib$ with ζ_i, a_1, b_1 are real. This gives

  ζ_i are real **vectors** in $\mathbb{R}^2$

  $$\begin{aligned} c_1 e^{\rho_1 t} v_1 \quad & + c_2 e^{\rho_2 t} v_2 = \\ & = c_1 e^{at} \left((\cos(bt) + i \sin(bt))(\zeta_1 + i\zeta_2) \right. \\ & \quad \left. + c_2 e^{at} (\cos(bt) - i \sin(bt))(\zeta_1 - i\zeta_2) \right) \end{aligned} \tag{8}$$

  By taking suitable choices of $c_1, c_2 \in \mathbb{C}$ one can rewrite this as

  $$d_1 e^{at} \left(\cos(bt)\zeta_1 - \sin(bt)\zeta_2 \right) + d_2 e^{at} \left(\sin(bt)\zeta_1 + \cos(bt)\zeta_2 \right) \tag{9}$$

  where $d_1, d_1 \in \mathbb{R}$. Indeed: the r.h.s. of (8) is equal to $(c_1 + c_2)e^{at}[(\cos(bt)\zeta_1 - \sin(bt)\zeta_2)] + (c_1 - c_2)ie^{at}[(\sin(bt)\zeta_1 + \cos(bt)\zeta_2)]$. For each d_1, d_2 real we can find complex c_1, c_2 so that $c_1 + c_2 = d_1$ and $(c_1 - c_2)i = d_2$. Thus we get equation (9).

An alternative way of seeing this goes as follows: $A(\zeta_1 + i\zeta_2) = (a + bi)(\zeta_1 + i\zeta_2) = (a\zeta_1 - b\zeta_2) + i(a\zeta_2 + b\zeta_1)$. So $A(\zeta_1) = a\zeta_1 - b\zeta_2$ and $A(\zeta_2) = (a\zeta_2 + b\zeta_1)$. It follows that if T is the matrix consisting of columns ζ_1, ζ_2 then

$$T^{-1}AT = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Indeed, $AT(e_1) = A(\zeta_1) = a\zeta_1 - b\zeta_2 = aT(e_1) - bT(e_2)$ and so $T^{-1}AT(e_1) = ae_1 - be_2$. Similarly $T^{-1}AT(e_2) = be_1 + ae_2$. So

$$\exp(tA) = T\exp(T^{-1}tAT)T^{-1} = T \begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix} T^{-1}.$$

Here we use Example 15. Now write $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = Tx_0$ and check that

$$\exp(At)x_0 = T \begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} =$$

$$d_1 e^{at}\left(\cos(bt)\zeta_1 - \sin(bt)\zeta_2\right) + d_2 e^{at}\left(\sin(bt)\zeta_1 + \cos(bt)\zeta_2\right).$$

2.5 Eigenvalues with higher multiplicity

- Case 1: Repeated eigenvalues If $\rho = \rho_1 = \cdots = \rho_k$ then we proceed as follows. Let us consider the case that $k = 2$ assume $\rho_1 = \rho_2$ and v_1 is an eigenvector w.r.t. ρ but there is not 2nd eigenvector. Then there exists a vector v_2 so that $(A - \rho I)v_2 = v_1$. (The general procedure is explained in Appendix D.)

  So
  $$x_1(t) = e^{\rho t}v_1 \text{ and } x_2(t) = te^{\rho t}v_1 + e^{\rho t}v_2$$
  is a solution: indeed
  $$\begin{aligned} \dot{x}_2 = e^{\rho t}v_1 + t\rho e^{\rho t}v_1 + \rho e^{\rho t}v_2 &= \rho t e^{\rho t}v_1 + (e^{\rho t}v_1 + \rho e^{\rho t}v_2) = \\ &= A(te^{\rho t}v_1 + e^{\rho t}v_2) \end{aligned}$$
  where we use that $Av_2 = \rho I v_2 + v_1$.

2.6 A worked example: 1

Computing solutions in several ways

Example 19. The matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}$ has eigenvalues $2, 1, -1$ with eigenvectors $(3, 2, 1), (1, 0, 0), (0, 1, 2)$. Set

$$T = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \text{ we get } T^{-1}AT \; = \; \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Indeed, $T^{-1}ATe_i = T^{-1}Av_i = \rho_i T^{-1}v_i = \rho_i e_i$ where e_i is the i-th unit vector. Hence

$$\begin{aligned} \exp(tA) \; &= \; \exp\left(tTT^{-1}ATT^{-1}\right) = T\exp\left(tT^{-1}AT\right)T^{-1} \\ &= T\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} T^{-1}. \end{aligned}$$

For each vector $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ there exists $x_0 \in \mathbb{R}^3$ so that $c = T^{-1}x_0$. Hence

$$\exp(tA)x_0 = T\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} T^{-1}x_0 = T\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} c =$$

$$T\begin{pmatrix} c_1e^{2t} \\ c_2e^{t} \\ c_3e^{-t} \end{pmatrix} = c_1e^{2t}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2e^{t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3e^{-t}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Notice that this agrees with the method suggested in Case below Example 18.

The previous example is an instance of the **diagonal Jordan Normal Form**:

Theorem: If an $n \times n$ matrix A has n **distinct eigenvalues** $\lambda_1, \dots, \lambda_n$ with eigenvectors v_i then

- The eigenvectors $v_1, \dots, v_n$ are linearly independent and span $\mathbb{R}^n$;

- If we take T the matrix with columns $v_1, \ldots, v_n$ then $T^{-1}AT = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.
- $e^{tA} = T \begin{pmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{pmatrix} T^{-1}$.

2.7 A second worked example

In the example below, we explain what to do when there is no basis of eigenvectors. As you will see, the example also explains what to do in the general situation.

Example 20. Take $A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$ and compute the solution of $x' = Ax$ with $x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\det(A - \rho I) = \begin{pmatrix} 1-\rho & 9 \\ -1 & -5-\rho \end{pmatrix} = (\rho+2)^2$ so the eigenvalue -2 appears with double multiplicity. $(A - \rho I)v = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} v = 0$ implies v is a multiple of $v_1 := \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ so there exists only one eigenvector. To find the 2nd 'generalised eigenvector' consider $(A - \rho I)v_2 = v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ which gives $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a solution. From this one can deduce, or 'guess' as in Section 2.5, equation ($\star$) in the next page. To see this more generally, note that the eigenvector v_1 and the corresponding generalised eigenvector v_2 satisfy

$$Av_1 = \rho v_1, Av_2 = \rho v_2 + v_1$$

and thus allow us to transform this matrix into what is called a Jordan normal form. Indeed, take T the matrix with columns v_1, v_2, i.e. $Te_i = v_i$. Then

$$\begin{aligned} T^{-1}AT(e_1) &= T^{-1}Av_1 = T^{-1}\rho v_1 = \rho e_1 \\ T^{-1}AT(e_2) &= T^{-1}Av_2 = T^{-1}(\rho v_2 + v_1) = \\ &= \rho T^{-1}(v_2) + T^{-1}(v_1) = \rho e_2 + e_1. \end{aligned}$$

This means that:

$$T^{-1}AT = N := \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

Hence $e^{tA} = Te^{tN}T^{-1} = T\begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}T^{-1}$. Remembering that $T = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ it follows that the solution $x(t) = e^{tA}x_0$ is of the form:

$$x(t) = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^{-2t} + c_2\left(t\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)e^{-2t}$$

where we take $T^{-1}x_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Of course for varying choice of c_1, c_2 this gives the general solution, and when we want that $x(0) = x_0$ then $c_1 = 1$, $c_2 = -2$ solves the initial value problem.

The previous example is an instance of the Jordan Normal Form Theorem:

If an $n \times n$ matrix A has only one eigenvector v (which implies that its eigenvalue λ appears with multiplicity n) then

- one can define inductively $v_1 = v$ and $(A - \lambda I)v_{i+1} = v_i$.
- $v_1, \dots, v_n$ are linearly independent and span $\mathbb{R}^n$.

- If we take T the matrix with columns $v_1, \ldots, v_n$ then
$$T^{-1}AT = \begin{pmatrix} \lambda & 1 & & & 0 \\ 0 & \lambda & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \lambda & 1 \\ 0 & & & & \lambda \end{pmatrix}.$$

- $e^{tA} = Te^{t\lambda}(I + tN + \cdots + \frac{t^{n-1}}{(n-1)!}N^{n-1})T^{-1}$ where $N =$
$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}.$$

2.8 Complex Jordan Normal Form (General Case)

Theorem 5. For each $n \times n$ matrix A there exists a (possibly complex) matrix T so that $T^{-1}AT$ takes the Jordan Normal Form: $T^{-1}AT = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix}$ where

$$J_j = \begin{pmatrix} \rho_j & 1 & 0 & & 0 \\ 0 & \rho_j & 1 & & \\ & & \ddots & & \\ 0 & & & \rho_j & 1 \\ 0 & & & & \rho_j \end{pmatrix}$$

and where ρ_j is an eigenvalue of A so that the dimension of $(A - \rho_j I)^k$ is equal to the dimension of J_j.

If J_j is a 1×1 matrix, then $J_j = (\rho_j)$. Associated to each block J_j, there exists an eigenvector v_j (with eigenvalue ρ_j). The dimension of J_j is equal to the maximal integer k_j so that there exist vectors $w_j^1, w_j^2, \ldots, w_j^{k_j} \neq 0$ (where $w_j^1 = v_j$)

inductively defined as $(A-\rho_j I)w_j^{i+1} = w_j^i$ for $i = 1, \ldots, k_j - 1$. The matrix T has columns $w_1^1, \ldots, w_1^{k_1}, \ldots, w_p^1, \ldots, w_p^{k_p}$.

In the computations above, we showed how to determine T so this holds.

2.9 Real Jordan Normal Form

Splitting real and complex parts we obtain:

For each real $n \times n$ matrix A there exists a real $n \times n$ matrix T so that $T^{-1}AT$ takes the real Jordan Normal Form:
$T^{-1}AT = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix}$ where J_j is either as in the complex Jordan Normal form when ρ_j real or if it is complex equal to

$$J_j = \begin{pmatrix} C_j & I & 0 & & 0 \\ 0 & C_j & I & & \\ & & \ddots & & \\ 0 & & & C_j & I \\ 0 & & & & C_j \end{pmatrix} \text{ where } C_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$$

where $\rho_j = a_j + ib_j$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If J_j is a 2×2 matrix, then $\begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$.

Proof: See appendix D.

3 Power Series Solutions

Theorem 6. If f is real analytic near $(x_0, 0)$, then $x' = f(t, x), x(0) = x_0$ has a real analytic solution, i.e. the solution $t \mapsto x(t)$ is a power series in t which converges for $|t| < h$.

To *prove* theorem one considers in the differential equation $x' = f(t, x)$ the time t be complex! We will not pursue this here.

Note that in this chapter we obtain take the derivative w.r.t. x, so write instead $y' = f(x, y)$ and look for solutions $x \mapsto y(x)$.

In this chapter we will consider some examples. Typically, one the coefficients appearing in the power series expansions of the solutions can be found inductively as in the next examples.

Example 21. $y' = y$. Then substitute $y = \sum_{i\geq 0} a_i x^i$ and $y' = \sum_{j\geq 1} j a_j x^{j-1} = \sum_{i\geq 0}(i+1)a_{i+1}x^i$. Comparing powers gives $\sum_{i\geq 0}(a_i x^i - (i+1)a_{i+1}x^i) = 0$ and so $a_{i+1} = a_i/(i+1)$. So $a_n = C/n!$ which gives $y(x) = C\sum_{n\geq 0} x^n/n! = C\exp(x)$.

3.1 Legendre equation

Example 22. Consider the Legendre equation at $x = 0$:

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0.$$

Write $y = \sum_{i\geq 0} a_i x^i$,

$$y' = \sum_{j\geq 1} j a_j x^{j-1} = \sum_{i\geq 0}(i+1)a_{i+1}x^i.$$

$$y'' = \sum_{j\geq 2} j(j-1)a_j x^{j-2} = \sum_{i\geq 0}(i+2)(i+1)a_{i+2}x^i.$$

We determine a_i as follows.

$$y'' - x^2 y'' - 2xy' + p(p+1)y =$$

$$\sum_{i\geq 0}(i+2)(i+1)a_{i+2}x^i - \sum_{i\geq 2} i(i-1)a_i x^i - 2\sum_{i\geq 1} ia_i x^i + p(p+1)\sum_{i\geq 0} a_i x^i$$

$$= \sum_{i\geq 2}\left[(i+2)(i+1)a_{i+2} - i(i-1)a_i - 2ia_i + p(p+1)a_i\right]x^i +$$

$$+(2a_2 + 6xa_3) - 2a_1 x + p(p+1)(a_0 + a_1 x)$$

$$\sum_{i\geq 0}(i+2)(i+1)a_{i+2}x^i - \sum_{i\geq 2} i(i-1)a_i x^i - 2\sum_{i\geq 1} ia_i x^i + p(p+1)\sum_{i\geq 0} a_i x^i$$

$$= \sum_{i\geq 2}\left[(i+2)(i+1)a_{i+2} - i(i-1)a_i - 2ia_i + p(p+1)a_i\right]x^i +$$

$$+(2a_2 + 6xa_3) - 2a_1 x + p(p+1)(a_0 + a_1 x)$$

So collecting terms with the same power of x together gives $a_2 = -\frac{p(p+1)}{2}a_0$ and $a_3 = \frac{(2-p(p+1))}{6}a_1$ and

$$a_{i+2} = \frac{[i(i-1)+2i-p(p+1)]a_i}{(i+1)(i+2)} = -\frac{(p-i)(p+i+1)}{(i+2)(i+1)}a_i.$$

If p is an integer, $a_{p+2j} = 0$ for $j \geq 0$. Convergence of $y = \sum_{i\geq 0} a_i x^i$ for $|x| < 1$ follows from the ratio test.

3.2 Second order equations with singular points

Sometimes one encounters a differential equation where the solutions are not analytic because the equation has a pole. For example

$$y'' + (1/x)y' - (1/x^2)y = 0.$$

Or more generally if the equation can be written in the form

$$y'' + p(x)y' - q(x)y = 0$$

where p has a pole of order 1 and q a pole of order 2. That is,

$$p(x) = \frac{a_{-1}}{x} + \sum_{n\geq 0} a_n x^n, q(x) = \frac{b_{-2}}{x} + \frac{b_{-1}}{x} + \sum_{n\geq 0} b_n x^n \quad (10)$$

and where the sums are convergent. Such systems are said to have a **regular singular point** at $x = 0$.

Even though the existence and uniqueness theorem from Chapter 2 no longer guarantees the existence of solutions, it turns out that a solution of the form $y = x^m \sum_{i\geq 0} a_i x^i$ exists. Here $m \in \mathbb{R}$ and $\sum a_i x^i$ converges near 0). For simplicity we always assume $a_0 \neq 0$.

Example 23.

$$2x^2 y'' + x(2x+1)y' - y = 0.$$

Substitute $y = \sum_{i\geq 0} a_i x^{m+i}$ where we CHOOSE m so that $a_0 \neq 0$. Then $y' = \sum_{i\geq 0}(m+i)a_i x^{m+i-1}$ and $y'' = \sum_{i\geq 0}(m+i)(m+i-1)a_i x^{m+i-2}$. Note that m may not be an integer so we always start with $i = 0$. Plugging this in gives

$$2\sum_{i\geq 0}(m+i)(m+i-1)a_i x^{m+i} + 2\sum_{i\geq 0}(m+i)a_i x^{m+i+1} +$$

$$+\sum_{i\geq 0}(m+i)a_i x^{m+i} - \sum_{i\geq 0} a_i x^{m+i} = 0.$$

Collecting the coefficient in front of x^m gives

$$(2m(m-1) + m - 1)a_0 = 0.$$

Since we assume $a_0 \neq 0$ we get the equation $2m(m-1)+m-1=0$ which gives $m=-1/2, 1$. The coefficient in front of all the terms with x^{m+i} gives

$2(m+i)(m+i-1)a_i + 2(m+i-1)a_{i-1} + (m+i)a_i - a_i = 0,$ i.e.

$[2(m+i)(m+i-1)+(m+i)-1]a_i = -2(m+i-1)a_{i-1}.$

If $m=-1/2$ this gives $a_j = \dfrac{3-2j}{-3j+2j^2}a_{j-1}$.

If $m=1$ then this gives $a_j = \dfrac{-2j}{3j+2j^2}a_{j-1}$.

So $y = Ax^{-1/2}\left(1-x+(1/2)x^2+\dots\right) + Bx\left(1-(2/5)x+\dots\right)$. The ratio test gives that $\left(1-x+(1/2)x^2+\dots\right)$ and $\left(1-(2/5)x+\dots\right)$ converge for all $x \in \mathbb{R}$.

Remark: The equation required to have the lowest order term vanish is called the **indicial equation** which has two roots m_1, m_2 (possibly of double multiplicity).

Theorem 7. Consider a differential equation $y'' + p(x)y' - q(x)y = 0$ where p, q are as in equation (10). Then

- If $m_1 - m_2$ is not an integer than we obtain two independent solutions of the form $y_1(x) = x^{m_1}\sum_{i\geq 0} a_i x^i$ and $y_2(x) = x^{m_2}\sum_{i\geq 0} a_i x^i$.

- If $m_1 - m_2$ is an integer than one either can find a 2nd solution in the above form, or - if that fails - a 2nd solution of the form $\log(x)y_1(x)$ where $y_1(x)$ is the first solution.

Certain families of this kind of differential equation with regular singular points, appear frequently in mathematical physics.

- Legendre equation

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

- Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

- Gauss' Hypergeometric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

For suitable choices of a, b solutions of this are the sine, cosine, arctan and log functions.

3.3 Computing invariant sets by power series

One can often obtain curves through certain points as convergent power series.

Example 24. Let $x' = x + y^2, y' = -y + x^2$.

- This is an autonomous differential equation in the plane.
- Solutions are unique (r.h.s. is locally Lipschitz).
- So solutions are of the form $t \mapsto \phi_t(x, y)$ where $\phi_t(x, y)$ has the flow property $\phi_{t+s}(x, y) = \phi_t\phi_s(x, y)$.
- Of course at $\phi_t(0, 0) = (0, 0)$ for all t, since the r.h.s. of the differential equation is zero (the speed is zero there).
- Nevertheless we can find a curve γ of the form $y = \psi(x)$ with the property that if $(x, y) \in \gamma$ then $\phi_t(x, y) \in \gamma$ for all t (for which $\phi_t(x, y)$ exists). typo corrected
- Later on we shall see that $x' = x + y^2, y' = -y + x^2$ locally behaves very much like the equation in which the higher order terms are removed: $x' = x, y' = -y$. For this linear equation the x-axis ($y = 0$) is invariant: on that line we have $x' = -x$ and so orbits go to 0 in the linear case.

- What about the non-linear case? Let us assume that one can write y as a function of x, i.e. $y = \psi(x)$. Then since, $x' = x + y^2, y' = -y + x^2$ we have

$$\psi'(x) = (\frac{dy}{dt})/(\frac{dx}{dt}) = \frac{-y + x^2}{x + y^2} = \frac{-\psi(x) + x^2}{x + [\psi(x)]^2}.$$

  Let us assume $0 \in \gamma$ and write $y = \psi(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots$. Comparing terms gives

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots = \frac{-[a_1 x + a_2 x^2 + a_3 x^3 + \dots] + x^2}{x + [a_1 x + a_2 x^2 + a_3 x^3 + \dots]^2}.$$

  Comparing terms of the same power, shows that $a_1 = 0$, $2a_2 = (1 - a_2 - a_1^2)$ and so on. This gives a curve which is tangent to the x-axis so that orbits remain in this curve.

