

LARGE DERIVATIVES, BACKWARD CONTRACTION AND INVARIANT DENSITIES FOR INTERVAL MAPS

HENK BRUIN; JUAN RIVERA-LETELIER;
WEIXIAO SHEN; SEBASTIAN VAN STRIEN

ABSTRACT. In this paper, we study the dynamics of a smooth multimodal interval map f with non-flat critical points and all periodic points hyperbolic repelling. Assuming that $|(f^n)'(f(c))| \rightarrow \infty$ as $n \rightarrow \infty$ holds for all critical points c , we show that f satisfies the so-called backward contracting property with an arbitrarily large constant, and that f has an invariant probability μ which is absolutely continuous with respect to the Lebesgue measure and the density of μ belongs to L^p for all $p < \ell_{\max}/(\ell_{\max} - 1)$, where ℓ_{\max} denotes the maximal critical order of f . In the appendix, we prove that various growth conditions on the derivatives along the critical orbits imply stronger backward contraction.

1. INTRODUCTION

The concept of absolutely continuous invariant measures plays an important role in studying the chaotic behavior of non-uniformly hyperbolic dynamical systems. In the area of interval dynamics, various conditions have been shown to guarantee the existence of an invariant probability which is absolutely continuous with respect to Lebesgue measure (acip). In [M], it was proved that an acip exists for an S-multimodal map without periodic attractors or recurrent critical points. In [CE], it was proved that an S-unimodal map f satisfying the following condition (the *Collet-Eckmann* condition) has an acip:

$$(CE) \quad \liminf_{n \rightarrow \infty} \frac{\log |(f^n)'(f(c))|}{n} > 0,$$

where c denotes the critical point of f . In [NS2], the following summability condition (the *Nowicki-van Strien condition*) was shown to imply the existence of an acip for an S-unimodal map:

$$(NS) \quad \sum_{n=0}^{\infty} \frac{1}{|(f^n)'(f(c))|^{1/\ell}} < \infty,$$

where ℓ is the order of the critical point. Moreover, it was proved that the density of the acip with respect to Lebesgue measure belongs to L^p for all $p < \ell/(\ell - 1)$. (Note that this regularity is the best possible since the density is never $L^{\ell/(\ell-1)}$.) Later on, this result was extended to the multimodal setting in [BS].

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For contact: Fax +86-551-3601005, Email wxshen@ustc.edu.cn; more in the last page.

In [BSS], we found that for S-unimodal maps, these conditions are far too restrictive for the existence of an acip. In fact, there exists a constant $C > 0$ depending on the critical order such that

$$\liminf_{n \rightarrow +\infty} |(f^n)'(f(c))| \geq C$$

implies the existence of an acip. In this paper we shall extend this last result to the multimodal setting.

Main Theorem. *Let $f : [0, 1] \rightarrow [0, 1]$ be a C^3 multimodal interval map with non-flat critical points and with all periodic points hyperbolic repelling. Assume that for each critical point c , we have*

$$\lim_{n \rightarrow +\infty} |(f^n)'(f(c))| = \infty,$$

then f has an acip μ whose density with respect to the Lebesgue measure belongs to L^p for all $1 \leq p < \ell_{\max}/(\ell_{\max} - 1)$, where m denotes the Lebesgue measure and ℓ_{\max} is the maximum of the orders of the critical points.

Here, as usual, by saying that f is of class C^3 with non-flat critical points, we mean that f is C^1 everywhere and satisfies the following:

- f is C^3 outside $\text{Crit}(f) = \{x \in [0, 1] : f'(x) = 0\}$;
- for each $c \in \text{Crit}(f)$, there exists a number $\ell_c > 1$ (called *the order of c*) and C^3 diffeomorphisms ϕ, ψ of \mathbb{R} with $\phi(c) = \psi(f(c)) = 0$ such that

$$|\psi \circ f(x)| = |\phi(x)|^{\ell_c}$$

holds in a neighborhood of c .

We shall use \mathcal{A} to denote the class of all C^3 interval maps with non-flat critical points and with all periodic points hyperbolic repelling.