4 Boundary Value Problems, Sturm-Liouville Problems and Oscillatory equations

Instead of initial conditions, in this chapter we will consider boundary values. Examples:

- $y'' + y = 0, y(0) = 0, y(\pi) = 0$. This has infinitely many solutions: $y(x) = c\sin(x)$.
- $y'' + y = 0, y(0) = 0, y(\pi) = \epsilon \neq 0$ has no solutions: $y(x) = a\cos(x) + b\sin(x)$ and $y(0) = 0$ implies $a = 0$ and $y(\pi) = 0$ has no solutions.
- Clearly boundary problems are more subtle.
- We will concentrate on equations of the form $u'' + \lambda u = 0$ with boundary conditions, where λ is a free parameter.
- This class of problems is relevant for a large class of physical problems: heat, wave and Schroedinger equations.
- This generalizes Fourier expansions.

4.1 Motivation: wave equation

Consider the **wave equation:**

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$$

where $x \in [0, \pi]$ and the end points are fixed:

$$u(0,t) = 0, u(\pi,t) = 0 \text{ for all } t\ ,$$

$$u(x,0) = f(x), \frac{\partial}{\partial t}u(x,t)|_{t=0} = 0.$$

Below we shall see that some additional condition is needed on f to ensure that one can find a solution u which is C^2. These technical conditions are **not examinable**.

- This is a model for a string of length π on a musical instrument such as a guitar; before the string is released the shape of the string is $f(x)$.

- As usual one solves this by writing $u(x,t) = w(x) \cdot v(t)$, substituting this into the wave equation and then obtaining $w''(x)/w(x) = v''(t)/v(t)$. Since the left hand does not depend on t and the right hand side not on x this expression is equal to some constant λ and we get

$$w'' = \lambda w \text{ and } v'' = \lambda v.$$

- We need to set $w(0) = w(\pi) = 0$ to satisfy the boundary conditions that $u(0,t) = u(\pi,t) = 0$ for all t.

Write $\lambda = -\mu^2$ where μ is not necessarily real.

- When $\lambda \neq 0$, $v'' = \lambda v$ has solution

$$v(t) = c_1 \cos(\mu t) + c_2 \sin(\mu t).$$

- Consider $w'' - \lambda w = 0$ and $w(0) = w(\pi) = 0$.
  - $\lambda = 0$ implies $w(x) = c_3 + c_4 x$ and because of the boundary condition $c_3 = c_4 = 0$. So can assume $\lambda \neq 0$.
  - If $\lambda \neq 0$, solution is $w(x) = c_3 \cos(\mu x) + c_4 \sin(\mu x)$.
    $w(0) = 0 \implies c_3 = 0$
    therefore $w(\pi) = c_4 \sin(\mu \pi) = 0$ implies $\mu = n \in \mathbb{N}$
    [check: μ is non-real $\implies \sin(\mu\pi) \neq 0$]. So

$$w(x) = c_4 \sin(nx) \text{ and } \lambda = -n^2 \text{ and } n \in \mathbb{N} \setminus \{0\}.$$

- So for any $n \in \mathbb{N}$ we obtain solution

  $$u(x,t) = w(x)v(t) = (c_1 \cos(nt) + c_2 \sin(nt)) \sin(nx).$$

- The string can only vibrate with frequencies which are a multiple of $\mathbb{N}$.

So $u(x,t) = \sum_{n \geq 1} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx)$ is solution provided the sum makes sense and is twice differentiable.

Lemma 6. $\sum n^2 |c_{1,n}| < \infty$ and $\sum n^2 |c_{2,n}| < \infty \implies u(x,t) = \sum_{n \geq 1} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx)$ is C^2.

Proof. That $\sum_{n=1}^{N} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx)$ converges follows from

Weierstrass test: if $M_n \geq 0$, $\sum M_n < \infty$ and $u_n \colon [a,b] \to \mathbb{R}$ is continuous with $\sup_{x \in [a,b]} |u_n(x)| \leq M_n$ then $\sum u_n$ converges uniformly on $[a,b]$ (and so the limit is continuous too!). Since the d/dx derivative of

$$\sum_{n=1}^{N} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx)$$

is equal to $\sum_{n=1}^{N} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) n \cos(nx)$, and the latter converges, $u(x,t)$ is differentiable w.r.t. x. $\square$

Next we need to make sure that the boundary conditions are satisfied. The **first boundary condition** is

$$\frac{\partial}{\partial t} u(x,t)|_{t=0} = 0 \text{ for all } x \in [0,\pi] \tag{11}$$

This implies that $\sum c_{2,n} n \sin(nx) \equiv 0 \implies c_{2,n} = 0$ for all $n \geq 0$. So we obtain that a solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_{1,n} \cos(nt) \sin(nx).$$

Note that in these notes, lemmas and theorems are numbered separately. So Theorem 8 follows Theorem 7 not Lemma 7.

The **second boundary condition** is

$$u(x,0)=\sum c_{1,n}\sin(nx)=f(x)\text{ for all }x\in[0,\pi]. \qquad (12)$$

This looks like a Fourier expansion, as in Theorem 8. The reason why it is possible to do this so that only the sin terms appear is explained in the margin. Note that we we to make sure that $u(x,0)=\sum c_{1,n}\sin(nx)=f(x)$ holds uniformly, and that in fact $u(x,t)$ is C^2. To make sure of this, we need to apply Theorem 9 and assume that f is C^3 and $f(0)=f(\pi)=f''(0)=f''(\pi)=0$ the assumptions in Lemma 6 are satisfied. For an explanation why these conditions on f implies $n^2|c_{1,n}|<\infty$, see the proof of Theorem 10 below.

The following theorem is quite straightforward:

Theorem 8. L^2 Fourier Theorem. If $f\colon[0,2\pi]\to\mathbb{R}$ is continuous (or continuous except at a finite number of points) then we one can coefficients $c_{1,n},c_{2,n}$ so that

$$f\sim\sum_{n=0}^{\infty}(c_{1,n}\cos(nx)+c_{2,n}\sin(nx))$$

in the sense that

$$\int_0^{2\pi}|f(x)-\sum_{n=0}^{N}(c_{1,n}\cos(nx)+c_{2,n}\sin(nx))|^2\,dx\to 0$$

as $N\to\infty$.

What we need here is a uniform convergence:

Theorem 9. Fourier Theorem with uniform convergence. Assume $f\colon[0,\pi]\to\mathbb{R}$ is C^2 (twice continuously differentiable) and $f(0)=f(\pi)=0$ then one can find $s_{1,n}$ so that

$$\sum_{n=1}^{N}s_{1,n}\sin(nx)\text{ converges \textbf{uniformly} to }f(x)\text{ as }N\to\infty.$$

Note that if $f\colon[0,\pi]\to\mathbb{R}$ and $f(0)=f(\pi)=0$ then we can define the function $g\colon[0,2\pi]\to\mathbb{R}$ so that $g(x)=f(x)$ for $x\in[0,\pi]$ and $g(x)=f(2\pi-x)$ for $x\in[\pi,2\pi]$. It follows that $g(\pi-x)=-g(\pi+x)$ for $x\in[0,\pi]$. So this means that $\int_0^{2\pi}g(x)\cos(nx)\,dx=0$ and therefore in the Fourier expansion of g the cosine terms vanish, and we have $g(x)=\sum_{n=1}^\infty s_{1,n}\sin(nx)$. In particular $f(x)=\sum_{n=1}^\infty s_{1,n}\sin(nx)$.

We will **not** prove this theorem here, but elaborate some of the ideas in the sketch of the proof of the next theorem (which is not examinable).

Theorem 10. If f is C^3 and $f(0) = f(\pi) = f''(0) = f''(\pi) = 0$, then the assumptions in Lemma 6 are satisfied.

Proof. (Non examinable). Let us assume that f is C^2, $f(0) = f(\pi) = 0$. According to the previous theorem (the Fourier Theorem) one can therefore write $f(x) = \sum_{n\geq 1} s_n \sin(nx)$. Let us now show that if f is C^3 and $f(0) = f(\pi) = f''(0) = f''(\pi) = 0$ the assumptions in Lemma 6 are satisfied, i.e. that f and f' can be written in the form $f(x) = \sum s_n(f') \sin(nx)$ and $f'(x) = \sum c_n(f') \cos(nx)$ and that $\sum n^2 s_n^2 < \infty$ and $\sum n^2 c_n^2 < \infty$. Let us prove that $\sum |s_n| < \infty$. (We change the notation from the coefficients c_n to s_n in the main text since the new notation is more natural here.) This remark and the proof below are **not examinable**, and will given in sketchy form only. First choose constants $s_n(f)$ and $c_n(f')$ so that $f(x) = \sum s_n(f) \sin(nx)$ and $f'(x) = \sum c_n(f') \cos(nx)$ (by the Fourier theorem one can write f' in this way since is C^2 and since $f''(0) = f''(\pi) = 0$). **Step 1:**

clarified the logic here a bit

$$(f', f') = \sum_{n,m\geq 0} c_n(f')c_m(f') \int_0^\pi \cos(nx)\cos(mx) =$$

$$= (\pi/2) \sum_{n\geq 1} |c_n(f')|^2 + \pi |c_0|^2.$$

It follows that $\sum_{n\geq 0} |c_n(f')|^2 < \infty$. **Step 2:** for $n \geq 1$ we have

$$s_n(f) = (2/\pi) \int_0^\pi f(x) \sin(nx)\, dx$$

and

$$c_n(f') = (2/\pi) \int_0^\pi f'(x) \sin(nx)\, dx.$$

Using partial integration on the last expression, and using that $f(0) = f(\pi) = 0$ gives for $n \geq 1$,

$$c_n(f') = (2/\pi)\int_0^\pi f'(x)\cos(nx)\,dx = (2/\pi)[f(x)\cos(nx)]_0^\pi +$$

$$+n(2/\pi)\int_0^1 f(x)\sin(nx)\,dx = (2n/\pi)s_n(f).$$

It follows from this, $f(0) = f(\pi) = 0$ and Step 1 that $\sum n^2|s_n(f)|^2 < \infty$. **Step 3:** Now we use the Cauchy inequality $\sum a_n b_n \leq \sum a_n^2 \sum b_n^2$. Taking $a_n = 1/n$ and $b_n = n|s_n(f)|$ we get that $\sum |s_n(f)| = \sum a_n b_n \leq \sum a_n^2 \sum b_n^2$. By Step 2, $\sum b_n^2 < \infty$ and since $\sum 1/n^2 < \infty$, it follows that $\sum |s_n(f)| < \infty$. In the same way, we can prove that if f is C^3 and $f(0) = f(\pi) = f'(0) = f'(\pi) = f''(0) = f''(\pi) = 0$ then $\sum n^2|s_n(f)| < \infty$. Therefore the assumptions in Lemma 6 are satisfied. If we assume $f(0) = f(\pi) = f''(0) = f''(\pi) = 0$ and consider $g(x) = f(x) - a_1 \sin x - a_2 \sin 2x$ with a_1, a_2 so that $g'(0) = g'(\pi) = 0$ then we can apply the above to g. It follows that $\sum n^2|s_n(g)| < \infty$. This also implies $\sum n^2|s_n(f)| < \infty$. This concludes the explanation of item 2 above Theorem 9. $\square$

In conclusion we get:

- Provided we assume that $f(0) = f(\pi) = f''(0) = f''(\pi)$ and that f is C^3 we can find

  $$u(x,t) = \sum_{n=1}^{\infty} c_{1,n}\cos(nt)\sin(nx)$$

  which is C^2 and solves the wave equation together with the boundary conditions.

- $c_{1,n}$ can be found by methods you have seen before.

- Since $w'' = \lambda w$, one calls λ an *eigenvalue* and w an eigenfunction.

4.2 A typical Sturm-Liouville Problem

In this subsection we will state the Sturm-Liouville Theorem which generalises the previous Fourier theorem.

Let us consider another example:

$$y'' + \lambda y = 0, y(0) + y'(0) = 0, y(1) = 0.$$

- If $\lambda = 0$ then $y(x) = c_1 + c_2 x$ and the boundary conditions give $y(x) = 1 - x$.
- If $\lambda \neq 0$ we write again $\lambda = \mu^2$. The equation $y'' + \lambda y = 0$ gives $y(x) = c_1 e^{i\mu x} + c_2 e^{-i\mu x}$.
- Plugging in $y(0) + y'(0) = 0, y(1) = 0$ gives
- $(c_1 + c_2) + i\mu(c_1 - c_2) = 0$ and $c_1 e^{i\mu} + c_2 e^{-i\mu} = 0$.
- So $c_2 = -c_1 e^{2i\mu}$ and $(1 + i\mu)e^{-i\mu} - (1 - i\mu)e^{i\mu} = 0$. $\tan\mu = \mu$ (see margin). This has infinitely many solutions $\mu_n \in [0, \infty)$, $n = 0, 1, \ldots$ with $\mu_n \to \infty$ and $\mu_n \approx (2n+1)\pi/2$.
- Eigenvalues: $\lambda_n = \mu_n^2 \approx (2n+1)^2(\pi/2)^2$, $n = 0, \ldots$; eigenfunctions: $y_0(x) = 1 - x$ and $y_n(x) = \sin(\sqrt{\lambda_n}(1 - x))$, $n \geq 1$. To see this, note that we have $y(x) = [c_1 e^{i\mu x} + c_2 e^{-i\mu x}] = c_1[e^{i\mu x} - e^{2i\mu} e^{-i\mu x}] = \tilde{c}_1[e^{-i\mu + i\mu x} - e^{i\mu - i\mu x}] = 2\tilde{c}_1 \sin(\mu x - \mu)$. Here $\tilde{c}_1$ is a new (complex) constant. So $y_n(x) = \sin(\mu_n(1 - x)$ is an eigenfunction.

Indeed, $0 = (1 + i\mu)(\cos\mu - i\sin\mu) - (1 - i\mu)(\cos\mu + i\sin\mu) = (2(\mu\cos\mu - \sin\mu)$, so $\tan\mu = \mu$. The easiest way to show that this has only real roots, is by using that eigenvalues are real, see Subsection 4.3. One can prove this also by elementary methods, but this is much more involved. For example, take $\mu = s + it$ with s, t real. Then you $(1 + i\mu)e^{-i\mu} - (1 - i\mu)e^{i\mu} = 0$ can be rewritten as $e^{2is}(\cos 2t + i\sin 2t) = \dfrac{1 + is - t}{1 - is + t}$ and do some geometric calculations...

These are special cases of following type of problem: given functions $p, q, r\colon [a,b] \to \mathbb{R}$ find $y\colon [a,b] \to \mathbb{R}$ so that

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0. \tag{13}$$

Theorem 11. Sturm-Louiville Theorem Assume that $p, r > 0$ are continuous and p is C^1 on $[a,b]$. Then (13) with the boundary conditions (14)

$$\alpha_0 y(a) + \alpha_1 y'(a) = 0, \beta_0 y(b) + \beta_1 y'(b) = 0. \tag{14}$$

(where α_i, β_i are assumed to be real and neither of the vectors $(\alpha_0, \alpha_1), (\beta_0, \beta_1)$ are allowed to be zero) has infinitely many solutions with the following properties:

1. The eigenvalues λ_n are real, distinct and of single multiplicity;

2. The eigenvalues λ_n tend to infinity, so $\lambda_1 < \lambda_2 < \dots$ and $\lambda_n \to \infty$.

3. If $n \neq m$ then corresponding eigenfunctions y_n, y_m are orthogonal in the sense that

$$\int_a^b y_m(x) y_n(x) r(x)\, dx = 0.$$

   In other words, one can find coefficients $c_n, n = 0, 1, 2, \dots$ so that f is the limit of the sequence of functions $\sum_{n=0}^N c_n y_n$. More precisely, if f is merely continuous than this convergence is in the L^2 sense, while if f is C^2 then this convergence is uniform (this sentence is not examinable, and in this course we will not cover a proof of this sentence).

4. Each continuous function can be expanded in terms of the eigenfunctions, as in the Fourier case!!

Let's make two additional remarks:

- Note that if y_n, y_m are solutions and we set

  $$W(y_m, y_n)(x) := \det\begin{pmatrix} y_m(x) & y_m'(x) \\ y_n(x) & y_n'(x) \end{pmatrix} = y_m(x)y_n'(x) - y_n(x)y_m'(x)$$

  then $W(a) = 0$ and $W(b) = 0$. To see this, note that the first boundary condition in equation (14) implies

  $$\begin{pmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = 0.$$

  Since $(\alpha_0, \alpha_1) \neq (0, 0)$ the determinant of the matrix is zero.

- How to find a_n so that $f(x) = \sum_{n\geq 0} a_n y_n(x)$? Just take

  $$\begin{aligned} (f, ry_k) &= (\textstyle\sum_{n\geq 0} a_n y_n(x), ry_k) = \sum_{n\geq 0} a_n (y_n, ry_k) \\ &= a_k (y_k, ry_k). \end{aligned}$$

  Here we used in the last equality that $(y_n, ry_k) \neq 0$ implies $n = k$. Hence

  $$a_k := \frac{(f, ry_k)}{(y_k, ry_k)}$$

  where (v, w) is the inner product: $(v, w) = \int_a^b v(t)\overline{w}(t)\, dt$.

4.3 A glimpse into symmetric operators

- Sturm-Liouville problems are solved using some operator theory: the 2nd order differential equation is equivalent to

  $$Ly(x) = \lambda r(x) y(x) \text{ where } L = \left(-\frac{d}{dx} p(x) \frac{d}{dx} - q(x)\right).$$

  This turns out to be a symmetric operator in the sense that $(Lv, w) = (v, Lw)$ where $(v, w) = \int_a^b v(x)\overline{w(x)}\, dx$ is as defined above.

- The situation for analogous to the finite dimensional case:
- L is a symmetric (and satisfies some additional properties) $\Longrightarrow$ its eigenvalues are real, and its eigenfunctions form a basis.

L **is symmetric (self-adjoint) on the space of functions satisfying the boundary conditions.** Let $Lu = -(pu')' - qu$ and $Lv = -(pv')' - qv$.

$\int_a^b L(u)\bar{v}\, dx = \int_a^b [-(pu')'\bar{v} - qu\bar{v}]\, dx$. $\int_a^b u\overline{L(v)}\, dx = \int_a^b [-u(p\bar{v}')' - qu\bar{v}]\, dx$.

$$\begin{aligned}\int_a^b & -(pu')'\bar{v}\, dx = -pu'\bar{v}|_a^b + \int_a^b pu'\bar{v}'\, dx \\ &= -pu'\bar{v}|_a^b + pu\bar{v}'|_a^b - \int_a^b u(p\bar{v}')'\, dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_a^b [L(u)\bar{v} - & u\overline{L(v)}]\, dx = -p(x)[u'\bar{v} - u\bar{v}']\Big|_a^b \\ &= -\left[p(b)W(u,\bar{v})(b) - p(a)W(u,\bar{v})(a)\right]\end{aligned}$$

If u, v satisfy the boundary conditions, then $\alpha_0 u(a) + \alpha_1 u'(a) = 0$ and $\alpha_0 v(a) + \alpha_1 v'(a) = 0$. Since α_0, α_1 are real, therefore $\alpha_0\bar{v}(a) + \alpha_1\bar{v}'(a) = 0$. Hence, $W(u,\bar{v})(a) = W(u,\bar{v})(b) = 0$, and therefore

$$\int_a^b [L(u)\bar{v} - u\overline{L(v)}]\, dx = 0.$$

In other words, L is **self-adjoint** is on the space H of functions satisfying the boundary conditions. That is,

$$\int_a^b [L(u)\bar{v} - u\overline{L(v)}]\, dx = 0,\ \ i.e.\ (Lu, v) = (u, Lv).$$

Proof that eigenvalues are real and orthogonality of eigenfunctions: Define $(u, v) = \int_a^b u(x)\overline{v(x)}\,dx$. Then the paragraph showed $(Lu, v) = (u, Lv)$.

- Suppose that $Ly = r\lambda y$. Then the eigenvalue λ is real: Indeed,

  $$\lambda(ry, y) = (\lambda ry, y) = (Ly, y) = (y, Ly) = \bar{\lambda}(y, ry) = \bar{\lambda}(ry, y)$$

  since r is real. Since $(ry, y) > 0$ it follows that $\lambda = \bar{\lambda}$.

- Suppose that $Ly = r\lambda y$ and $Lz = r\mu z$.
  $\lambda \neq \mu \implies \int_a^b r(x)y(x)\overline{z(x)}\,dx = (ry, z) = 0.$
  So the eigenfunctions y, z are orthogonal. Indeed,

  $$\begin{aligned} \lambda(ry, z) &= (\lambda ry, z) = (Ly, z) = (y, Lz) = (y, \mu rz) \\ &= \bar{\mu}(y, rz) = \bar{\mu}(ry, z) = \mu(ry, z). \end{aligned}$$

  where we have used that r and μ are real. Since $\lambda \neq \mu$ it follows that $(ry, z) = 0$.

Remark 1. In this remark (which is not examinable) we discuss what is required for the proof of the above theorem:

- to define Hilbert space H: this is a Banach space with an inner product for which $(v, w) = \overline{(w, v)}$ where $\overline{z}$ is complex conjugation;

- to define the norm $||v|| = \sqrt{(v, v)}$ (generalizing $||z|| = \sqrt{(z, z)} = \sqrt{z\overline{z}}$ on $\mathbb{C}$); Note $||v|| = \sqrt{\int_a^b |v(x)|^2\,dx}$, the so-called L^2 norm. typo corrected

- to associate to a linear $A\colon H \to H$ the operator norm $||A|| = \sup_{f \in H, ||f||=1} ||Af||$;

- to call a linear operator A is **compact** if for each sequence $||f_n|| \leq 1$, there exists a convergent subsequence of Af_n.
- to show that if $A\colon H \to H$ is compact, then there exists a sequence of eigenvalues $\alpha_n \to 0$ and eigenfunctions u_n. These eigenvalues are all real and the eigenfunctions are orthogonal. If the closure of $A(H)$ is equal to H, then for each $f \in H$ then one can write $f = \sum_{j=0}^{\infty}(u_j, f)u_j$.
- The operator L in Sturm-Liouville problems is **not** compact, and that is why one considers some related operator (the resolvent).
- This related operator is compact.
- The above theorem then follows.

The Sturm-Liouville Theorem is fundamental in

- quantum mechanics;
- in large range of boundary value problems;
- and related to geometric problems describing properties of geodesics.

4.4 Oscillatory equations

Consider $(py')' + ry = 0$ where $p > 0$ and C^1 as before.

Theorem 12. Let y_1, y_2 be solutions. Then the Wronskian $x \mapsto W(y_1, y_2)(x) := y_1(x)y_2'(x) - y_2(x)y_1'(x)$ has constant sign.

Proof. $(py_1')' + ry_1 = 0$ and $(py_2')' + ry_2 = 0$. Multiplying the first equation by y_2 and the second one by y_1 and subtract:

$$0 = y_2(py_1')' - y_1(py_2')' = y_2p'y_1' + y_2py_1'' - y_1p'y_2' - y_1py_2''.$$

Differentiating W and substituting the last equation in

$$pW' = p[y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1''] = py_1y_2'' - py_2y_1''$$

gives

$$pW' = -p'W.$$

This implies that if $W(x) = 0$ for some $x \in [a, b]$ then $W(x) = 0$ for all $x \in [a, b]$. □

Lemma 7. $W(y_1, y_2) \equiv 0 \implies \exists c \in \mathbb{R}$ with $y_1 = cy_2$ (or $y_2 = 0$).

Proof. Since $W(y_1, y_2) = 0$, $y_2 \neq 0$ implies (y_1, y_1') is a multiple of (y_2, y_2'). Can this multiple depend on x? No: if

$$\begin{aligned} y_1(x) &= c(x)y_2(x) \text{ and } y_1'(x) = c(x)y_2'(x) \ \ \forall x \\ &\implies c(x)y_2'(x) = y_1'(x) = c'(x)y_2(x) + c(x)y_2'(x) \ \ \forall x. \end{aligned}$$

Hence $c' \equiv 0$. □

Theorem 13. Sturm Separation Theorem Let y_1, y_2 be two solutions which are independent (one is not a constant multiple of the other). Then zeros are interlaced: between consecutive zeros of y_1 there is a zero of y_2 and vice versa.

Proof. Assume $y_1(a) = y_1(b) = 0$. $y_1'(a) \neq 0$ (otherwise $y_1 \equiv 0$) and $y_2'(b) \neq 0$. We may choose a, b so that $y_1(x) > 0$ for $x \in (a, b)$. Then $y_1'(a)y_1'(b) < 0$. (Draw a picture.) Also,

$$W(y_1, y_2)(a) = -y_2(a)y_1'(a) \text{ and } W(y_1, y_2)(b) = -y_2(b)y_1'(b).$$

Since $y_1'(a)y_1'(b) < 0$ and $W(a)W(b) > 0$ (W does not change sign), we get $y_2(a)y_2(b) < 0$, which implies that y_2 has a zero between a and b. □

5 Calculus of Variations

Many problems result in differential equations. In this chapter we will consider the situation where these arise from a minimisation (variational) problem. Specifically, the problems we will consider are of the type

- Minimize

$$I[y] = \int_0^1 f(x, y(x), y'(x))\, dx \tag{15}$$

  where f is some function and y is an unknown function.

- Minimize (15) conditional to some restriction of the type $J[y] = \int_0^1 f(x, y(x), y'(x))\, dx = 1$.

5.1 Examples (the problems we will solve in this chapter):

Example 25. Let $A = (0, 0)$ and $B = (1, 0)$ with $l, b > 0$ and consider a path of the form $[0, 1] \ni\mapsto c(t) = (c_1(t), c_2(t))$, connecting A and B. What is the shortest path?

Task: Choose $[0, 1] \ni t \mapsto c(t) = (c_1(t), c_2(t))$ with $c(0) = (0, 0)$ and $c(1) = (1, 0)$ which minimises

$$L[c] = \int_0^1 \sqrt{c_1'(t)^2 + c_2'(t)^2}\, dt.$$

Of course this is a line segment, but how to make this precise?

If we are not in a plane, but in a surface or a higher dimensional set, these shortest curves are called **geodesics**, and these are studied extensively in mathematics.

Example 26. Let $A = (0, 0)$ and $B = (l, -b)$ with $l, b > 0$ and consider a path of the form $(x, y(x))$, $x \in [0, l]$, connecting

A and B. Take a ball starting at A and rolling along this path under the influence of gravity to B. Let T be the time this ball will take. Which function $x \mapsto y(x)$ which will minimise T?

The sum of kinetic and potential energy is constant

$$(1/2)mv^2 + mgh = const.$$

Since the ball rolls along $(x, y(x))$ we have $v(x) = \sqrt{-2gy(x)}$. Let $s(t)$ be the length travelled at time t. Then $v = ds/dt$. Hence $dt = ds/v$ or

$$T[y] := \int_0^l \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-2gy(x)}}\, dx.$$

Task: minimise $T[y]$ within the space of functions $x \mapsto y(x)$ for which y and y' continuous and $y(0) = 0$ and $y(l) = -b$. This is called the **Brachisotochrome**, going back to Bernouilli in 1696.

Example 27. Take a closed curve in the plane without self-intersections and of length one. What is the curve c which maximises the area D it encloses? Again, let $[0, 1] \ni\mapsto c(t) = (c_1(t), c_2(t))$ with $c(0) = c(1)$ and so that $s, t \in [0, 1)$ and $s \neq t$ implies $c(s) \neq c(t)$.

The length of the curve is again $L[c] = \int_0^1 \sqrt{c_1'(t)^2 + c_2'(t)^2}\, dt$. To compute the area of D we use the Green theorem:

$$\int_c P dx + Q dy = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx dy$$

Take $P \equiv 0$ and $Q = x$. Then

$$\int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx dy = \int\int_D 1\, dx dy = \text{ area of } D.$$

So

$$A[c] = \int\int_D 1\, dx dy = \int_c x dy = \int_0^1 c_1(t) c_2'(t)\, dt.$$

This is an *isoperimetric problem:* find the supremum of $A[c]$ given $L[c] = 1$.

5.2 Extrema in the finite dimensional case

We say that $F\colon \mathbb{R}^n \to \mathbb{R}$ take a local minimum at $\tilde{x}$ if there exists $\delta > 0$ so that

$$F(x) \geq F(\tilde{x}) \text{ for all } x \text{ with } |x - \tilde{x}| < \delta.$$

Theorem 14. Assume that F is differentiable at a and also has a minimum at $\tilde{x}$ then $DF(\tilde{x}) = 0$.

Proof. Let us first assume that $n = 1$. Then that f has a minimum means that $F(\tilde{x} + h) - F(\tilde{x}) \geq 0$ for all h near zero. Hence

$$\frac{F(\tilde{x} + h) - F(\tilde{x})}{h} \geq 0 \text{ for } h > 0 \text{ near zero and}$$

$$\frac{F(\tilde{x} + h) - F(\tilde{x})}{h} \leq 0 \text{ for } h < 0 \text{ near zero.}$$

Therefore

$$F'(\tilde{x}) = \lim_{h \to 0} \frac{F(\tilde{x} + h) - F(\tilde{x})}{h} = 0$$

Let us consider the case that $n > 1$ and reduce to the case that $n = 1$. So take a vector v at $\tilde{x}$, define $l(t) = \tilde{x} + tv$ and $g(t) := F \circ l(t)$. So we can use the first part of the proof and thus we get $g'(0) = 0$. Applying the chain rule $0 = g'(0) = Dg(0) = DF(l(0))Dl(0) = DF(\tilde{x})v$ and so

$$\frac{\partial F}{\partial x_1}(\tilde{x})v_1 + \cdots + \frac{\partial F}{\partial x_n}(\tilde{x})v_n = 0.$$

Hence $DF(\tilde{x})v = 0$ where $DF(\tilde{x})$ is the Jacobian matrix at $\tilde{x}$. Since this holds for all v, we get $DF(\tilde{x}) = 0$. $\square$

Remember we also wrote sometimes $DF_{\tilde{x}}$ for the matrix $DF(\tilde{x})$ and that $DF(\tilde{x})v$ is the *directional derivative* of f at $\tilde{x}$ in the direction v.

5.3 The Euler-Lagrange equation

The infinite dimensional case: the Euler-Lagrange equation

- In the infinite dimensional case, we will take $F\colon H \to \mathbb{R}$ where H is some function space. The purpose of this chapter is to generalise the previous result to this setting, and show that the solutions of this problem gives rise to differential equations.

- Mostly the function space is the space $C^1[a,b]$ of C^1 functions $y\colon [a,b] \to \mathbb{R}^n$. This space is an infinite dimensional vector space (in fact, a Banach space) with norm $|y|_{C^1} = \sup_{x\in[a,b]}(|y(x)|, |Dy(x)|)$.

- Choose some function $f\colon [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$. Take $(x,y,y') \in [a,b]\times\mathbb{R}^n\times\mathbb{R}^n$ **denote by $f_y, f_{y'}$ the corresponding partial derivatives**. So $f_y(x,y,y')$ and $f_{y'}(x,y,y')$ vectors. Attention: here y and y' are just the names of vectors in $\mathbb{R}^n$ (and not - yet - functions or derivatives of functions).

- Here f_y is the part of the $1\times(1+n+n)$ vector Df which concerns the y derivatives.

Assume $f\colon [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ with $f_y, f_{y'}$ continuous and define $I\colon C^1[a,b] \to \mathbb{R}$ by,

$$I[y] = \int_a^b f(x, y(x), y'(x))\,dx.$$

Given $\tilde{y}\colon [a,b] \to \mathbb{R}^n$, let's denote

$f_y[\tilde{y}](x) = f_y(x, \tilde{y}(x), \tilde{y}'(x))$ and $f_{y'}[\tilde{y}](x) = f_{y'}(x, \tilde{y}(x), \tilde{y}'(x))$

where $f_y, f_{y'}$ are the corresponding partial derivatives of f. Fix $y_a, y_b \in \mathbb{R}^n$ and define

$$\mathcal{A} = \{y;\ y\colon [a,b] \to \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b\}.$$

Theorem 15. If $\mathcal{A} \ni y \mapsto I[y]$ has a minimum at $\tilde{y}$ then

1. for every $v \in C^1[a,b]$ with $v(a) = v(b) = 0$ we get $\int_a^b (f_y[\tilde{y}] \cdot v + f_{y'}[\tilde{y}]v')\, dx = 0$.
2. $f_{y'}[\tilde{y}]$ exists, is continuous on $[a,b]$ and
$$\frac{d}{dx} f_{y'}[\tilde{y}] = f_y[\tilde{y}].$$

Proof. Remember that

$$\mathcal{A} = \{y;\ y\colon [a,b] \to \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b\}.$$

Hence $v \in C^1[a,b]$ with $v(a) = v(b) = 0$, then $y + hv \in \mathcal{A}$ for each h. So the space $\mathcal{A}$ is affine.

Assume that $I\colon C^1[a,b] \to \mathbb{R}$ has a minimum at $\tilde{y}$, which means that

$$I[\tilde{y} + hv] \geq I[\tilde{y}] \ \forall v \in C^1[a,b], v(a) = v(b) = 0 \ \forall h \in \mathbb{R}$$

$$I[\tilde{y} + hv] - I[\tilde{y}] =$$

$$= \int_a^b f(x, (\tilde{y} + hv)(x), (\tilde{y} + hv)'(x)) - f(x, \tilde{y}(x), \tilde{y}'(x))dx.$$

By Taylor's Theorem,

$$f(x, (\tilde{y} + hv)(x), (\tilde{y} + hv)'(x)) - f(x, \tilde{y}(x), \tilde{y}'(x)) =$$
$$f_y[\tilde{y}]hv + f_{y'}[\tilde{y}]hv' + o(h).$$

So

$$I[\tilde{y} + hv] - I[\tilde{y}] = h \cdot \left[\int_a^b \left[f_y[\tilde{y}]v + f_{y'}[\tilde{y}]v'\right]\, dx\right] + o(h).$$

Hence a necessary condition for $\tilde{y}$ to be a minimum of I is

$$\int_a^b \left[f_y[\tilde{y}]v + f_{y'}[\tilde{y}]v'\right] \, dx = 0$$

for each $v \in C^1[a,b]$ with $v(a) = v(b) = 0$.

Partial integration gives

$$\int_a^b f_{y'}[\tilde{y}]v' \, dx = \left(f_{y'}[\tilde{y}]v\right)\Big|_a^b - \int_a^b \frac{d}{dx} f_{y'}[\tilde{y}]v \, dx.$$

Remember $v(a) = v(b) = 0$, so $\left(f_{y'}[\tilde{y}]v\right)\Big|_a^b = 0$. Therefore a necessary condition for $\tilde{y}$ to be a minimum of I is:

$$v \in C^1[a,b] \quad \text{with } v(a) = v(b) = 0 \implies \int_a^b \left[f_y[\tilde{y}] - \frac{d}{dx} f_{y'}[\tilde{y}]\right] v \, dx = 0.$$

This prove first assertion of Theorem and also the 2nd assertion because of the following lemma: $\square$

Lemma 8. If $G\colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^b Gv \, dx = 0$ for each $v \in C^1[a,b]$ with $v(a) = v(b) = 0$, then $G \equiv 0$.

Proof. If $G(x_0) > 0$ then $\exists \delta > 0$ so that $G(x) > 0$, $\forall x$ with $|x - x_0| < \delta$. Choose $v \in C^1[a,b]$ with $v(a) = v(b) = 0$, so that $v > 0$ on $x \in (x_0 - \delta, x_0 + \delta) \cap (a,b)$ and zero outside. Then $\int_a^b G(x)v(x) \, dx > 0$. $\square$

Quite often x does not appear in f. Then it is usually more convenient to rewrite the Euler-Lagrange equation:

Lemma 9. If x does not appear explicitly in f, then $\dfrac{d}{dx} f_{y'}[\tilde{y}] = f_y[\tilde{y}]$ implies $f_{y'}[\tilde{y}]\tilde{y}' - f[\tilde{y}] = C$.

Proof.

$$\begin{aligned}\frac{d}{dx}(f_{y'}[\tilde{y}]\tilde{y}' - f[\tilde{y}]) &= (\frac{d}{dx}f_{y'}[\tilde{y}])\tilde{y}' + f_{y'}[\tilde{y}]\tilde{y}'' \\ &\quad -(f_x[\tilde{y}] + f_y[\tilde{y}]\tilde{y}' + f_{y'}[\tilde{y}]\tilde{y}'') \\ &= y'\left\{\frac{d}{dx}f_{y'}[\tilde{y}] - f_y[\tilde{y}]\right\} - f_x[\tilde{y}].\end{aligned}$$

Since $f_x = 0$, and by the E-L equation, the term $\{\cdot\} = 0$ this gives the required result. $\square$

Example 28. Shortest curve connecting two points $(0,0)$ and $(1,0)$. Let us consider curves of the form $x \mapsto (x, y(x))$ and minimise the length: $I[c] = \int_a^b \sqrt{1 + y'(x)^2}\, dx$. The Euler-Lagrange equation is $\frac{d}{dx}f_{y'}[\tilde{y}] = f_y[\tilde{y}] = 0$. Note $f_{y'} = \frac{y'}{\sqrt{1+(y')^2}}$, so $\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = 0$. Hence the EL equation gives $\frac{\tilde{y}'}{\sqrt{1+(\tilde{y}')^2}} = C$. This means that $\tilde{y}' = C_1$. Hence $\tilde{y}(x) = C_1 x + C_2$. With the boundary conditions this gives $\tilde{y}(x) = 0$.

5.4 The brachistochrone problem

Example 29. (See Example 26) The curve $x \to (x, y(x))$ connecting $(0,0)$ to $(l, -b)$ with the shortest travel time **(brachistochrone)**. Then $f(x, y, y') = \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}$. Since $y < 0$, we orient the vertical axis downwards, that is we write $z = -y$ and $z' = -y'$, i.e. take $f(x, z, z') = \frac{\sqrt{1+(z')^2}}{\sqrt{2gz}}$. Note that

$$f_{z'} = (1/2)\frac{1}{\sqrt{1+(z')^2}\sqrt{2gz}}2z'.$$

The EL equation from the previous lemma gives $f'_{z'}[\tilde{z}]\tilde{z}' - f[\tilde{z}] = const$, i.e. (writing z instead of $\tilde{z}$):

$$\frac{(z')^2}{\sqrt{1+(z')^2}\sqrt{z}} - \frac{\sqrt{1+(z')^2}}{\sqrt{z}} = const.$$

Rewriting this gives

$$z[1+(z')^2] = const.$$

Rewriting this again gives the differential equation

$$\frac{dz}{dx} = \sqrt{\frac{C-z}{z}} \text{ or } \frac{dx}{dz} = \sqrt{\frac{z}{C-z}}$$

with $C > 0$. As usual we solve this by writing $dx = \sqrt{\frac{z}{C-z}}dz$ and so

$$x = \int \sqrt{\frac{z}{C-z}}\,dz.$$

Substituting $z = C\sin^2(s)$, where $s \in [0,\pi]$, gives

$$x = \int \sqrt{\frac{\sin^2(s)}{1-\sin^2(s)}}(2C)\sin(s)\cos(s)\,ds =$$

$$2C\int \sin^2(s)\,dt = C\int(1-\cos(2s))\,dt = (C/2)(2s-\sin(2s))+A$$

Since the curve starts at $(0,0)$ we have $A = 0$.

So we get

$$\begin{aligned} x(s) &= \frac{C}{2}(2s-\sin(2s)), \\ z(s) &= C\sin^2(s) = \frac{C}{2}(1-\cos(2s)). \end{aligned} \tag{16}$$

Here we choose C so that $z = b$ when $x = L$. This is called a cycloid, an evolute of the circle. This is the path of a fixed point on a bicycle wheel, as the bicycle is moving forward.

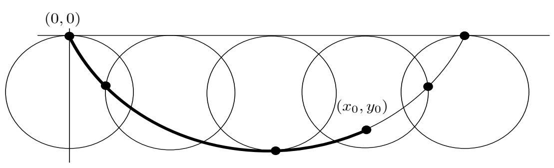
Substituting $2s$ to ϕ and taking $a = C/2$ we get

$$\begin{aligned} x(\phi) &= a(\phi - \sin(\phi)), \\ z(\phi) &= a(1 - \cos(\phi)). \end{aligned} \tag{17}$$

What is a? Given $L = x_0$ and $-b = y_0$ we need to choose a, ϕ so that $x(\phi) = L$ and $z(\phi) = b$. This amounts two equations and two unknowns.