In order to state a more precise version of our main theorem, we need to introduce more notation. For a fixed positive integer N and a positive number $\ell_{\max} > 1$, denote by $\mathcal{A}(N, \ell_{\max})$ the collection of all maps $f \in \mathcal{A}$ having exactly N critical points, and for which the maximum of the orders of the critical points is ℓ_{\max} . For $K > 0$ denote by $\mathcal{A}(N, \ell_{\max}, K)$ the class of all maps $f \in \mathcal{A}(N, \ell_{\max})$ for which the following holds:

(*) *there exists a neighborhood \mathcal{V} of $f(\text{Crit}(f))$ such that if $f^n(f(c)) \in \mathcal{V}$ for some critical point $c \in \text{Crit}(f)$, then*

$$(1) \quad |Df^n(f(c))| > K.$$

We shall actually prove the following Main Theorem', which clearly implies the Main Theorem.

Main Theorem'. *Given a positive integer N , real numbers $\ell_{\max} > 1$ and $p \in [1, \ell_{\max}/(\ell_{\max} - 1))$, there exists a constant $K = K(N, \ell_{\max}, p)$ such that if $f \in \mathcal{A}(N, \ell_{\max}, K)$, then f has an acip μ with $d\mu/dm \in L^p$, where m denotes the Lebesgue measure.*

The result of the density is new even in the unimodal setting. We note that the proof in this paper is quite different from and significantly simpler than that of [BSS]. In fact, the proof follows much closer the proof in [NS2] but is simpler; the main reason for this is that we use here the notion of nice intervals which significantly simplifies the proof of formula (8) compared to the corresponding inductive statement in [NS2].

It is probably difficult to improve these results. It is not possible to give a topological condition equivalent to the existence of acip since the last property is not a topological invariant, not even a quasisymmetric invariant, see [B1]. Moreover, although no invariant density needs to exist if $|Df^n(f(c))|$ remains small [BKNS], $\liminf_{n \rightarrow +\infty} |Df^n(f(c))| = 0$ does not rule out the existence of acip [B2].

Clearly, the condition (*) is a C^1 invariant in the unimodal case, i.e., if for a given K two unimodal maps $f, g \in \mathcal{A}$ are C^1 conjugate, and f satisfies (*), then so does g . However, for multimodal maps, this is not the case. So our Main Theorem' applies to a slightly more general case, i.e., it holds for maps $g \in \mathcal{A}$ which is C^1 conjugate to a map f satisfying the requirements in the Main Theorem'.

To prove the Main Theorem', we shall use the following terminology, which was first introduced in [R]. Let $CV = CV(f) = f(\text{Crit}(f))$ be the set of critical values of f . We say that f satisfies *the backward contracting property with constant r* ($BC(r)$ in short) if the following holds: there exists $\varepsilon_0 > 0$ such that for each $\varepsilon < \varepsilon_0$, and any component W of $f^{-s}(B(CV, r\varepsilon))$, $s \geq 1$,

$$(2) \quad \text{dist}(W, CV) < \varepsilon \text{ implies } |W| < \varepsilon.$$

We say that f satisfies $BC(\infty)$ if it satisfies $BC(r)$ for all $r > 1$. Clearly, for any $r > 1$, the property $BC(r)$ implies that f has no critical relation, i.e., no critical point is mapped into the critical set under forward iteration.

The proof of the Main Theorem' breaks into the following two propositions.

Proposition 1. *For real number $\ell_{\max} > 1$ and $r > 1$, there exists $K = K(r, \ell_{\max})$ such that if f is a map in the class $\mathcal{A}(N, \ell_{\max}, K)$ for some $N \geq 1$, then it satisfies the property $BC(r)$.*

We shall provide two proofs of this proposition. The first one is given in Sect. 2 as a consequence of Lemma 2 and Proposition 4. The second one is given in the appendix, see Theorem 1. The reason that K depends only on f through N and ℓ_{\max} , is that all constants related to f (such as those related to non-flatness) vanish when looking at sufficiently small scales.

Proposition 2. *For any positive integer N , real number $\ell_{\max} > 1$ and $\kappa \in (0, 1)$, there exists $r = r(N, \ell_{\max}, \kappa)$ such that if $f \in \mathcal{A}(N, \ell_{\max})$ satisfies the $BC(r)$ property then there exists a constant M such that for every Borel set A we have*

$$(3) \quad |f^{-n}(A)| \leq M|fA|^{\kappa/\ell_{\max}}.$$

It follows from the inequality (3) that any weak limit μ of

$$\frac{1}{n} \sum_{i=0}^{n-1} (f^i)_* m$$

is an acip of f . By the argument above Theorem A in [NS2], the density of μ with respect to m is in L^p , where $p \rightarrow \ell_{\max}/(\ell_{\max} - 1)$ as $\kappa \rightarrow 1$.

The proof of Proposition 2 is given in Sect. 3 and 4.