Two special cases:

- The right endpoint is $(L, 0)$, the top of the curve: then take $\phi = 2\pi$ and we have $x(2\pi) = 2\pi a$ and $z(2\pi) = a$.
- The right endpoint is $(L, 2a)$ and this is the bottom of the curve: then $\phi = \pi$ and $x(\pi) = a\pi$ and $y(\pi) = 2a$.



A remarkable property of the brachistochrone: Take an initial point $(\hat{x}, \hat{y})$ on this curve, and release it from rest. Then the time to hit the lower point of the curve is independent of the choice of the initial point!!!

Theorem 16. For any initial point $(\hat{x}, \hat{y})$ (i.e. for any initial $\hat{\phi}$)

$$T = \int_{\hat{x}}^{L} \sqrt{\frac{1 + (z')^2}{2g(z - z_0)}}\, dx = \sqrt{\frac{a}{g}} \int_{\phi=\hat{\phi}}^{\pi} \sqrt{\frac{1 - \cos(\phi)}{\cos(\hat{\phi}) - \cos(\phi)}}\, d\phi$$

is equal to = $\pi\sqrt{a/g}$. Wow!

Proof. Not examinable Let us first show the integrals are equal:

$$x(\phi) = a(\phi - \sin(\phi)), z(\phi) = a(1 - \cos(\phi)) \implies$$

$$z' = \frac{dz}{dx} = \frac{\frac{dz}{d\phi}}{\frac{dx}{d\phi}} = \frac{a\sin(\phi)}{a(1-\cos(\phi))} \implies$$

$$\sqrt{1+(z')^2} = \sqrt{\frac{(1-\cos(\phi))^2+\sin^2(\phi)}{(1-\cos(\phi))^2}} = \sqrt{\frac{2(1-\cos(\phi))}{(1-\cos(\phi))^2}}.$$

$\square$

$$x(\phi) = a(\phi - \sin(\phi)), z(\phi) = a(1-\cos(\phi)) \implies$$

$$z' = \frac{dz}{dx} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}} = \frac{a\sin(\phi)}{a(1-\cos(\phi))} \implies$$

$$\sqrt{1+(z')^2} = \sqrt{\frac{(1-\cos(\phi))^2+\sin^2(\phi)}{(1-\cos(\phi))^2}} = \sqrt{\frac{2(1-\cos(\phi))}{(1-\cos(\phi))^2}}.$$

Since $dx = a(1-\cos(\phi))\,d\phi$ this gives

$$\sqrt{\frac{1+(z')^2}{2g(z-z_0)}}dx = \frac{\sqrt{a}}{\sqrt{g}}\sqrt{\frac{1-\cos(\phi)}{\cos(\hat{\phi})-\cos(\phi)}}\,d\phi.$$

Showing the two integrals the same.

Claim: the following integral does not depend on $\hat{\phi}$:

$$\int_{\phi=\hat{\phi}}^{\pi} \sqrt{\frac{1-\cos(\phi)}{\cos(\hat{\phi})-\cos(\phi)}}\,d\phi$$

Substitute $\sin(\phi/2) = \sqrt{1-\cos\phi}/\sqrt{2}$ and $\cos\phi = 2\cos^2(\phi/2) - 1$ gives:

$$\sqrt{\frac{1-\cos(\phi)}{\cos(\hat{\phi})-\cos(\phi)}} = \sqrt{2}\frac{\sin(\phi/2)}{\sqrt{2[\cos^2(\hat{\phi}/2)-\cos^2(\phi/2)]}}$$

Substitute $u = \cos(\phi/2)/\cos(\hat{\phi}/2)$, then as ϕ varies between $[\hat{\phi}, \pi]$ then u varies from 1 to 0.

$$\int_{\hat{\phi}}^{\pi} \frac{\sin(\phi/2)}{\sqrt{\cos^2(\hat{\phi}/2) - \cos^2(\phi/2)}}\, d\phi$$

Substitute $u = \cos(\phi/2)/\cos(\hat{\phi}/2)$ gives

$$\frac{\sin(\phi/2)}{\sqrt{\cos^2(\hat{\phi}/2) - \cos^2(\phi/2)}} = \frac{\sin(\phi/2)}{\cos(\hat{\phi}/2)\sqrt{1-u^2}}.$$

Since $du = -(1/2)\dfrac{\sin(\phi/2)}{\cos(\hat{\phi}/2)}\, d\phi$ and since u varies from 1 to 0 the integral is equal to

$$\int_0^1 \frac{2}{\sqrt{1-u^2}}\, du = 2\arcsin(u)\Big|_0^1 = \pi$$

So the time to decent from any point is $\pi\sqrt{a/g}$.
For history and some movies about this problem:

- http://www.sewanee.edu/physics/TAAPT/TAAPTTALK.html
- http://www-history.mcs.st-and.ac.uk/HistTopics/Brachistochrone.html
- http://www.youtube.com/watch?v=li-an5VUrIA
- http://www.youtube.com/watch?v=gb81TxF2R_4&hl=ja&gl=JP
- http://www.youtube.com/watch?v=k6vXtjne5-c
- Check out this book: Nahin: When Least Is Best. Great book!

5.5 Are the critical points of the functional I minima?

Are the critical points of I minima?

- In general we **cannot** guarantee that the solutions of the Euler-Lagrange equation gives a minimum.
- This is of course is not surprising: a minimum $\tilde{x}$ of $F\colon \mathbb{R}^n \to \mathbb{R}$ satisfies $DF(\tilde{x}) = 0$, but the latter condition is not enough to guarantee that $\tilde{x}$ is a minimum.
- It is also not always the case that a functional of the form $I[y] = \int_a^b f(x, y(x), y'(x))\,dx$ over the set $\mathcal{A} = \{y;\ y\colon [a,b] \to \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b\}$ does have a minimum.
- Additional considerations are often required.

5.6 Constrains in finite dimensions

Often one considers problems where one has a constraint. Let us first consider this situation in finite dimensions:

5.6.1 Curves, surfaces and manifolds

Definition: We define a subset M of $\mathbb{R}^n$ to be a **manifold** (of codimension k) if $M = \{x \in \mathbb{R}^n; g(x) = 0\}$ where $g\colon \mathbb{R}^n \to \mathbb{R}^k$ and $k < n$ where the matrix $Dg(x)$ has rank k for each $x \in M$.

Remark: There are other, equivalent, definitions of manifolds and also some more general definitions of the notion of a manifold, but this goes outside the scope of this course.

Theorem 17. Let $M \subset \mathbb{R}^n$ be a manifold of codimension k. Then near every $x \in M$ one can write M as the graph of a function of $(n-k)$ of its coordinates.

Examples. $M = \{(x,y) \in \mathbb{R}^2; x^2+y^2=1\}$ can be described locally in the form $x \mapsto (x, y(x))$ or in the form $y \mapsto (x(y), y)$.

Proof. Consider $x_0 \in M$ and for simplicity assume that the last k columns of the $k \times n$ matrix $Dg(x)$ are linearly independent. Then the $k \times k$ matrix made up of the last k columns of the $k \times n$ matrix $Dg(x)$ is invertible. This puts us in the position of the Implicit Function Theorem. Indeed, write $x = (u,v) \in \mathbb{R}^{n-k} \oplus \mathbb{R}^k$. The Implicit Function Theorem implies that there exists a function $G\colon \mathbb{R}^{n-k} \to \mathbb{R}^k$ so that

$$g(u,v) = 0 \iff v = G(u).$$

So M is locally a graph of a function G: the set is locally of the form $M = \{(u, G(u)); u \in \mathbb{R}^{n-k}\}$. (If some other combination of columns of $Dg(x)$ are linearly independent then we argue similarly.) $\square$

Examples:

- Assume that $g\colon \mathbb{R}^3 \to \mathbb{R}$ and consider $M = \{x \in \mathbb{R}^n; g(x) = 0\}$. Moreover assume that $Dg(x) \neq 0$ for each $x \in M$. Then M is a surface. Any (orientable) surface can be written in this form.

- The set $x^2+2y^2=1$, $x^2+y^4+z^6=1$ is a codimension-two manifold (i.e. a curve) in $\mathbb{R}^3$.

Definition: The **tangent plane** at $\hat{x} \in M$ is defined as the collection of vectors $v \in \mathbb{R}^n$ (based at $\hat{x}$) so that $Dg_{\hat{x}}(v) = 0$.

To motivate this definition consider a C^1 curve $\gamma\colon [0,1] \to M \subset \mathbb{R}^n$ with $\gamma(0) = \hat{x}$. Since $\gamma(t) \in M$, it follows that $g \circ \gamma(t) = 0$ for all t and therefore

$$\frac{\partial g}{\partial x_1}(\hat{x})\gamma_1'(0) + \cdots + \frac{\partial g}{\partial x_n}(\hat{x})\gamma_n'(0) = 0.$$

This if we write $v = \gamma'(0)$ then $Dg(\hat{x})v = 0$. Hence $0 = Dg(\hat{x})v = \nabla g(\hat{x}) \cdot v$ where $\cdot$ is the usual dot product in $\mathbb{R}^n$. So the vector $\nabla g(\hat{x})$ is orthogonal to $v := \gamma'(0)$ for **each** such curve γ.

5.6.2 Minima of functions on constraints (manifolds)

Suppose $\tilde{x}$ is minimum of $F\colon M \to \mathbb{R}$ where $M = \{x \in \mathbb{R}^n; g(x) = 0\}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$. What does this imply? Write $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{n-1}, \tilde{x}_n)$.

Theorem 18. (Lagrange multiplier) If $Dg(\tilde{x}) \neq 0$ and $\tilde{x}$ is minimum of $F\colon M \to \mathbb{R}$, then $\exists \lambda \in \mathbb{R}$ with $DF(\tilde{x}) = \lambda Dg(\tilde{x})$.

Proof. Since $Dg(\tilde{x}) \neq 0$, we get that $\dfrac{\partial g}{\partial x_i}(\tilde{x}) \neq 0$ for some $i = 1, \dots, n$. In order to be definite assume $\dfrac{\partial g}{\partial x_n}(\tilde{x}) \neq 0$ and write $\tilde{w} = (\tilde{x}_1, \dots, \tilde{x}_{n-1})$. By the Implicit Function Theorem, locally near $\tilde{w}$ there exits h so that $g(x) = 0 \iff x_n = h(x_1, \dots, x_{n-1})$. So $\tilde{w}$ is minimum of $(x_1, \dots, x_{n-1}) \mapsto F \circ (x_1, x_2, \dots, x_{n-1}, h(x_1, \dots, x_{n-1}))$. This means for all $i = 1, \dots, n-1$:

$$\frac{\partial F}{\partial x_i}(\tilde{x}) + \frac{\partial F}{\partial x_n}(\tilde{x}) \frac{\partial h}{\partial x_i}(\tilde{w}) = 0.$$

Since $g(x_1, \dots, x_{n-1}, h(x_1, \dots, x_{n-1})) = 0$ we also get

$$\frac{\partial g}{\partial x_i}(\tilde{x}) + \frac{\partial g}{\partial x_n}(\tilde{x}) \frac{\partial h}{\partial x_i}(\tilde{w}) = 0 \ \forall i = 1, \dots, n-1.$$

Substituting these into the previous equation and writing

$$\lambda = \frac{\dfrac{\partial F}{\partial x_n}(\tilde{x})}{\dfrac{\partial g}{\partial x_n}(\tilde{x})}$$

gives

$$\frac{\partial F}{\partial x_i}(\tilde{x}) - \lambda \frac{\partial g}{\partial x_i}(\tilde{x}) = 0 \ \forall i = 1, \dots, n-1.$$

(For $i = n$ the last equation also holds, by definition.) $\square$

5.7 Constrained Euler-Lagrange Equations

Let $I[y] = \int_a^b f(x, y(x), y'(x))\, dx$ and $J[y] = \int_a^b g(x, y(x), y'(x))\, dx$ be functionals on

$$\mathcal{A} = \{y;\ y\colon [a,b] \to \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b\}.$$

as before. Define

$$M = \{y;\ y \in \mathcal{A} \text{ with } J[y] = 0\}.$$

Theorem 19. If $M \ni y \mapsto I[y]$ has a minimum at $\tilde{y}$ then there exists $\lambda \in \mathbb{R}$ so that the E-L condition hold for $F = f - \lambda g$. That is,

$$\frac{d}{dx} F_{y'}[\tilde{y}] = F_y[\tilde{y}].$$

The idea of the proof combines the Lagrange multiplier approach with the proof of the previous Euler Lagrange theorem.

Example 30. Maximize the area bounded between the graph of y and the line segment $[-1, 1] \times \{0\}$, conditional on the length of the arc being L. (This is a special case of Dido's problem.)

Let $\mathcal{A}$ be the set of C^1 functions $y\colon [-1,1] \to \mathbb{R}$ with $y(-1) = y(1) = 0$. Fix $L > 0$ and let

$$I[y] = \int_1^1 y(x)\, dx \text{ and } J[y] = \int_{-1}^1 \sqrt{1 + (y')^2}\, dx - L = 0.$$

Write

$$f = y, g = \sqrt{1+(y')^2}, F = f - \lambda g = y - \lambda\sqrt{1+(y')^2}.$$

The Euler Lagrange equation in the version of Lemma 9 gives $F_{y'}[\tilde{y}]\tilde{y}' - F[\tilde{y}[= C$ which amounts to (writing y instead of $\tilde{y}$):

$$\frac{-\lambda(y')^2}{\sqrt{1+(y')^2}} - [y - \lambda\sqrt{1+(y')^2}] = C.$$

Rewriting this gives

$$1 = \frac{(y+C)^2}{\lambda^2}(1+(y')^2).$$

Substituting $y + C = \lambda\cos\theta$ gives $y' = -\lambda\sin\theta\dfrac{d\theta}{dx}$. Substituting this in the previous equation gives

$$1 = \cos^2\theta\left(1 + \lambda^2\sin^2\theta\left(\frac{d\theta}{dx}\right)^2\right).$$

Since $\cos^2\theta + \sin^2\theta = 1$, this implies

$$\lambda\cos\theta\frac{d\theta}{dx} = \pm 1, \text{ i.e. } \frac{dx}{d\theta} = \pm\lambda\cos\theta$$

which means $x = \pm\lambda\sin\theta$ and $y + C = \lambda\cos\theta$: a circle segment!

6 Nonlinear Theory

In the remainder of this course we will study initial value problems associated to autonomous differential equations

$$x' = f(x), x(0) = x_0 \tag{18}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^∞. We saw:

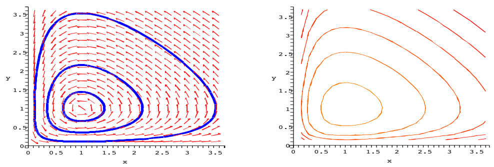
- There exists $\delta(x_0) > 0$ so that this has a unique solution $x\colon (-\delta, \delta) \to \mathbb{R}^n$;
- There exists a unique **maximal domain of existence** $I(x_0) = (\alpha(x_0), \beta(x_0))$ and a unique **maximal** solution $x\colon I(x_0) \to \mathbb{R}^n$.
- If $\beta(x_0) < \infty$ then $|x(t)| \to \infty$ when $t \uparrow \beta(x_0)$.
- If $\alpha(x_0) > -\infty$ then $|x(t)| \to \infty$ when $t \downarrow \alpha(x_0)$.
- The solution is often denoted by $\phi_t(x_0)$.
- One has the **flow property**: $\phi_{t+s}(x_0) = \phi_t\phi_s(x_0)$, $\phi_0(x_0) = x_0$.
- Solutions do not intersect. One way of making this precise goes as follows: $t > s$ and $\phi_t(x) = \phi_s(y)$ implies $\phi_{t-s}(x) = y$. (So $\phi_s(\phi_{t-s}(x)) = \phi_t(x) = \phi_s(y)$ implies $\phi_{t-s}(x) = y$.)

6.1 The orbits of a flow

Rather than studying each initial value problem separately, it makes sense to study the **flow** ϕ_t associated to $x' = f(x), x(0) = x_0$. The curves $t \mapsto \phi_t(x)$ are called the **orbits**. For example we will show that the flow of

$$\begin{aligned} \dot{x} &= Ax - Bxy \\ \dot{y} &= Cy + Dxy \end{aligned}$$

is equal to



6.2 Singularities

Consider $x' = f(x), x(0) = x_0$.

If $f(x_0) = 0$ then we say that x_0 is a *rest point* or *singularity*. In this case $x(t) \equiv x_0$ is a solution, and by uniqueness **the** solution. So $\phi_t(x_0) = x_0$ for all $t \in \mathbb{R}$.

This notion is so important that several alternative names are used for this: **rest point**, **fixed point**, **singular point** or **critical point**.

Near such points usually a linear analysis suffices.

Since $f(x_0) = 0$, and assuming that f is C^1 we obtain by Taylor's Theorem

$$f(x) = f(x_0)+A(x-x_0)+o(|x-x_0|^1) = A(x-x_0)+o(|x-x_0|)$$

where $o(|x-x_0|)$ is so that $o(|x-x_0|)/|x-x_0| \to 0$ as $x \to x_0$. (By the way, if f is C^2 we have $f(x) = A(x - x_0) + O(|x - x_0|^2)$.)

$A = Df(x_0)$ is called the **linear part** of f at x_0.

6.3 Stable and Unstable Manifold Theorem

A matrix A is called **hyperbolic** if its eigenvalues $\lambda_1, \ldots, \lambda_n$ have non-zero real part, i.e. satisfy $\Re(\lambda_i) \neq 0$, $i = 1, \ldots, n$. Order the eigenvalues so that

$$\Re(\lambda_i) < 0 \text{ for } i = 1, \ldots, s \text{ and } \Re(\lambda_i) > 0 \text{ for } i = s+1, \ldots, n.$$

Let E^s (resp. E^u) be the eigenspace associated to the eigenvalues $\lambda_1, \ldots, \lambda_s$ (resp. $\lambda_{s+1}, \ldots, \lambda_n$).

A singular point x_0 of f is called **hyperbolic** if the matrix $Df(x_0)$ is hyperbolic.

Theorem 20. Stable and Unstable Manifold Theorem Let x_0 be a singularity of f and assume x_0 is hyperbolic. Then there exist a manifold $W^s(x_0)$ of dimension s and a manifold $W^u(x_0)$ of dimension $n-s$ both containing x_0 so that

$$x \in W^s(x_0) \implies \phi_t(x) \to x_0 \text{ as } t \to \infty,$$

$$x \in W^u(x_0) \implies \phi_t(x) \to x_0 \text{ as } t \to -\infty.$$

$W^s(x_0), W^u(x_0)$ are tangent to $x_0 + E^s$ resp. $x_0 + E^u$ at x_0.

Remarks:

- Remember the notion of a manifold was defined in the previous chapter. Most of the time we will consider the case dimension one (then it is a **curve**) or of dimension two (then it is a **surface**.
- If $s = n$ then the singularity is called a **sink**.
- If $1 \leq s < n$ then it is called a **saddle**.
- $s = 0$ then it called a **source**.
- $W^s(x_0)$ is called the **stable manifold**.
- $W^u(x_0)$ is called the **unstable manifold**.

Example 31. Take $x' = x + y^2, y' = -y + x^2$. By Theorem 20 there is supposed to an invariant manifold $W^u(0)$ (a curve) which is tangent to the x-axis. How to find the power series expansion of $W^u(0)$? Of course this is the same example as described in Chapter 3, but let us redo it here:

$$y'(x) = (\frac{dy}{dt})/(\frac{dx}{dt}) = \frac{-y + x^2}{x + y^2}. \tag{19}$$

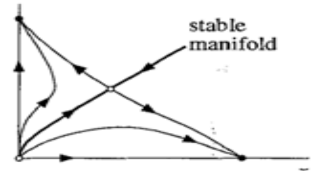


Figure 1: An example of a differential equation which will be studied later on in which there are several singularities: with a sink, source and saddle.

Since $0 \in W^u(0)$ and $W^u(0)$ is tangent to the horizontal axis, we can describe this curve by $y(x) = a_2x^2 + a_3x^3 + \dots$. That this power series converges follows from the stable and unstable manifold theorem). Substituting this in (19) gives

$$2a_2x + 3a_3x^2 + \cdots = \frac{-[a_2x^2 + a_3x^3 + \dots] + x^2}{x + [a_2x^2 + a_3x^3 + \dots]^2}.$$

Comparing terms of the same power, shows that $2a_2 = (1 - a_2)$ and so on. Thus we determine the power series expansion of $y(x)$.

Proof of Theorem 20. We will only prove this theorem in the case that $s = n$ and when the matrix $A = Df(x_0)$ has n real eigenvalues $\lambda_i < 0$ and n eigenvectors $v_1, \dots, v_n$. **For simplicity also assume** $x_0 = 0$. Consider x near $x_0 = 0$ and denote the orbit through x by $x(t)$.

Let T be the matrix consisting of the vectors $v_1, \dots, v_n$ (that is $Te_j = v_j$). Then $T^{-1}AT = \Lambda$ where Λ is a diagonal matrix (with $\lambda_1, \dots, \lambda_n$ on the diagonal).

Let us show that $\lim_{t\to\infty} x(t) = 0$ provided x is close to 0.

Let us write $y(t) = T^{-1}(x(t))$. It is sufficient to show that $y(t) \to 0$. Instead we will show $|y(t)|^2 = T^{-1}(x(t)) \cdot$

$T^{-1}(x(t)) \to 0$ as $t \to \infty$.

$$\begin{aligned} \frac{d|y(t)|^2}{dt} &= \frac{d}{dt}(T^{-1}x(t) \cdot T^{-1}(x(t)) = 2T^{-1}x \cdot T^{-1}\dot{x} \\ &= 2T^{-1}x \cdot T^{-1}f(x) \\ &= 2T^{-1}x \cdot T^{-1}Ax + 2T^{-1}x \cdot T^{-1}[f(x) - Ax]. \end{aligned}$$

Let us first estimate the **first term** in this sum under the assumption that all eigenvalues of A are real. Then

$$T^{-1}x \cdot T^{-1}Ax = y \cdot \Lambda y \le -\rho|y|^2 \tag{20}$$

where $\rho = \min_{i=1,\dots,n} |\lambda_i|$. Here we use that Λ is diagonal with all eigenvalues real (and therefore the eigenvectors are real and so T and y are also real).

The **second term** can be estimated as follows: Since $f(x) - Ax = o(|x|)$ for any $\epsilon > 0$ there exists $\delta > 0$ so that $|f(x) - Ax| \le \epsilon|x|$ provided $|x| \le \delta$. Hence using the Cauchy inequality and the matrix norm we get

$$T^{-1}x \cdot T^{-1}[f(x) - Ax] \le \epsilon|y| \cdot |T^{-1}[f(x) - Ax]| \le |y| \cdot ||T^{-1}|| \cdot \epsilon|x|.$$

provided $|x| \le \delta$. Of course we have that $|x| = |TT^{-1}x| = |Ty| \le ||T|| \cdot |y|$. Using this in the previous inequality gives

$$2T^{-1}x \cdot T^{-1}[f(x) - Ax] \le 2\epsilon||T|| \cdot ||T^{-1}|| \cdot |y|^2. \tag{21}$$

Using (20) and (21) in the estimate for $\frac{d|y(t)|^2}{dt}$ gives

$$\frac{d|y(t)|^2}{dt} \le -2\rho|y(t)|^2 + 2\epsilon||T|| \cdot ||T^{-1}|| \cdot |y(t)|^2 \le -\rho'|y(t)|^2$$

where $\rho' = (2\rho - 2\epsilon||T|| \cdot ||T^{-1}||)$. Provided we take $\epsilon > 0$ sufficiently small we get that $\rho' > 0$. That is if we write $z(t) = |y(t)|^2$ then we get $z' \le -\rho' z$ which means

$$|y(t)|^2 = z(t) \le z(0)e^{-t\rho'} \le |y(0)|^2 e^{-t\rho'}$$

and therefore that $|y(t)| \to 0$ as $t \to \infty$ (with a rate which is related to $\rho'/2$ (which is close to ρ if we choose $\epsilon > 0$ small).

If A is diagonalisable but the eigenvalues are no longer real, then the estimate in the inequality in (20) needs to be altered slightly. Let us explain the required change by considering an example. Take $A = \begin{pmatrix} -a & b \\ -b & -a \end{pmatrix}$. Note A has eigenvalues $-a \pm bi$ and that A is already in the real Jordan normal form. Moreover,

$$y \cdot Ay = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \cdot \begin{pmatrix} -ay_1 + by_2 \\ -by_1 - ay_2 \end{pmatrix}$$

$$= -a\left([y_1(t)]^2 + [y_2(t)]^2\right) = -a|y|^2.$$

so the argument goes through. Using the real Jordan normal form theorem, the same method applies as long as A has a basis of n eigenvectors. This concludes the proof of **Theorem 20** in this setting. We will skip the prove in the general setting, but the next example shows what happens if there is no basis of eigenvectors.

In fact, when we prove that $x(t)$ by showing that $|y(t)|^2$ tends to zero, we use the function $U(x) := |T^{-1}(x)|^2$. Later we will call this a **Lyapounov function**.

Example 32. Let us consider a situation when the matrix does not have a basis of eigenvectors. Let $A = \begin{pmatrix} -1 & Z \\ 0 & -1 \end{pmatrix}$ where $Z \in \mathbb{R}$. This has eigenvalues -1 (with double multiplicity). Take $U(x, y) = ax^2 + bxy + cy^2$. Then

$$\begin{aligned} \dot{U} &= 2ax\dot{x} + b\dot{x}y + bx\dot{y} + 2cy\dot{y} \\ &= 2ax(-x + Zy) + b(-x + Zy)y + bx(-y) + 2cy(-y) \\ &= -2ax^2 + (2Za - b - b)xy + (Zb - 2c)y^2. \end{aligned}$$

Case 1: If $Z \approx 0$, then we can take $a = 1, b = 0, c = 1$ because then $\dot{U} = -2x^2 + (2Z)xy - 2y^2 \leq 0$ (since $Z \approx 0$).

Case 2: If Z is large and $a = 1, b = 0, c = 1$ then we definitely don't get $\dot{U} \leq 0$. However, in this case we can set $b = 0$, and write

$$\begin{aligned}\dot{U} &= -2ax^2 + (2Za)xy - 2cy^2 \\ &= -2a[x - (Z/2)y]^2 + (aZ^2/2 - 2c)y^2 \\ &= -2[x - (Z/2)y]^2 - y^2 < 0.\end{aligned}$$

where in the last line we substitutes $a = 1$ and $c = 1/2 + Z^2/4$. Thus $U = c$ corresponds to a 'flat' ellipse when Z is large.

General case: This all seems rather ad hoc, but the Jordan normal form suggests a general method. Indeed A has an eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (i.e. $(A + I)v_1 = 0$) and we can choose a 2nd vector v_2 so that $(A + I)v_2 = \epsilon v_1$ where $\epsilon > 0$ is small. So $v_2 = \begin{pmatrix} 0 \\ \epsilon/Z \end{pmatrix}$. Taking $T = (v_1 v_2)$ gives $T^{-1}AT = \begin{pmatrix} -1 & \epsilon \\ 0 & -1 \end{pmatrix}$. In this new coordinates we are in the same position as if $Z \approx 0$. So we can argue as in the first case.

6.4 Hartman-Grobman

Theorem 21. Hartman-Grobman Let x_0 be a singularity and that $A = Df(x_0)$ is a hyperbolic matrix. Then there exists a continuous bijection (a homeomorphism) $h\colon \mathbb{R}^n \to \mathbb{R}^n$ so that $h(x_0) = 0$ and so that near x_0,

h sends orbits of $x' = f(x)$ to orbits of $x' = Ax$.

Remark: In other words, there exists an open set $U \ni x_0$ so that

$$h \circ \phi_t(x) = \phi_t^A \circ h(x)$$

for each x, t so that

$$\cup_{0\le s\le t}\phi_s(x) \subset U.$$

Here ϕ_t^A is the flow associated to $x' = Ax$ and ϕ_t the flow for $x' = f(x)$.

Remark: A **homeomorphism** is a continuous bijection whose inverse is also continuous. In Euclidean space (and 'manifolds'), this is the same as saying that it is continuous bijection.

6.5 Lyapounov functions

Sometimes one applies a method which is similar to the proof given in Theorem 20, namely one uses a so-called Lyapounov function:

Definition: Let $W \subset \mathbb{R}^n$ be an open set containing x_0. $V : W \to \mathbb{R}$ is a **Lyapounov** function for x_0 if it is C^1 and

- $V(x_0) = 0$, $V(x) > 0$ for $x \in W \setminus \{x_0\}$;
- $\dot{V} \le 0$ for $x \in W$.

Here $\dot{V} := \dfrac{dV(x(t))}{dt} = DV_{x(t)}\dfrac{dx}{dt} = DV_{x(t)}f(x(t))$.

In actual fact, we will use the notion of Lyapounov function more loosely, and for example sometimes give the same name to a function for which merely $\dot{V} \le 0$.

Remarks: V should be thought of as a way to measure the distance to x_0. That $\dot{V} \le 0$ means that this 'distance' is non-increasing. In quite a few textbooks a Lyapounov function is one which merely satisfies the first property; let's call such functions weak-Lyapounov functions.

Warning: In some cases one calls a function Lyapounov even if it does not satisfies all its properties.

Definitions:

- x_0 is called **asymptotically stable** if, for each x near x_0, one has $\phi_t(x) \to x_0$.

- x_0 is called **stable** if for each $\epsilon > 0$ there exists $\delta > 0$ so that if $x \in B_\delta(x_0)$ implies $\phi_t(x) \in B_\epsilon(x_0)$ for all $t \geq 0$. (So you nearby points don't go far.)

Lemma 10. Lyapounov functions

1. If $\dot{V} \leq 0$ then x_0 is stable. Moreover, $\phi_t(x)$ exists for all $t \geq 0$ provided $d(x, x_0)$ is small.
2. If $\dot{V} < 0$ for $x \in W \setminus \{x_0\}$ then $\forall x$ is close to x_0 one gets $\phi_t(x) \to x_0$ as $t \to \infty$, i.e. x_0 is asymptotically stable.

Proof. (1) Take $\epsilon > 0$ so that $B_{2\epsilon}(x_0) \subset W$. Let

$$\delta := \inf_{y \in \partial B_\epsilon(x_0)} V(y).$$

Since $V > 0$ except at x_0 we get $\delta > 0$. It follows that

$$V^{-1}[0, \delta) \cap \partial B_\epsilon(x_0) = \emptyset. \tag{22}$$

Take $x \in V^{-1}[0, \delta) \cap B_\epsilon(x_0)$ (this holds for all x near x_0 by continuity of V and since $V(0) = 0$). Since $\phi_0(x) = x$ and $t \to V(\phi_t(x))$ is non-increasing, $\phi_t(x) \in V^{-1}[0, \delta)$ for all $t \geq 0$. Since $t \to \phi_t(x)$ is continuous curve, $\phi_0(x) = x \in B_\epsilon(x_0)$ and (22), it follows that $\phi_t(x) \in B_\epsilon(x_0)$ for all $t \geq 0$. In particular $\phi_t(x)$ remains bounded, and so $\phi_t(x)$ exists $\forall t$.

(2) $\dot{V} < 0$ implies that $t \to V(\phi_t(x))$ is strictly decreasing. Take $x \in V^{-1}[0, \delta) \cap B_\epsilon(x_0)$ and suppose by contradiction that $V(\phi_t(x))$ does not tend to 0 as $t \to \infty$. Then, since $t \mapsto V(\phi_t(x))$ is decreasing, there exists $V_0 > 0$ so that $V(\rho_t(x)) \geq V_0 > 0$. Hence $\exists \rho > 0$ with $\phi_t(x) \notin B_\rho(x_0)$ $\forall t \geq 0$. Combining this with part (1) gives that

$$\phi_t(x) \in \overline{B_\epsilon(x_0)} \setminus B_\rho(x_0) \text{ for all } t \geq 0.$$

But $\dot{V} < 0$, $\dot{V}$ is only zero at x_0 and $\dot{V}$ attains its maximum in a compact set $\overline{B_\epsilon(x_0)} \setminus B_\rho(x_0)$ it follows that $\exists \kappa > 0$ so that

$$\dot{V} \leq -\kappa \text{ whenever } x(t) \in \overline{B_\epsilon(x_0)} \setminus B_\rho(x_0).$$

But since $x(t)$ is in this compact set for all $t \geq 0$,

$$\dot{V} \leq -\kappa, \forall t \geq 0.$$

Hence

$$V(\phi_t(x)) - V(x) \leq -\kappa t \to -\infty \text{ as } t \to \infty,$$

contradicting $V \geq 0$. □

Example 33.

$$\begin{aligned} x' &= 2y(z-1) \\ y' &= -x(z-1) \\ z' &= xy \end{aligned}$$

Its linearisation is $A := \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note A has eigenvalues $\pm\sqrt{2}i$ and 0. So A is not hyperbolic, and theorem 20 does not apply.

Take $V(x, y, z) = ax^2 + by^2 + cz^2$. Then

$$\dot{V} = 2(ax\dot{x} + by\dot{y} + cz\dot{z}) = 4axy(z-1) - 2bxy(z-1) + 2cxyz.$$

We want $V \geq 0$ and $\dot{V} \leq 0$. We can achieve this by setting $c = 0, 2a = b$. This makes $\dot{V} = 0$. It follows that solutions stay on level sets of the function $V = x^2 + 2y^2$. $x_0 = (0, 0, 0)$ is **not** asymptotically stable. Strictly speaking V is **not** a Lyapounov function because $V(0, 0, z) = 0$: more work needed to check if x_0 is stable.

Example 34. Consider the system $x' = -y - xy^2, y' = x - yx^2$. The only singularity of this system is at $(0, 0)$. Indeed, if $x' = 0$, then either $y = 0$ or $1 + xy = 0$; if $y = 0$ then $x(1 - xy) = 0$ implies $x = 0$; if $1 + xy = 0$ then $0 = x(1 - xy) = 2x$ implies $x = 0$ which contradicts $1 + xy = 0$.

Let us show that $(0, 0)$ is asymptotically stable. To do this, take the quadratic function $V(x, y) = x^2 + y^2$. Then $\dot{V} = 2x\dot{x}+$

$2y\dot{y} = -2x^2y^2 \leq 0$, so $(0,0)$ is stable. Since V is decreasing (non-increasing), this implies that there exists $V_0 \geq 0$ so that $V(x(t), y(t)) \downarrow V_0$. If $V_0 = 0$ then the solution converges to $(0,0)$ as claimed. If $V_0 > 0$ then the solution converges to the circle $\{(x,y); x^2 + y^2 = V_0\}$, and in particular remains bounded. Note that the set $V(x,y) = x^2 + y^2 = V_0$ does not contain singular points and also is not a periodic orbit, since $\dot{V} < 0$ except when $x = 0$ or $y = 0$. (By looking at the arrows, one concludes that the orbits are tangent to circles when $x = 0$ or when $y = 0$ but otherwise spiral inwards.) It follows that $V_0 = 0$ and so we are done.

6.6 The pendulum

Consider a pendulum moving along a circle of radius l, with a mass m and friction k. Let $\theta(t)$ be the angle from the vertical at time t. The force tangential to the circle is $-(kl\dfrac{d\theta}{dt} + mg\sin(\theta))$. So Newton's law gives

$$ml\theta'' = -kl\theta' - mg\sin\theta \quad \text{i.e.} \quad \theta'' = -(k/m)\theta' - (g/l)\sin\theta.$$

Taking $\omega = \theta'$ gives

$$\begin{aligned}
\theta' &= \omega \\
\omega' &= \frac{-g}{l}\sin(\theta) - \frac{k}{m}\omega.
\end{aligned}$$

Singularities are $(n\pi, 0)$ which corresponds to the pendulum being in vertical position (pointing up or down). Linearizing this at $(0,0)$ gives

$$\begin{pmatrix} 0 & 1 \\ -g/l & -k/m \end{pmatrix}$$

which gives eigenvalues $(-k/m \pm \sqrt{(k/m)^2 - 4g/l})/2$.

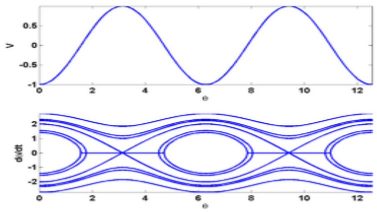


Figure 2: The phase portrait of the pendulum (no friction).

Note that, as $l > 0$, the real part of $(-k/m \pm \sqrt{(k/m)^2 - 4g/l})/2$ is negative. (If $(k/m)^2 - 4g/l < 0$ then both e.v. are complex and if $(k/m)^2 - 4g/l > 0$ then both e.v. are real and negative.)

Let us construct a Lyapounov function for this:

$$\begin{aligned} E &= \text{kinetic energy } + \text{ potential energy} \\ &= (1/2)mv^2 + mg(l - l\cos(\theta)) \\ &= (1/2)ml^2\omega^2 + mgl(1 - \cos(\theta)). \end{aligned}$$

Then $E \geq 0$ and $E = 0$ if and only if $\omega = 0$ and $\theta = n\pi$. Moreover,

$$\begin{aligned} \dot{E} &= ml(l\omega\omega' + g\theta' \sin\theta) \\ &= ml(l\omega(\frac{-g}{l}\sin(\theta) - \frac{k}{m}\omega) + g\omega\sin\theta) \\ &= -kl^2\omega^2 \end{aligned} \cdot$$

If the friction $k > 0$ then $\dot{E} < 0$ except when $\omega = 0$. If the friction $k = 0$ then $\dot{E} = 0$ and so solutions stay on level sets of E.

6.7 Hamiltonian systems

When the friction $k = 0$ we obtain an example of a Hamiltonian system.

This is a system for which there exists a function $H\colon \mathbb{R}^2 \to \mathbb{R}$ so that the equation of motion (i.e. the differential equation):

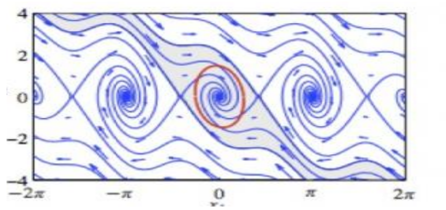


Figure 3: The phase portrait of the pendulum (with friction). The labels in the axis of this figure should have been $-4\pi, -2\pi, 0, 2\pi, 4\pi$.

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x,y) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x,y) \end{aligned}$$

For such systems

$$\begin{aligned} \dot{H} &= \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \\ &= \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}(-\frac{\partial H}{\partial x}) \\ &= 0. \end{aligned}$$

6.8 Van der Pol's equation

In electrical engineering the following equation often arrises

$$\begin{aligned} \dot{x} &= y - x^3 + x \\ \dot{y} &= -x. \end{aligned} \tag{23}$$

This system has a singularity at $(x,y) = (0,0)$. Its linear part at $(0,0)$ is $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. This has eigenvalues $(1 \pm \sqrt{3}i)/2$ and therefore $(0,0)$ is a source. What happens with other orbits?

Theorem 22. There is one periodic solution of this system and every non-equilibrium solution tends to this periodic solution.

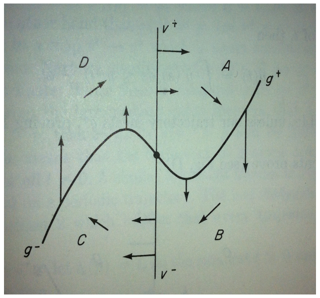


Figure 4: The phase portrait of the van der Pol equation.

The proof of this theorem will occupy the remainder of this section.

Define

$$v^{\pm} = \{(x, y); \pm y > 0, x = 0\} \text{ and } g^{\pm} = \{(x, y); \pm x > 0, y = x^3 - x\}.$$

This splits up $\mathbb{R}^2$ in regions A, B, C, D where horizontal and vertical speed is positive/negative.

$$\begin{aligned} \dot{x} &= y - x^3 + x \\ \dot{y} &= -x. \end{aligned}$$

Lemma 11. For any $p \in v^+$, $\exists t > 0$ with $\phi_t(p) \in g^+$.