In the Appendix we state and prove a result which is somewhat related to the proof of Proposition 1. It shows that various growth conditions of the derivatives along the critical orbits implies stronger backward contraction.

Remark. If f is unimodal, then the acip is ergodic and unique. A general multimodal interval map may have more than one acips. For a map satisfying $BC(r)$ for a large r , the estimate provided by Proposition 2 rules out the existence of Cantor attractors. Thus any compact forward invariant set of positive measure contains a cycle of periodic intervals [SV], each supporting an acip.

Notation. Unless otherwise stated, $X = [0, 1]$ and $f : X \rightarrow X$ is a map in the class \mathcal{A} . We will assume, without loss of generality, that $f(\partial X) \subset \partial X$ and that $f'(x) \neq 0$ for $x \in \partial X$.

If J is an interval and $\lambda > 0$, we use λJ to denote the concentric open interval which has length $\lambda|J|$. We say that J is λ -well inside another interval I or that I contains the λ -scaled neighborhood of J , if $I \supset (1+2\lambda)J$.

2. REAL BOUNDS

We shall use the following result throughout our analysis.

Proposition 3. *Let $f \in \mathcal{A}$, let $s \geq 1$ be an integer and let $T = (a, b)$ be an interval. Assume that $f^s|_T$ is a diffeomorphism onto its image and that $|f^s(T)|$ is sufficiently small. Then*

- (i) *(the Minimum Principle) for every $x \in T$,*

$$|(f^s)'(x)| \geq 0.9 \min(|(f^s)'(a)|, |(f^s)'(b)|);$$

- (ii) *(the One-sided Koebe Principle) Let $x \in T$ be such that $|f^s(a) - f^s(x)| \geq \tau|f^s(x) - f^s(b)|$. Then*

$$|(f^s)'(x)| \geq 0.9 \left(\frac{\tau}{1+\tau} \right)^2 |(f^s)'(b)|,$$

- (iii) *(the Koebe Principle) If J is a subinterval of T such that $f^s(J)$ is τ -well inside $f^s(T)$, then for any $x, y \in J$,*

$$0.9 \left(\frac{\tau}{1+\tau} \right)^2 \leq \frac{|(f^s)'(x)|}{|(f^s)'(y)|} \leq \frac{1}{0.9} \left(\frac{1+\tau}{\tau} \right)^2;$$

- (iv) *(the Macroscopic Koebe Principle) If J is a subinterval of T such that $f^s(J)$ is τ -well inside $f^s(T)$, then J is τ' -well inside T , where $\tau' = 0.9\tau^2/(1+2\tau)$.*

Proof. If f has negative Schwarzian derivative, then so does $f^s|T$. In this case, it is well-known that the statements hold with 0.9 being replaced by 1 (and without the assumption that $f^s(T)$ has a small length), see for example [MS]. For the general case, we first note that $|f^s(T)|$ small implies that $|T|$ is small as well since f has no wandering interval, see [MS]. Then we apply a theorem of Graczyk and Sands [GS] which states that any $f \in \mathcal{A}$ is real-analytically conjugate to a map with negative Schwarzian derivative. \square

Remark. An alternative to prove the proposition – again for maps f without negative Schwarzian derivative – is by using Theorem C of [SV].

2.1. Backward contraction. A sequence of open intervals $\{G_j\}_{j=0}^s$ is called a *chain* if for each $0 \leq j < s$, G_j is a component of $f^{-1}(G_{j+1})$. The *order* of the chain is defined to be the number of j 's with $0 \leq j < s$ and such that G_j contains a critical point.

For each critical point c and $\varepsilon > 0$, let $\hat{B}_\varepsilon(c)$ be the connected component of $f^{-1}((f(c) - \varepsilon, f(c) + \varepsilon))$ containing c . Moreover, let

$$\hat{B}_\varepsilon = \bigcup_{c \in \text{Crit}(f)} \hat{B}_\varepsilon(c).$$

Note that provided that ε is small enough,

$$\hat{B}_{r\varepsilon}(c) \approx r^{1/\ell_c} \hat{B}_\varepsilon(c).$$

Lemma 1. *For any $\rho > 0$ and $\ell_{\max} > 1$, there exists $K > 1$, and for each $f \in \bigcup_{N=1}^{\infty} \mathcal{A}(N, \ell_{\max}, K)$ there exists $\varepsilon_0 > 0$ with the following property. Let $c, c' \in \text{Crit}(f)$ and $\varepsilon \in (0, \varepsilon_0)$. If $f^s(c) \in \hat{B}_\varepsilon(c')$ for some $s \geq 1$, and if J is the component of $f^{-s}(\hat{B}_\varepsilon(c'))$ containing c , then*

$$J \subset \hat{B}_{\rho\varepsilon}(c).$$

Proof. We may assume that $\rho \in (0, 1)$. Consider the chains $\{G_j\}_{j=0}^s$ and $\{H_j\}_{j=0}^s$ with $G_s = \hat{B}_{2\varepsilon}(c') \supset H_s = \hat{B}_\varepsilon(c')$ and $G_0 \supset H_0 = J$.