Proof. Define $(x_t, y_t) = \phi_t(p)$.

- Since $x'(0) > 0$, $\phi_t(p) \in A$ for $t > 0$ small.
- $x' > 0, y' < 0$ in A. So the only way the curve $\phi_t(p)$ can leave the region $A \cap \{(x, y); y < y_0\}$ is via g^+.
- So $\phi_t(p)$ cannot go to infinity before hitting g^+.
- Hence $T = \inf\{t > 0; \phi_t(p) \in g^+\}$ is well-defined.
- We need to show $T < \infty$.

- Choose $t_0 \in (0,T)$ and let $a = x_{t_0}$. Then $a > 0$ and $x_t \geq a$ for $t \in [t_0, T]$.

- Hence $\dot{y} \leq -a$ for $t \in [t_0, T]$ and therefore $y(t) - y(t_0) \leq -a(t - t_0)$ for $t \in [t_0, T]$.

- $T = \infty \implies \lim_{t\to\infty} y(t) \to -\infty$ which gives a contradiction since $(x(t), y(t)) \in A$ for $t \in (0,T)$.

$\square$

Similarly

Lemma 12. For any $p \in g^+,\ \exists t > 0$ with $\phi_t(p) \in v^-$.

For each $y > 0$ define $F(y) = \phi_t(0,y)$ where $t > 0$ is minimal so that $\phi_t(0,y) \in v^-$. Similarly, define for $y < 0$ define $F(y) = \phi_t(0,y)$ where $t > 0$ is minimal so that $\phi_t(0,y) \in v^+$. By symmetry $F(-y) = -F(y)$.

Define the **Poincaré first return** map to v^+ as

$$P\colon v^+ \to v^+ \text{ by } (0,y) \mapsto (0, F^2(y)).$$

$P(p) = \phi_t(p)$ where $t > 0$ is minimal so that $\phi_t(p) \in v^+$.

Lemma 13.
1. $P\colon v^+ \to v^+$ is increasing (here we order v^+ as $(0,y_1) < (0,y_2)$ when $y_1 < y_2$);
2. $P(p) > p$ when $p \approx 0$;
3. $P(p) < p$ when p is large;
4. $P\colon v^+ \to v^+$ has a unique attracting fixed point.

Proof. The proof of (1): Uniqueness of solns $\implies$ orbits don't cross $\implies$ P is increasing.

Instead of (2), (3) and (4) we shall prove the following statement:

$$\begin{aligned} p \mapsto \quad & \delta(p) := |F(p)|^2 - |p|^2 \text{ is strictly decreasing} \\ & \delta(p) > 0 \text{ for } p > 0 \text{ small and} \qquad (24) \\ & \delta(p) \to -\infty \text{ as } p \to \infty \end{aligned}$$

Note that this implies (2) and (3) and (4) (see lecture). So Theorem 22 follows from (24).

Step 1: A useful expression for $\delta(p)$. Define

$$p^* = (0, y^*) \in v^+ \quad \text{so that } \exists t \text{ with } \phi_t(p^*) = (1,0) \text{ and } \phi_s(p^*) \in A \text{ for } 0 < s < t.$$

Define $U(x,y) = x^2 + y^2$. Pick $p \in v^+$ and let $\tau > 0$ be minimal so that $\phi_\tau(p) \in v^-$. (So $\phi_\tau(p) = F(p)$.) Hence

$$\begin{aligned} \delta(p) : \; &= |F(p)|^2 - |p|^2 = U(\phi_\tau(p)) - U(\phi_0(p)) \\ &= \textstyle\int_0^\tau \dot{U}(\phi_t(p))\, dt. \end{aligned}$$

Note

$$\begin{aligned} \dot{U} \; &= 2x\dot{x} + 2y\dot{y} = \\ &= 2x(y - x^3 + x) + 2y(-x) = -2x(x^3 - x) = 2x^2(1 - x^2) \end{aligned}.$$

Hence

$$\delta(p) = 2\int_0^\tau [x(t)]^2(1 - [x(t)]^2) dt = 2\int_\gamma x^2(1 - x^2)\, dt.$$

Here γ is the curve $[0,\tau] \ni t \to \phi_t(p)$. If $p < p^*$ then $\delta(p) > 0$ because then $(1 - [x(t)]^2) \geq 0$ for all $t \in [0, \tau]$.

Step 2: $\delta(p)$ when $p > p^*$. We can decompose the curve γ in three pieces as γ meets the line $x = 1$ twice

- the piece of γ with both endpoint on this line we call γ_2;
- γ_1 is the curve which connects $p \in v^+$ to the line $x = 1$.
- γ_3 is the curve which connects $F(p) \in v^-$ to the line $x = 1$.

Now consider

$$\delta_i(p) := 2 \int_{\gamma_i} x^2(1-x^2)\, ds \text{ for } i = 1,2,3.$$

Step 3: $\delta_1(p)$ is decreasing when $p > p^*$.

- γ_1 is a curve which can be regarded as function of x.
- Hence we can write
$$\int_{\gamma_1} x^2(1-x^2)\, dt = \int_{\gamma_1} \frac{x^2(1-x^2)}{dx/dt}\, dx = \int_{\gamma_1} \frac{x^2(1-x^2)}{y-(x^3-x)}\, dx.$$
- As p moves up, the curve γ_1 (connecting $p \in v^+$ to a point on the line $x = 1$) moves up and so $y - (x^3 - x)$ (along this curve) increases.
- Hence $p \to \delta_1(p) = 2\int_{\gamma_1} x^2(1-x^2)\, dt$ decreases as p increases.

Step 4: $\delta_2(p)$ is decreasing when $p > p^*$.

- Along γ_2, $x(t)$ is a function of $y \in [y_1, y_2]$ (where $(1, y_1)$, $y_1 > 0$ and $(1, y_2)$, $y_2 < 0$) are the intersections points of γ with the line $x = 1$.
- Since $-x = dy/dt$ we get
$$\begin{aligned} \int_{\gamma_1} x^2(1-x^2)\, dt &= \int_{y_1}^{y_2} -x(y)(1-[x(y)]^2)\, dy \\ &= \int_{y_2}^{y_1} x(y)(1-[x(y)]^2)\, dy \end{aligned}$$

in the 2nd integral one has $\int_{y_1}^{y_2}$ because that corresponds to the way the curve γ_1 is oriented.

- Since $x(y) \geq 1$ along γ_2 (and $y_2 < y_1$), this integral is negative.

- As p increases, the interval $[y_1, y_2]$ gets larger, and the curve γ_2 moves to the right and so $x(y)(1 - [x(y)]^2)$ decreases. It follows that $\delta_2(p)$ decreases as p increases.

- It is not hard to show that $\delta_2(p) \to -\infty$ as $p \to \infty$, see lecture.

Exactly as for $\delta_1(p)$, one also gets that $\delta_3(p)$ decreases as p increases. This completes the proof of the equation (24) and therefore the proof of Lemma 13 and Theorem 22. $\square$

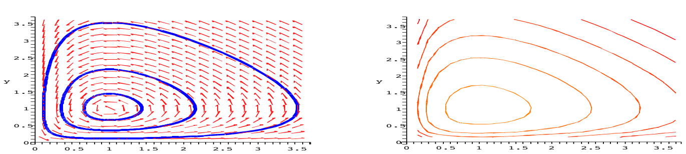
6.9 Population dynamics

A common predator-prey model is the equation

$$\begin{aligned} \dot{x} &= (A - By)x \\ \dot{y} &= (Cx - D)y. \end{aligned} \quad \text{where } A, B, C, D > 0$$

Here x are the number of rabbits and y the number of foxes. For example, $x' = Ax - Bxy$ expresses that rabbits grow with speed A but that the proportion that get eaten is a multiple of the number of foxes.

Let us show that the orbits look like the following diagram:



- Singularities are $(x,y) = (0,0)$ and $(x,y) = (D/C, A/B)$.
- If p is on the axis, then $\phi_t(x)$ is on this axis for all $t \in \mathbb{R}$.
- At $(0,0)$ the linearisation is $\begin{pmatrix} A & 0 \\ 0 & -D \end{pmatrix}$, so eigenvalues are $A, -D$ and $(0,0)$ is a saddle point.
- At $(x,y) = (D/C, A/B)$ the linearisation is $\begin{pmatrix} A-By & -Bx \\ Cy & Cx-D \end{pmatrix} =$ $\begin{pmatrix} 0 & -BD/C \\ CA/B & 0 \end{pmatrix}$ which has eigenvalues $\pm ADi$ (purely imaginary).

$$\begin{aligned} \dot{x} &= (A-By)x \\ \dot{y} &= (Cx-D)y. \end{aligned} \quad \text{where } A,B,C,D > 0$$

- Analysing the direction field, suggests that orbits cycle around $(D/C, A/B)$ (see lecture).
- Try to find Lyapounov of the form $H(x,y) = F(x) + G(y)$.
- $\dot{H} = F'(x)\dot{x} + G'(y)\dot{y} = xF'(x)(A-By) + yG'(y)(Cx-D)$.
- If we set (that is, insist on) $\dot{H} = 0$ then we obtain

$$\frac{xF'}{Cx-D} = \frac{yG'}{By-A} \tag{25}$$

- LHS of (25) only depends on x and RHS only on y. So expression in (25) $= const$.
- We may as well set $const = 1$. This gives $F' = C - D/x$ and $G' = B - A/y$.

- So $F(x) = Cx - D\log x, G(y) = By - A\log y$ and $H(x,y) = Cx - D\log x + By - A\log y$.

Summarising:

Theorem 23. Take $(x,y) \neq (D/C, A/B)$ with $x, y > 0$ and consider its orbits under

$$\begin{aligned} \dot{x} &= (A - By)x \\ \dot{y} &= (Cx - D)y. \end{aligned} \quad \text{where } A, B, C, D > 0.$$

Then $t \mapsto \phi_t(x,y)$ is periodic (i.e. is a closed curve).

Proof. Take $H_0 = H(x,y)$ and let $\Sigma = \{(u,v); H(u,v) = H_0\}$.

- The orbit $\phi_t(x,y)$ stays on the level set Σ of H.
- It moves with positive speed.
- So it returns in finite time.
- Orbits exist for all time, because it remains on Σ (and therefore cannot go to infinity).

$\square$

7 Dynamical Systems

So far we saw:

- Most differential equations cannot be solved explicitly.
- Nevertheless in many instances one can still prove many properties of its solutions.
- The point of view taken in the field dynamical systems is to concentrate on
  - attractors and limit sets: what happens *eventually*;
  - statistical properties of orbits.

In this chapter we will discuss a result which describes the planar case (i.e. the two-dimensional case).

Throughout the remainder of this notes, we will tacitly assume the solution $\phi_t(x)$ through x exists for all $t \geq 0$.

7.1 Limit Sets

Let ϕ_t be the flow of a dynamical system and take a point x. Then the ω-limit set of x, denoted by $\omega(x)$, is the set of limit points of the curve $[0, \infty) \ni t \mapsto \phi_t(x)$. More specifically, $y \in \omega(x)$ if and only if there exists a sequence $t_n \to \infty$ so that $\phi_{t_n}(x) \to y$.

So $\omega(x)$ describes where the point x eventually goes. It is easy to prove that $\omega(x)$ is a closed set (see assignments). (But, possibly, $\omega(x) = \emptyset$.)

We say that x lies on a periodic orbit if $\phi_T(x) = x$ for some $T > 0$. The smallest such $T > 0$ is called the period of x. Note that then

- $\gamma = \cup_{t\in[0,T)}\phi_t(x)$ is closed curve without self-intersections, and
- $\omega(x) = \gamma$.

7.2 Local sections

Definition: We say that a manifold $S \ni p$ of codimension-one in $\mathbb{R}^n$ is a **local section at** p for the autonomous differential equation $x' = f(x)$ if:

1. $S = \{x \in \mathbb{R}^n; g(x) = 0\}$ contains p and $Dg(p) \neq 0$ (hence is a manifold containing p);
2. $f(p) \neq 0$ and that $Dg(p)f(p) \neq 0$ (this means that $f(p)$ does not lie in the tangent space of S at p).

Theorem 24 (Flow Box Theorem). Assume S is a local section at p and assume q is that $\phi_{t_0}(q) = p$ for some $t_0 > 0$. Then

- there exists a neighbourhood U of q;
- a smooth function $\tau\colon U \to \mathbb{R}$ so that $\tau(q) = t_0$ so that for each $x \in U$, $\phi_{\tau(x)}(x) \in S$.

If $t_0 > 0$ is the minimal time so that $\phi_{t_0}(q) \in S$ then we will also have that $\tau(x) > 0$ is minimal so that $\phi_{\tau(x)}(x) \in S$. $\tau(x)$ is then called the first arrival time and the map $P(x) = \phi_{\tau(x)}(x)$ the Poincaré entry map to S.

Proof. Define $G(x,t) = g(\phi_t(x))$. Then $G(q,t_0) = g(\phi_{t_0}(q)) = g(p) = 0$. Moreover,

$$\begin{aligned}\frac{\partial G}{\partial t}(q,t_0) &= Dg(\phi_{t_0}(q))\frac{\partial \phi_t}{\partial t}(q)\Big|_{t=t_0} = Dg(p)f(\phi_{t_0}(q)) \\ &= Dg(p)f(p) \neq 0 \text{ (because } S \text{ is a section at } p).\end{aligned}$$

Hence by the implicit function theorem there exists $x \mapsto \tau(x)$ so that $G(x, \tau(x)) = 0$ for x near q. Hence $\phi_{\tau(x)} \in S$ for x near q. $\square$

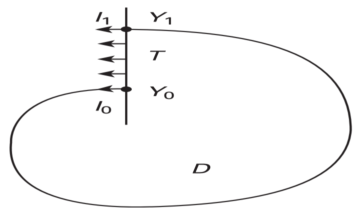
Remarks:

1. If S is a section at p and $\phi_{t_n}(x) \to p$ for some $t_n \to \infty$ then there exists $t'_n \to \infty$ so that $\phi_{t'_n}(x) \to p$ and $\phi_{t'_n}(x) \in S$.

2. If $f(p) \neq 0$ then one can find a local section at p: just take $g\colon \mathbb{R}^n \to \mathbb{R}$ affine (of the form $x \mapsto A(x-p)$ where A is a $1 \times n$ matrix with $Af(p) \neq 0$. Then $S = \{x; g(x) = 0\}$ is a codimension-one hyperplane with the required properties.

3. If p lies on a periodic orbit and S a local section at p, then $\phi_T(p) = p$ and then there exists a neighbourhood U of p and a map $P\colon S \cap U \to S$ so that $P(p) = p$. This is called the Poincaré return map.

4. As in the example of the van der Pol equation, one can use this map to check whether the periodic orbit is attracting.



7.3 Planar Systems

Theorem 25. Let S be a local section for a planar differential equation, so S is an arc c. Let $\gamma = \cup_{t\geq 0}\phi_t(x)$ and let $y_0, y_1, y_2 \in S \cap \gamma$. Then y_0, y_1, y_2 lie ordered on γ if and only if they lie ordered on S.

Proof. Take $y_0, y_1, y_2 \in \gamma \cap c$. Assume that y_0, y_1, y_2 are *consecutive* points on γ, i.e. assume $y_2 = \phi_{t_2}(y_0)$, $y_1 = \phi_{t_1}(y_0)$ with $t_2 > t_1 > 0$. Let $\gamma' = \cup_{0\leq s\leq t_1}\phi_s(y_0)$ and consider the arc c' in c between y_0, y_1. Then

- $c' \cup \gamma'$ is a closed curve which bounds a compact set D (here we use a special case of a deep result namely the Jordan theorem);
- Either all orbits enter D along c' or they all leave D along c'.
- Either way, since the orbit through y does not have self-intersections and because of the orientation of $x' = f(x)$ along c, $\phi_{t_2}(y_0)$ cannot intersect c', see figure.

□

In this chapter we tacitly assume that if γ is a closed curve in $\mathbb{R}^2$ without self-intersections, then the complement of γ has two connected components: one bounded one and the other unbounded. This result is called the *Jordan curve theorem* which looks obvious, but its proof is certainly not easy. It can be proved using algebraic topology.

Lemma 14. If $y \in \omega(x)$. Then the orbit through y intersects any local section at most once.

Proof.

1. Assume by contradiction that $y_1 = \phi_u(y)$ and $y_2 = \phi_v(y)$ (where $v > u$) are contained on a local section S.

2. Since $y \in \omega(x)$ where exists $t_n \to \infty$ so that $\phi_{t_n}(x) \to y$. Hence $\phi_{t_n+u}(x) \to y_1$ and $\phi_{t_n+v}(x) \to y_2$.

3. Since $y_1, y_2 \in S$, (2) implies that for n large there exists $u_n, v_n \to 0$ so that $\phi_{t_n+u+u_n}(x) \in S$, $\phi_{t_n+u+u_n}(x) \to y_1$ and $\phi_{t_n+v+v_n}(x) \in S$, $\phi_{t_n+v+v_n}(x) \to y_2$.

4. Take $n' > n$ so that

$$t_n + u + u_n < t_n + v + v_n < t_{n'} + u + v_{n'}. \tag{26}$$

Then

$$\phi_{t_n+u+u_n}(x), \phi_{t_n+v+v_n}(x), \phi_{t_{n'}+u+v_{n'}}(x)$$

do not lie ordered on S: the first and last one are close to y_1 and the middle one close to y_2. This and (26) contradict the previous theorem.

□

7.4 Poincaré Bendixson

Theorem 26 (Poincaré-Bendixson Theorem)**.** Consider a planar differential equation, take $x \in \mathbb{R}^2$ and assume that $\omega := \omega(x)$ is non-empty, bounded and does not contain a singular point of the differential equation. Then ω is a periodic orbit.

That is, we have an autonomous differential equation in $\mathbb{R}^2$, $\dot{x} = f(x)$ with $x \in \mathbb{R}^2$.

Proof.

- Assume that ω does not contain a singular point.

- Take $y \in \omega$. Then there exists $s_m \to \infty$ so that $\phi_{s_m}(x) \to y$. Hence for each fixed $t > 0$, $\phi_{s_m+t}(x) \to \phi_t(y)$ as $m \to \infty$. It follows that the forward orbit $\gamma = \cup_{t \geq 0} \phi_t(y)$ is contained in ω. Since ω is compact, *any* sequence $\phi_{t_n}(y)$ has a convergent subsequence. Hence $\omega(y) \neq \emptyset$ and $\omega(y) \subset \omega$.

- Take $z \in \omega(y)$. Since z is not a singular point, there exists a local section S containing z. Since $z \in \omega(y)$, there exists $t_n \to \infty$ so that $\phi_{t_n}(y) \to z$ and $\phi_{t_n}(y) \in S$.

- By the previous lemma, $\phi_{t_n}(y) = \phi_{t_{n'}}(y)$ for all n, n'. So $\exists T > 0$ so that $\phi_T(y) = y$ and y lies on a periodic orbit.

- We will skip the proof that ω is *equal* to the orbit through y (but see lecture).

□

We say that A is a forward invariant domain in $\mathbb{R}^2$ if $x \in A$ implies that $\phi_t(x) \in A$ for $t > 0$. The following theorem follows from the previous one:

Theorem 27. Consider a planar differential equation and assume that A is a bounded forward invariant set so that either A does not contain any singularities or so that the stable manifold each singularity in A has dimension ≤ 1. Then A contains a periodic orbit.

7.5 Consequences of Poincaré-Bendixson

Theorem 28. Let γ be a periodic orbit of a differential equation $x' = f(x)$ in the plane surrounding an open region D. Then D contains a singularity or another periodic orbit.

We could have also restated this theorem in the following way: Let γ be a periodic orbit of a differential equation $x' = f(x)$ in the plane surrounding a region D. Then D contains a singularity or a periodic orbit in the interior of D.

The proof is **not examinable**:

Proof: Assume by contradiction that D contains no other singularity or periodic orbit. Note that the D is invariant. Take $x \in D$ and consider the orbit $\phi_t(x)$. Then $\omega(x) = \gamma$ by Poincaré-Bendixson. Now consider instead of $x' = f(x)$ the differential equation $x' = -f(x)$ (so just run time backwards). Again γ is a periodic orbit, and applying Poincaré-Bendixson again, the accumulation points of $\phi_t(x)$ as $t \to -\infty$ again must be equal to γ. Now take a section S at a point $z \in \gamma$. Then the previous assertions imply that there exist sequences $t_n \to \infty$ and $s_n \to -\infty$ so that $\phi_{t_n}(x), \phi_{s_n}(x) \in S$. This contradicts Theorem 23 in the notes.

Theorem 29. Let γ be a periodic orbit of a differential equation $x' = f(x)$ in the plane surrounding a region D. Then D contains a singularity.

This theorem strengthens the previous one.

Again the proof is **not examinable**:

Sketch of proof: assume that D does not contain a singularity. Then by the previous theorem it contains another periodic orbit γ'. Let $A \geq 0$ be the greatest lower bound of the areas of regions surrounded by periodic orbits. If $A = 0$ then one can show this implies there exists a singularity. If $A > 0$ then one can show that there exists a periodic orbit $\hat{\gamma}$ in D which does not surround another periodic orbit. But this then contradicts the previous lemma.

Other ways of proving the previous corollary are related to the Brouwer fixed point theorem, are rather use index arguments related to the Euler characteristic. To describe this one needs to discuss some ideas from algebraic topology.

7.6 Further Outlook

- The Poincaré Bendixson theorem implies that planar differential equations cannot have 'chaotic' behaviour.

- Differential equations in dimension ≥ 3 certainly can have chaotic behaviour, see the 3rd year course *dynamical systems* (M3PA23) and for example `http://www.youtube.com/watch?v=ByH8_nKD-ZM`.

- To describe their statistical behaviour one uses probabilistic arguments; this area of mathematics is called *ergodic theory*. This is a 4th year course (M4PA36). For more information see for example, `http://en.wikipedia.org/wiki/Ergodic_theory`

- The geometry of attractors is often fractal like, see the 3rd year course *chaos and fractals* (M3PA46). For more information see for example, `http://en.wikipedia.org/wiki/Fractal`.

- Instead of differential equations one also studies discrete dynamical systems, $x_{n+1} = f(x_n)$. When $f\colon \mathbb{C} \to \mathbb{C}$ is a polynomial this leads to the study of *Julia sets* using tools from complex analysis. For more information, see `http://en.wikipedia.org/wiki/Julia_set`.

Dynamical systems is an extremely active area, and is both interesting for people focusing on pure as well as those more interested in applied mathematics.

For example, Fields Medalists whose work is in or related to this area, include: Avilla (2014, complex dynamics), Lindenstrauss (2010, ergodic theory), Smirnov (2010, part of his work relates to complex dynamics), Tao (2006, part of his work related to ergodic theory), McMullen (1998, complex dynamics), Yoccoz (1994, complex dynamics), Thurston (1982, a significant amount of work was about low and complex dynamics), Milnor (1962, his current work is in complex dynamics).

Applied dynamicists often aim to understand specific dynamical phenomena, related to for example biological systems, network dynamics, stability and bifurcation issues etc.

One of the appeals of dynamical systems that it uses mathematics from many branches of mathematics, but also that it is so relevant for applications.

Appendix A Multivariable calculus

Some of you did not do multivariable calculus. This note provides a crash course on this topic and includes some very important theorems about multivariable calculus which are not included in other 2nd year courses.

A.1 Jacobian

Suppose that $F\colon U \to V$ where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$. We say that F is differentiable at $x \in U$ if there exists a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$ (i.e. a $m \times n$ matrix A)

$$\frac{|(F(x+u) - F(x)) - Au|}{|u|} \to 0$$

as $u \to 0$. In this case we define $DF_x = A$.

- In other words $F(x+u) = F(x) + Au + o(|u|)$. ($A$ is the linear part of the Taylor expansion of F).
- How to compute DF_x? This is just the Jacobian matrix, see below.
- If $f\colon \mathbb{R}^n \to \mathbb{R}$ then Df_x is a $1 \times n$ matrix which is also called grad(f) or $\nabla f(x)$.

Example 35. Let $F(x,y) = \begin{pmatrix} x^2 + yx \\ xy - y \end{pmatrix}$ then

$$DF_{x,y} = \begin{pmatrix} 2x+y & x \\ y & x-1 \end{pmatrix}.$$

Usually one denotes by $(Df_\xi)u$ is the directional derivative of f (in the direction u) at the point ξ.

Example 36. If $F(x,y) = \begin{pmatrix} x^2+yx \\ xy-y \end{pmatrix}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $(DF_{x,y})e_1 = \begin{pmatrix} 2x+y & x \\ y & x-1 \end{pmatrix} e_1 = \begin{pmatrix} 2x+y \\ y \end{pmatrix}$. This is what you get when you fix y and differentiate w.r.t. x in $F(x,y)$.

For each fixed y one has a curve $x \mapsto F(x,y) = \begin{pmatrix} x^2+yx \\ xy-y \end{pmatrix}$ and $(DF_{x,y})e_1 = \begin{pmatrix} 2x+y \\ y \end{pmatrix}$ gives its speed vector.

Remark: Sometimes one writes $DF(x,y)u$ instead of $DF_{x,y}u$.

If u is the i-th unit vector e_i then one often writes $D_iF_{x,y}$ and if $i=1$ something like $D_xF(x,y)$.

Theorem 30 (Multivariable Mean Value Theorem)**.** If $f\colon \mathbb{R} \to \mathbb{R}^m$ is continuously differentiable then $\forall x,y \in \mathbb{R}$ there exists $\xi \in [x,y]$ so that $|f(x)-f(y)| \leq |Df_\xi||x-y|$.

Proof. By the Main Theorem of integration, $f(y)-f(x) = \int_x^y Df_s\,ds$ (where Df_t is the $n \times 1$ matrix (i.e. vertical vector) of derivatives of each component of f. So

$$\begin{aligned} |f(x)-f(y)| &= |\int_x^y Df_s\,ds| \leq \int_x^y |Df_s|\,ds \\ &\leq \max_{s\in(x,y)} |Df_s|\,|x-y| \leq |Df_\xi||x-y| \end{aligned}$$

for some $\xi \in [x,y]$. □

Corollary: If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable then for each $x,y \in \mathbb{R}^n$ there exists ξ in the arc $[x,y]$ connecting x and y so that $|f(x)-f(y)| \leq |Df_\xi(u)||x-y|$ where $u = (x-y)/|x-y|$. **Proof:** just consider f restricted to the line connecting x,y and apply the previous theorem.

A.2 The statement of the Inverse Function Theorem

Theorem 31 (The Inverse Function Theorem)**.** Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F\colon U \to \mathbb{R}^n$ be continuously differentiable and suppose that the matrix DF_p is invertible. Then there exist open sets $W \subset U$ and $V \subset \mathbb{R}^n$ with $p \in W$ and $F(p) \in V$, so that $F\colon W \to V$ is a bijection and so that its inverse $G\colon V \to W$ is also differentiable.

Definition A differentiable map $F\colon U \to V$ which has a differentiable inverse is called a **diffeomorphism**.

Proof: Without loss of generality we can assume that $p = 0 = F(p)$ (just apply a translation). By composing with a linear transformation we can even also assume $DF_0 = I$. Since we assume that $x \mapsto DF_x$ is continuous, there exists $\delta > 0$ so that

$$|| I - DF_x|| \leq 1/2 \text{ for all } x \in \mathbb{R}^n \text{ with } |x| \leq 2\delta. \qquad (27)$$

Here, as usual, we define the norm of a matrix A to be

typo corrected

$$||A|| = \sup\{|Ax|; |x| = 1\}.$$

Given y with $|y| \leq \delta/2$ define the transformation

$$T_y(x) = y + x - F(x).$$

Note that

$$T_y(x) = x \Longleftrightarrow F(x) = y.$$

So finding a fixed point of T_y gives us the point x for which $G(y) = x$, where G is the inverse of F that we are looking for.

We will find x using the Banach Contraction Mapping Theorem.

(**Step 1**) By (27) we had $|| I - DF_x|| \leq 1/2$ when $|x| \leq 2\delta$. Therefore, the Mean Value Theorem applied to $x \mapsto x - F(x)$ gives

$$|x - F(x) - (0 - F(0))| \leq \frac{1}{2}|x - 0| \text{ for } |x| \leq 2\delta$$

Therefore if $|x| \leq \delta$ (and since $|y| \leq \delta/2$),

$$|T_y(x)| \leq |y| + |x - F(x)| \leq \delta/2 + \delta/2 = \delta.$$

So T_y maps the closed ball $B := B_\delta(0)$ into itself.

(**Step 2**) $T_y \colon B \to B$ is a contraction since if $x, z \in B_\delta(0)$ then $|x-z| \leq 2\delta$ and so we obtain by the Mean Value Theorem again

$$|T_y(x) - T_y(z)| = |x - F(x) - (z - F(z))| \leq \frac{1}{2}|x - z|. \quad (28)$$

(**Step 3**) Since $B_\delta(0)$ is a complete metric space, there exists a unique $x \in B_\delta(0)$ with $T_y(x) = x$. That is, we find a unique x with $F(x) = y$.

(**Step 4**) The upshot is that for each $y \in B_{\delta/2}(0)$ there is precisely one solution $x \in B_\delta(0)$ of the equation $F(x) = y$. Hence there exists $W \subset B_\delta(0)$ so that the map

$$F \colon W \to V := B_{\delta/2}(0)$$

is a bijection. So $F \colon W \to V$ has an inverse, which we denote by G.

(**Step 5**) G **is continuous:** Set $u = F(x)$ and $v = F(z)$. Applying the triangle inequality in the first inequality and equation (28) in the 2nd inequality we obtain,

$$|x - z| = |(x - z) - (F(x) - F(z)) + (F(x) - F(z))| \leq$$

$$\leq |(x - z) - (F(x) - F(z))| + |F(x) - F(z)| \leq$$

$$\leq \frac{1}{2}|x - z| + |F(x) - F(z)|.$$

So $|G(u) - G(v)| = |x - z| \leq 2|F(x) - F(z)| = 2|u - v|$.

(Step 6) G **is differentiable:**

$|(G(u)-G(v))-(DF_z)^{-1}(u-v)| = |x-z-(DF_z)^{-1}(F(x)-F(z))| \leq$

$||(DF_z)^{-1}||\cdot|DF_z(x-z)-(F(x)-F(z))| = o(|x-z|) = 2o(|u-v|)$.

as $||(DF_z)^{-1}||$ is bounded, using the definition and the last inequality in step 5. Hence

$$|G(u) - G(v) - (DF_z)^{-1}(u - v)| = o(|u - v|)$$

proving that G is differentiable and that $DG_v = (DF_z)^{-1}$.

Example 37. Consider the set of equations

$$\frac{x^2 + y^2}{x} = u, \sin(x) + \cos(y) = v.$$

Given (u, v) near $(u_0, v_0) = (2, \cos(1) + \sin(1))$ is it possible to find a unique (x, y) near to $(x_0, y_0) = (1, 1)$ satisfying this set of equations? To check this, we define

$$F(x, y) = \begin{pmatrix} \frac{x^2+y^2}{x} \\ \sin(x) + \cos(y) \end{pmatrix}.$$

The Jacobian matrix is

$$\begin{pmatrix} \frac{x^2-y^2}{x^2} & \frac{2y}{x} \\ \cos(x) & -\sin(y) \end{pmatrix}.$$

The determinant of this is $\frac{y^2-x^2}{x^2}\sin(y) - \frac{2y}{x}\cos(x)$ which is non-zero near $(1, 1)$. So F is invertible near $(1, 1)$ and for every (u, v) sufficiently close to (u_0, v_0) one can find a unique solution near to (x_0, y_0) to this set of equations. Near $(\pi/2, \pi/2)$ the map F is probably not invertible.

A.3 The Implicit Function Theorem

Theorem 32 (Implicit Function Theorem)**.** Let $F\colon \mathbb{R}^p\times\mathbb{R}^n\to\mathbb{R}^n$ be differentiable and assume that $F(0,0)=0$. Moreover, assume that $n\times n$ matrix obtained by deleting the first p columns of the matrix $DF_{0,0}$ is invertible. Then there exists a function $G\colon \mathbb{R}^p\to\mathbb{R}^n$ so that for all (x,y) near $(0,0)$

$$y=G(x)\iff F(x,y)=0.$$

The proof is a fairly simple application of the inverse function theorem, and won't be given here. The $\mathbb{R}^p$ part in $\mathbb{R}^p\times\mathbb{R}^n$ can be thought as parameters.

Example 38. Let $f(x,y)=x^2+y^2-1$. Then one can consider this as locally as a function $y(x)$ when $\partial f/\partial y=2y\neq 0$.

Example 39. Consider the following equations:

$$\begin{aligned} x^2-y^2-u^3+v^2+4 &= 0,\\ 2xy+y^2-2u^2+3v^4+8 &= 0. \end{aligned}$$

Can one write u,v as a function of x,y in a neighbourhood of the solution $(x,y,y,v)=(2,-1,2,1)$? To see this, define

$$F(x,y,u,v)=(x^2-y^2-u^3+v^2+4,2xy+y^2-2u^2+3v^4+8).$$

We have to consider the part of the Jacobian matrix which concerns the derivatives w.r.t. u,v at this point. That is

$$\begin{pmatrix} -3u^2 & 2v\\ -4u & 12v^3 \end{pmatrix}\Bigg|_{(2,-1,2,1)}=\begin{pmatrix} -12 & 2\\ -8 & 12 \end{pmatrix}$$

which is an invertible matrix.

So locally, near $(2,-1,2,1)$ one can write

$$(u,v)=G(x,y) \text{ that is } F(x,y,G_1(x,y),G_2(x,y))=0.$$

It is even possible to determine $\partial G_1/\partial x$ (i.e. $\partial u/\partial x$). Indeed, writing $u = G_1(x,y)$ and $v = G_2(x,y)$ and differentiate:

$$\begin{aligned} x^2 - y^2 - u^3 + v^2 + 4 &= 0, \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 &= 0, \end{aligned}$$

with respect to x. This gives

$$\begin{aligned} 2x - 3u^2\tfrac{\partial u}{\partial x} + 2v\tfrac{\partial v}{\partial x} &= 0, \\ 2y - 4u\tfrac{\partial u}{\partial x} + 12v^3\tfrac{\partial v}{\partial x} &= 0. \end{aligned}$$

So

$$\begin{aligned} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} &= \begin{pmatrix} 3u^2 & -2v \\ 4u & -12v^3 \end{pmatrix}^{-1} \begin{pmatrix} 2x \\ 2y \end{pmatrix} \\ &= \frac{1}{8uv - 36u^2v^2} \begin{pmatrix} -12v^3 & 2v \\ -4u & 3u^2 \end{pmatrix} \begin{pmatrix} 2x \\ 2y \end{pmatrix} \end{aligned}$$

Hence $\dfrac{\partial u}{\partial x} = \dfrac{(-24xv^3 + 4vy)}{8uv - 36u^2v^2}$.

Appendix B Prerequisites

B.1 Function spaces

1. Let $f\colon [0,1] \to \mathbb{R}$ be a function and $f_n\colon [0,1] \to \mathbb{R}$ be a sequence functions. Define what it means to say that $f_n \to f$ **uniformly**.

   Answer: for all $\epsilon > 0$ there exists n_0 so that for all $n \geq n_0$ and all $x \in [0,1]$ one has $|f_n(x) - f(x)| < \epsilon$.

   Answer 2: $||f_n - f||_\infty \to 0$ as $n \to \infty$ where $||f_n - f||_\infty = \sup_{x\in[0,1]} |f_n(x) - f(x)|$.

2. Let $f\colon [0,1] \to \mathbb{R}$ be a function and $f_n\colon [0,1] \to \mathbb{R}$ be a sequence functions. Define what it means to say that $f_n \to f$ **pointwise**.

   Answer: for all $\epsilon > 0$ and all $x \in [0,1]$ there exists n_0 so that for all $n \geq n_0$ one has $|f_n(x) - f(x)| < \epsilon$.

3. Let $f\colon [0,1] \to \mathbb{R}$ be a function and $f_n\colon [0,1] \to \mathbb{R}$ be a sequence functions. Assume that $f_n \to f$ uniformly and that f_n is continuous. Show that f is continuous.

   Answer: Take $\epsilon > 0$, $x \in [0,1]$. Choose n_0 so that $|f_n - f|_\infty < \epsilon/3$ for $n \geq n_0$ and pick $\delta > 0$ so that $|f_{n_0}(x) - f_{n_0}(y)| < \epsilon/3$ for all y with $|y - x| < \delta$. Then for all y with $|y - x| < \delta$, $|f(x) - f(y)| < |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + f_{n_0}(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

4. Let $f\colon [0,1] \to \mathbb{R}$ be a function and $f_n\colon [0,1] \to \mathbb{R}$ be a sequence functions. Assume that $f_n \to f$ pointwise and that f_n is continuous. Show that f is not necessarily continuous.

   Answer: Take $f_n(x) = (1 - nx)$ for $x \in [0, 1/n]$ and $f_n(x) = 0$ elsewhere. Then $f_n \to f$ pointwise, where

$f(0) = 1$ and $f(x) = 0$ for $x \in (0, 1]$.

Appendix C Explicit methods for solving ODE's

This Appendix summarises explicit methods for solving ODE's. Since most of the material is already covered in first year material, it will not be covered in the lectures.

C.1 State independent

- This section summarises techniques for **solving** ODE's.
- The first subsections are about finding $x \colon \mathbb{R} \to \mathbb{R}$ so that $x' = f(x, t)$ and $x(0) = x_0$ where $f \colon \mathbb{R}^2 \to \mathbb{R}$.
- So the issue is to find curves with prescribed tangents.
- Let us first review methods for explicitly solving such equations (in part reviewing what you already know).

C.2 State independent $\dot{x} = f(t)$.

In this case, each solution is of the form $x(t) = \int_0^t f(s)\, ds + x(0)$.

Example 40. Assume the graph $t \mapsto (t, x(t))$ has tangent vector $(1, \sin(t))$ at t. Then $x'(t) = \sin(t)$ and so $x(t) = -\cos(t) + c$. So the solution of the ODE $x'(t) = \sin(t)$ finds a curve which is tangent to the arrows of the vector field.

C.3 Separation of variables

Separation of variables: $\dot{x} = f(t)g(x)$. Then one can find solutions as follows.

$$\int_{x(0)}^{x(T)} \frac{dy}{g(y)} = \int_0^T \frac{1}{g(x(t))}\frac{dx}{dt}dt = \int_0^T f(t)\,dt.$$

Here the first equality follows from the substitution rule (taking $y = x(t)$) and 2nd from $\frac{1}{g(x(t))}\frac{dx}{dt} = f(t)$.

Example 41. $\frac{dx}{dt} = ax + b, x(t) = x_0$. Then $\frac{dx}{ax+b} = dt, x(0) = x_0$ which gives, when $a \neq 0$,

$$(1/a)[\log(ax+b)]_{x_0}^{x(T)} = T,$$
$$\log((ax(T)+b)/(ax_0+b)) = aT$$

and therefore

$$x(T) = x_0 e^{aT} + \frac{e^{aT}-1}{a}b \text{ for } T \in (-\infty, \infty)$$

Example 42. $\frac{dx}{dt} = x^2, x(0) = x_0$. Then $\frac{dx}{x^2} = dt, x(0) = x_0$. Hence $[-1/x]_{x_0}^{x(t)} = t$ and so $x(t) = \frac{1}{1/x_0 - t}$. Note that $x(t)$ is well-defined for $t \in (-\infty, 1/x_0)$ but that $x(t) \to \infty$ as $t \uparrow 1/x_0$. *The solution goes to infinity in finite time.*

Example 43. $\frac{dx}{dt} = \sqrt{|x|}, x(0) = x_0$. If $x_0 > 0$ and $x(t) > 0$ then we obtain $\frac{dx}{\sqrt{x}} = dt, x(0) = x_0$ and so $2\sqrt{x(t)} - 2\sqrt{x_0} = t$. Thus $x(t) = (\sqrt{x_0} + t/2)^2$ for $t \in (-2\sqrt{x_0}, \infty)$. When $t = -2\sqrt{x_0}$ then $x(t) = 0$, so need to analyse this directly.