Let $s_1 < s$ be maximal such that G_{s_1} contains a critical point c_1 . Let H'_{s_1+1} be the convex hull of $H_{s_1+1} \cup \{f(c_1)\}$.

Claim. Provided that ε is small enough and that K is large enough, we have

$$(4) \quad H_{s_1} \subset \hat{B}_{\rho\varepsilon}(c_1).$$

In fact, since

$$f^{s-s_1-1} : G_{s_1+1} \rightarrow G_s$$

is a diffeomorphism with $|G_s|$ small, it follows from the One-sided Koebe Principle (Proposition 3 (ii)) that for each $x \in H'_{s_1+1}$, we have

$$(5) \quad |Df^{s-s_1-1}(x)| \geq C |Df^{s-s_1-1}(f(c_1))|,$$

where C is a universal constant. Provided that ε is small enough, we have

$$|Df^{s-s_1}(f(c_1))| \geq K$$

by the hypothesis. Moreover, by non-flatness of the critical points, we have

$$|Df(f^{s-s_1}(c_1))| \leq 2\ell_c \frac{|fG_s|}{|G_s|}.$$

Thus

$$|Df^{s-s_1-1}(f(c_1))| = \frac{|Df^{s-s_1}(f(c_1))|}{|Df(f^{s-s_1}(c_1))|} \geq KC_1 \frac{|G_s|}{|fG_s|},$$

where $C_1 = 1/2\ell_c$. This, equation (5) and the mean value theorem imply

$$\frac{|G_s|}{|H'_{s_1+1}|} \geq CC_1K \frac{|G_s|}{|fG_s|},$$

which implies that

$$|H'_{s_1+1}| \leq \rho\varepsilon$$

provided that K is sufficiently large. The claim follows.

If $s_1 = 0$ then the proof of the lemma is completed. For the general case, the lemma follows by an easy induction on s . \square

Let us say that f satisfies *the property* $BC^*(r)$ if the following holds: for any $\varepsilon > 0$ small enough, $c, c' \in \text{Crit}(f)$, and $s \geq 1$, if $f^s(c) \in \hat{B}_{r\varepsilon}(c')$ and J is the component of $f^{-s}(\hat{B}_{r\varepsilon}(c'))$ which contains c , then $J \subset \hat{B}_\varepsilon(c)$. (So the difference with the property $BC(r)$ is that in equation (2), the assumption $\text{dist}(W, CV) < \varepsilon$ is replaced by $W \cap CV \neq \emptyset$.)

The above lemma can be reformulated as

Proposition 4. *For any $\ell_{\max} > 1$ and $r > 1$ there exists $K \geq 1$ such that each $f \in \bigcup_{N=1}^{\infty} \mathcal{A}(N, \ell_{\max}, K)$ satisfies the property $BC^*(r)$.*

The property $BC^*(r)$ is closely related to $BC(r)$. Clearly the latter implies the former. The other direction is shown in the following proposition.

Lemma 2. *For any $f \in \mathcal{A}$, $BC^*(8^{\ell_{\max}r})$ implies $BC(r)$, where ℓ_{\max} is the maximum of the order of the critical points of f .*

Proof. Let $\varepsilon > 0$ be a small constant. Then for all $c \in \text{Crit}(f)$, $\hat{B}_{8^{\ell_{\max}r\varepsilon}}(c)$ contains the 3-scaled neighborhood of $\hat{B}_{r\varepsilon}(c)$.

Let $c, c' \in \text{Crit}(f)$ and $x \in \hat{B}_\varepsilon(c)$. Let $s \geq 1$ be such that $f^s(x) \in \hat{B}_{r\varepsilon}(c')$ and let J_k be the component of $f^{-(s-k)}(\hat{B}_{r\varepsilon}(c'))$ which contains $f^k(x)$. We want to show that $|J_1| < \varepsilon$.