When $x_0 = 0$ then there are many solutions (*non-uniqueness*). For any $-\infty \leq t_0 \leq 0 \leq t_1 \leq \infty$

$$x(t) = \begin{cases} -(t-t_0)^2/4 & \text{for } t \in (-\infty, t_0) \\ 0 & \text{for } t \in [t_0, t_1] \\ (t-t_1)^2/4 & \text{for } t \in (t_1, \infty) \end{cases}$$

is a solution.

So, without imposing some assumptions, solutions **need not be unique**.

C.4 Linear equations $x' + a(t)x = b(t)$.

To solve this, first consider the **homogeneous case** $x' + a(t)x = 0$. This can be solved by separation of variables: dx/x=-a(t)dt and so $x(t) = x_0 \exp[-\int_0^t a(s)\,ds]$.

To find the solution of the ODE, apply the *variation of variables 'trick'*: substitute $x(t) = c(t)\exp[-\int_0^t a(s)\,ds]$ in the equation and obtain an equation for $c(t)$.

Example 44. $x' + 2tx = t$. The homogeneous equation $x' + 2tx = 0$ has solution $x(t) = ce^{-t^2}$.

Substituting $x(t) = c(t)e^{-t^2}$ into $x' + 2tx = t$ gives $c'(t)e^{-t^2} + c(t)(-2t)e^{-t^2} + 2tc(t)e^{-2t^2} = t$, i.e. $c'(t) = te^{t^2}$. Hence $c(t) = c_0 + (1/2)e^{t^2}$ and therefore $x(t) = c_0e^{-t^2} + (1/2)$. That the equation is of the form

$$c_0 \cdot \text{solution of hom.eq} + \text{ special solution}$$

is due to the fact that the space of solutions $x' + 2tx = 0$ is linear (linear combination of solutions are again solutions).

C.5 Exact equations $M(x,y)dx + N(x,y)dy = 0$ when $\partial M/\partial y = \partial N/\partial x$.

Suppose $f(x,y) \equiv c$ is a solution. Then $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy = 0$ and this corresponds to the ODE if $\partial f/\partial x = M$ and $\partial f/\partial y = N$. But if f is twice differentiable we have

$$\partial M/\partial y = \partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x = \partial N/\partial x.$$

It turns out that this necessary condition for 'exactness' is also sufficient if the domain we consider has no holes (is simply connected).

Example 45. $(y - x^3)dx + (x + y^2)dy = 0$. The exactness condition is satisfied (check!). How to find f with $\partial f/\partial x = y - x^3$ and $\partial f/\partial y = x + y^2$? The first equation gives $f(x, y) = yx - (1/4)x^4 + c(y)$. The second equation then gives $x + c'(y) = \partial f/\partial y = x + y^2$. Hence $c(y) = y^3/3 + c_0$ and $f(x) = yx - (1/4)x^4 + y^3/3 + c_0$ is a solution.

Sometimes you can rewrite the equation to make it exact.

Example 46. $ydx + (x^2y - x)dy = 0$. This equation is **not** exact (indeed, $\dfrac{\partial y}{\partial y} \neq \dfrac{\partial(x^2y - x)}{\partial x}$). If we rewrite the equation as $y/x^2dx + (y - 1/x)dy = 0$ then it becomes exact.

Clearly this was a lucky guess. Sometimes one can guess that by multiplying by a function of (for example) x the ODE becomes exact.

Example 47. The equation $(xy-1)dx+(x^2-xy)dy = 0$ is not exact. Let us consider the equation $\mu(x)(xy-1)dx+\mu(x)(x^2-xy)dy = 0$. The exactness condition is $\mu x = \mu'(x^2 - xy) + \mu(2x-y)$. Rewriting this gives $\mu'(x)x(x-y)+\mu(x)(x-y) = 0$, and so $x\mu' + \mu = 0$ implies the exactness condition. So we can take $\mu(x) = 1/x$. So instead of the original ODE we solve $(y - 1/x)dx + (x - y)dy = 0$ as in the previous example.

C.6 Substitutions

- Sometimes one can simplify the ODE by a substitution.
- One instance of this method, is when the ODE is of the form $M(x, y)dx + N(x, y)dy = 0$ where M, N are homogeneous polynomials of the same degree.

  In this case we can simplify by substituting $z = y/x$.

Example 48. $(x^2-2y^2)dx + xydy = 0$. Rewrite this as $\frac{dy}{dx} = \frac{-x^2+2y^2}{xy}$. Substituting $z = y/x$, i.e. $y(x) = z(x)x$ gives

$$x\frac{dz}{dx} + z = \frac{dy}{dx} = \frac{-1+2z^2}{z}.$$

Hence

$$\frac{dz}{dx} = \frac{-1}{z} + z.$$

This can be solved by separation of variables.

C.7 Higher order linear ODE's with constant coefficients

Note that each y_1 and y_2 are solutions of

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0 \tag{29}$$

then linear combinations of y_1 and y_2 are also solutions.

Substituting $y(x) = e^{rx}$ in this equation gives:

$$e^{rn}\left(r^n + a_{n-1}r^{n-1} + \cdots + a_0\right) = 0.$$

Of course the polynomial equation $r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$ has n solutions $r_1, \dots, r_n \in \mathbb{C}$.

Case 1: If these r_i's are all different (i.e. occur with single multiplicity), then we obtain as a solution:

$$y(x) = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}.$$

Case 2: What if, say, r_1 is complex? Then $\bar{r}_1$ is also a root, so we may (by renumbering) assume $r_2 = \bar{r}_1$ and write $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ with $\alpha, \beta \in \mathbb{R}$. So

$$e^{r_1 x} = e^{\alpha x}(\cos(\beta x) + \sin(\beta x)i), e^{r_2 x} = e^{\alpha x}(\cos(\beta x) - \sin(\beta x)i),$$

and $c_1e^{r_1x}+c_2e^{r_2x} = (c_1+c_2)e^{\alpha x}\cos(\beta x)+(c_1-c_2)ie^{\alpha x}\sin(\beta)$. Taking $c_1 = c_2 = A/2 \in \mathbb{R} \implies c_1e^{r_1x} + c_2e^{r_2x} = Ae^{\alpha x}\cos(\beta x)$ On the other hand, taking $c_1 = -(B/2)i = -c_2 \implies c_1e^{r_1x} + c_2e^{r_2x} = Be^{\alpha x}\sin(\beta x)$ (nothing prevents us choosing c_i non-real!!).

So if $r_1 = \bar{r}_2$ is non-real, we obtain as a general solution

$$y(x) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x) + c_3e^{r_3x} + \cdots + c_ne^{r_nx}.$$

Case 3: Repeated roots: If $r_1 = r_2 = \cdots = r_k$ then one can check that $c_1e^{r_1x} + c_2xe^{r_2x} + \cdots + c_kx^ke^{r_1x}$ is a solution.

Case 4: Repeated complex roots: If $r_1 = r_2 = \cdots = r_k = \alpha + \beta i$ are non-real, then we have corresponding roots $r_{k+1} = r_{k+2} = \cdots = r_{2k} = \alpha - \beta i$ and we obtain as solution

$$c_1e^{\alpha x}\cos(\beta x) + \cdots + c_kx^ke^{\alpha x}\cos(\beta x)+$$

$$+c_{k+1}e^{\alpha x}\sin(\beta x) + \cdots + c_{2k}x^ke^{\alpha x}\sin(\beta x).$$

Example 49. **Vibrations and oscillations of a spring**

One can model an object attached to a spring by $Mx'' = F_s + F_d$ where F_d is a damping force and F_s a spring force. Usually one assumes $F_d = -cx'$ and $F_s = -kx$. So

$$Mx'' + cx' + kx = 0 \text{ or } x'' + 2bx' + a^2x = 0$$

where $a = \sqrt{k/M} > 0$ and $b = c/(2M) > 0$.

Using the previous approach we solve $r^2 + 2br + a^2$, i.e. $r_1, r_2 = \frac{-2b\pm\sqrt{4b^2-4a^2}}{2} = -b \pm \sqrt{b^2 - a^2}$.

Case 1: If $b^2 - a^2 > 0$ then both roots are real and negative. So $x(t) = x_0(e^{r_1t} + Be^{r_2t})$ is a solution and as $t \to \infty$ we get $x(t) \to 0$.

Case 2: If $b^2 - a^2 = 0$ then we obtain $r_1 = r_2 = -a$ and $x(t) = Ae^{-at} + Bte^{-at}$. So $x(t)$ still goes to zero as $\to \infty$, but when B is large, $x(t)$ can still grow for t not too large.

Case 3: If $b^2 - a^2 < 0$. Then $x(t) = e^{-bt}(A\cos(\alpha t) + B\sin(\alpha t))$ is a solution. Solutions go to zero as $t \to \infty$ but oscillate.

Example 50. Vibrations and oscillations of a spring with forcing

Suppose one has external forcing

$$Mx'' + cx' + kx = F_0 \cos(\omega t).$$

If $b^2 - a^2 < 0$ (using the notation of the previous example) then

$$e^{-bt}(A\cos(\alpha t) + B\sin(\alpha t))$$

is still the solution of the homogeneous part and one can check

$$\frac{F_0}{\sqrt{(k-\omega^2 M)^2 + \omega^2 c^2}}(\omega c \sin(\omega t) + (k - \omega^2 M)\cos(\omega t) =$$

$$\frac{F_0}{\sqrt{(k-\omega^2 M)^2 + \omega^2 c^2}} \cos(\omega t - \phi)$$

is a particular solution where $\omega = \arctan(\omega c/(k - \omega^2 M))$.

$$\frac{F_0}{\sqrt{(k-\omega^2 M)^2 + \omega^2 c^2}} \cos(\omega t - \phi)$$

is a particular solution where $\omega = \arctan(\omega c/(k - \omega^2 M))$.

Here c is the damping, M is the mass and k is the spring constant.

- If damping $c \approx 0$ and $\omega \approx k/M$ then the denominator is large, and the oscillation has large amplitude.
- $(k - \omega^2 M)^2 + \omega^2 c^2$ is minimal for $\omega = \sqrt{\frac{k}{M} - \frac{c^2}{2M^2}}$ and so this is the natural frequency (or eigen-frequency).

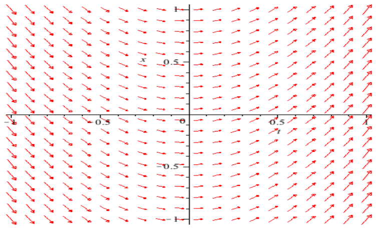


Figure 5: The vector field $(1, \sin(t))$ drawn with the Maple command: with(plots):fieldplot([1, sin(t)], t = -1 .. 1, x = -1 .. 1, grid = [20, 20], color = red, arrows = SLIM);

- This is important for bridge designs (etc), see
  - `http://www.ketchum.org/bridgecollapse.html`
  - `http://www.youtube.com/watch?v=3mclp9QmCGs`
  - `http://www.youtube.com/watch?v=gQK21572oSU`

C.8 Solving ODE's with maple

Example 51. `> ode1 := diff(x(t), t) = x(t)^2;`

```
                         d              2
                        --- x(t) = x(t)
                         dt
> dsolve(ode1);
                                     1
                        x(t) = --------
                                -t + _C1
> dsolve({ode1, x(0) = 1});
                                      1
                         x(t) = - -----
                                   t - 1
```

Example 52. Example: $y'' + 1 = 0$.

```
> ode5 := diff(y(x), x, x)+1 = 0;
                    / d  / d      \\
                    |--- |--- y(x)|| + 1 = 0
                    \ dx \ dx     //
> dsolve(ode5);
                           1   2
                  y(x) = - - x   + _C1 x + _C2
                           2
```

C.9 Solvable ODE's are rare

It is not that often that one can solve an ODE explicitly. What then?

- Use approximation methods.
- Use topological and qualitative methods.
- Use numerical methods.

This module will explore all of these methods.

In fact, we need to investigate whether we can even speak about solutions. Do solutions exist? Are they unique? Did we find all solutions in the previous subsections?

C.10 Chaotic ODE's

Very simple differential equations can have complicated dynamics (and clearly cannot be solved analytically). For example the famous Lorenz differential equation

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{30}$$

with $\sigma = 10, r = 28, b = 8/3$.

has solutions which are chaotic and have sensitive dependence (the *butterfly effect*).

http://www.youtube.com/watch?v=ByH8_nKD-ZM

Appendix D A proof of the Jordan normal form theorem

In this section we will give a proof of the Jordan normal form theorem.

Lemma 15. Let $L_1, L_2 \colon V \to V$ where V is a finite dimensional vector space. Assume $L_1 L_2 = 0$ and $\ker(L_1) \cap \ker(L_2) = \{0\}$. Then $V = \ker(L_1) \oplus \ker(L_2)$.

Proof. Let n be the dimension of V and let $\Im(L_2)$ stands for the range of L_2. Note that $\dim\ker(L_2) + \dim\Im(L_2) = n$, Since $L_1 L_2 = 0$ it follows that $\ker(L_1) \supset \Im(L_2)$ and therefore $\dim\ker(L_1) \geq \dim\Im(L_2) = n - \dim\ker(L_2)$. As $\ker(L_1) \cap \ker(L_2) = \{0\}$, equality holds in $\dim\ker(L_1) + \dim\ker(L_2) \geq n$ and the lemma follows. $\square$

Proposition 1. Let $L \colon V \to V$ where V is a finite dimensional vector space. Let $\lambda_1, \dots, \lambda_s$ be its eigenvalues with (algebraic) multiplicity m_i. Then one can write $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ where $V_i = \ker((L - \lambda_i I)^{m_i})$ and so $L(V_i) \subset V_i$.

Proof. Consider the polynomial $p(t) = \det(tI - L)$. This is a polynomial of degree n, where n is the dimension of the vector space and with leading term t^n. By the Cayley-Hamilton theorem one has $p(L) = 0$ and of course $p(L)$ is also of the form $L^n + c_1 L^{n-1} + \cdots + c_n = 0$. This can be factorised as

$$(L - \lambda_1 I)^{m_1} (L - \lambda_2 I)^{m_2} \cdots (L - \lambda_s I)^{m_s} = 0,$$

where all λ_i's are distinct - here we use that the factors $(L - \lambda_i I)$ commute.

We claim that $\ker((L-\lambda_i I)^{m_i}) \cap \ker(L-\lambda_j I)^{m_j} = 0$. Indeed, if not then we can take a vector $v \neq 0$ which is in the intersection. We may assume $m_i \geq m_j$. Choose $1 \leq m'_j \leq m_j$ minimal so that $(L-\lambda_j I)^{m'_j} v = 0$ and $(L-\lambda_j I)^{m'_j-1} v \neq 0$. Since $v \in \ker((L-\lambda_i I)^{m_i})$ we have that $w := (L-\lambda_j I)^{m'_j-1}(L-\lambda_i I)^{m_i} v$ is equal to 0, but on the other hand w is equal to $(L-\lambda_j I)^{m'_j-1}((L-\lambda_j I)+(\lambda_j-\lambda_i))^{m_i} v$ which, by expanding the latter expression (and using that $v \in \ker((L-\lambda_i I)^{m_i}))$ is equal to $(L-\lambda_j I)^{m'_j-1}(\lambda_j-\lambda_i)^{m_i} v \neq 0$. This contradiction proves the claim.

This means that we can apply the previous lemma inductively to the factors $(L-\lambda_i I)^{m_i}$, and thus obtain the proposition. □

It follows that if we choose T so that it sends the decomposition $\mathbb{R}^{n_1} \oplus \ldots \mathbb{R}^{n_k}$, where $n_i = \dim V_i$, to $V_1 \oplus \cdots \oplus V_k$ then $T^{-1}LT$ is of the form $\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix}$ where A_i are square matrices corresponding to V_i (and the remaining entries are zero). The next theorem gives a way to find a more precise description for a linear transformation T so that $T^{-1}LT$ takes the Jordan form. Indeed, we apply the next theorem to each matrix A_i separately. In other words, for each choice of i, we take $W = V_i$, $A = (L-\lambda_i I)|V_i$ and $m = m_i$ in the theorem below.

Theorem 33. Let $A\colon W \to W$ be a linear transformation of a finite dimensional vector space so that $A^m = 0$ for some $m \geq 1$. Then there exists a basis W of the form

$$u_1, Au_1, \ldots, A^{a_1-1}u_1, \ldots, u_s, \ldots, A^{a_s-1}(u_s)$$

where $a_i \geq 1$ and $A^{a_i}(u_i) = 0$ for $1 \leq i \leq s$.

Remark: Note that $A^{a_j-1}(u_j) = (T-\lambda_i)^{a_j-1}(u_j)$ is in the kernel of $A = T - \lambda_i I$, so is an eigenvector of A corresponding to eigenvalues 0 (i.e. an eigenvector of T corresponding to eigenvalue λ_i. The vector $w_j^1 = A^{a_j-2}(u_j) = (T-\lambda_i I)^{a_j-2}(u_j)$ corresponds to a vector so that $Aw_j^1 = w_j$, so $Tw_j^1 = \lambda w_j^1 + w_j$, and so on. So as in Chapter 2, if we take the matrix T with columns

$$A^{a_1-1}u_1, \dots, u_1, A^{a_2-1}u_2, \dots, u_2, A^{a_s-1}u_s, \dots, u_s$$

then $T^{-1}LT$ will have the required Jordan form with λ on the diagonal, and 1's in the off-diagonal except in columns $a_1, a_1 + a_2, \dots, a_1 + a_2 + \cdots + a_s$.

Proof. The proof given below goes by induction with respect to the dimension of W. When $\dim W = 0$ the statement is obvious. Assume that the statement holds for dimensions $<$ $\dim(W)$. Note that $A(W) \neq W$ since otherwise $AW = W$ and therefore $A^m(W) = A^{m-1}(W) = \cdots = W$ which is a contradiction. So $\dim A(W) < \dim W$ and by induction there exists $v_1, \dots, v_l \in A(W)$ so that

$$v_1, Av_1, \dots, A^{b_1-1}(v_1), \dots, v_l, Av_l, \dots, A^{b_l-1}v_l \qquad (31)$$

is a basis for $A(W)$ and $A^{b_i}(v_i) = 0$ for $1 \leq i \leq l$. Since $v_i \in A(W)$ one can choose u_i so that $Au_i = v_i$. The vectors $A^{b_1-1}v_1, \dots, A^{b_l-1}v_l$ are linearly independent and are contained in $\ker(A)$ and so we can find vectors $u_{l+1}, \dots, u_m$ so that

$$A^{b_1-1}v_1, \dots, A^{b_l-1}v_l, u_{l+1}, \dots, u_m \qquad (32)$$

forms a basis of $\ker(A)$. But then

$$u_1, Au_1, \dots, A^{b_1}(u_1), \dots, u_l, \dots, A^{b_l}u_l, u_{l+1}, \dots, u_m \qquad (33)$$

is the required basis of W. Indeed, consider a linear combination of vectors from (33) and apply A. Then, because $v_i = Au_i$, we obtain a linear combination of the vectors from (31) and so the corresponding coefficients are zero. The remaining vectors are in the kernel of A and are linearly independent because they correspond to (32). This proves the linear independence of (33). That (33) spans W holds, because the number of vectors appearing in (33) is equal to $\dim \ker(A) + \dim AW$. Indeed, $A^{b_1}(u_1), \ldots, A^{b_l} u_l, u_{l+1}, \ldots, u_m$ are all in $\ker(A)$ (they are the same vectors as the vectors appearing in (32)). The remaining number of vectors is $b_1 + \cdots + b_l$ which is the same as the dimension of AW, as (31) forms a basis of this space. It follows that the total number of vectors in (33) is the same as $\dim(\ker(A)) + \dim(AW)$ and so together with their linear independence this implies that (33) forms a basis of W. $\square$