Let us prove this by induction on s . For $s = 1$ the statement is trivially true. Fix s_0 and assume that the statement holds if $s < s_0$. To prove the statement for $s = s_0$, consider the chain $\{G_j\}_{j=0}^s$ with $G_s = \hat{B}_{8^{\ell_{\max}r\varepsilon}}(c')$ and $G_0 \ni x$. We distinguish two cases:

Case 1. There exists $0 \leq s_1 < s$ such that G_{s_1} contains a critical point c_1 . By the definition of the BC^* property, it follows that $G_{s_1} \subset \hat{B}_\varepsilon(c_1)$. If $s_1 = 0$, then $c_1 = c$ and $J_0 \subset G_0 \subset \hat{B}_\varepsilon(c)$. Otherwise, the statement follows by the induction hypothesis.

Case 2. For any $0 \leq k < s$, G_k contains no critical point. Then $f^{s-1} : G_1 \rightarrow G_s$ is a diffeomorphism. By the Macroscopic Koebe Principle (Proposition 3 (iii)), we obtain that G_1 contains the 1-scaled neighborhood of J_1 . Since $c \notin G_0$, $f(c) \notin G_1$ and $f(x) \in B_\varepsilon(f(c))$, it follows that $|J_1| < \varepsilon$. \square

Proof of Proposition 1. Combine Lemma 2 and Proposition 4. \square

2.2. Nice sets. An open set $V \subset [0, 1]$ is called *nice* if for each $x \in \partial V$ and for any $k \geq 1$, $f^k(x) \notin V$.

Proposition 5. *For any $f \in \mathcal{A}$ satisfying $BC(2)$ and any $\varepsilon > 0$ sufficiently small, the following holds: for each $c \in \text{Crit}(f)$, there exists an open interval V_c such that $\bigcup_{c \in \text{Crit}(f)} V_c$ is nice and such that*

$$\hat{B}_\varepsilon(c) \subset V_c \subset \hat{B}_{2\varepsilon}(c).$$

Proof. The proof follows from the following argument due to Rivera-Letelier, see Lemma 6.2 in [R]. For $\varepsilon > 0$ small, define the open set $V^n = \bigcup_{i=0}^n f^{-i}(\hat{B}_\varepsilon)$. Clearly V^∞ is nice. Take V_c^n to be the connected component of V^n which contains c , and let $V_c = V_c^\infty$. It remains to show that $V_c^n \subset \hat{B}_{2\varepsilon}(c)$ for each n . We do this by induction. For $n = 0$ this holds by definition, so assume it holds for n . Consider $Z = f(V_c^{n+1}) \setminus B(f(c), \varepsilon)$. For $z \in Z$, there exists $m(z) \in \{0, 1, \dots, n\}$ and $c_0(z) \in \text{Crit}(f)$ so that $f^{m(z)}(z) \in \hat{B}_\varepsilon(c_0(z))$. Now choose $z_0 \in Z$ so that $m_0 = m(z_0)$ is minimal among $m(z)$ for points $z \in Z$ and let $\hat{c}_0 = c_0(z_0)$. Since $f^{m_0}(z_0) \in \hat{B}_\varepsilon(\hat{c}_0)$, and since $f^{m_0}(Z) \subset V_c^n$, the induction hypothesis implies

$$f^{m_0}(Z) \subset \hat{B}_{2\varepsilon}(\hat{c}_0).$$

Since f satisfies $BC(2)$ and Z has distance ε to $f(c)$, it follows that $|f(Z)| < \varepsilon$ and $Z \subset \hat{B}_{2\varepsilon}(c)$. This completes the induction step. \square

For $\lambda > 0$ we say that a nice open set V is λ -*nice* if for each return domain J of V , we have $(1 + 2\lambda)J \subset V$.

A nice open set $V \supset \text{Crit}(f)$ will be called a *puzzle neighborhood* of $\text{Crit}(f)$ if each component of V contains exactly one critical point of f .

Lemma 3. *For any $\lambda > 0$ and real number $\ell_{\max} > 1$, there exists $r > 1$ such that for any N and any $f \in \mathcal{A}(N, \ell_{\max})$ which satisfies $BC(r)$ the following holds. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a puzzle neighborhood $V = \bigcup_c V_c$ with the following properties:*

- V is λ -nice;
- for each $c \in \text{Crit}(f)$ we have

$$\hat{B}_\varepsilon(c) \subset V_c \subset \hat{B}_{2\varepsilon}(c).$$

Proof. By Proposition 5, there exists a puzzle neighborhood W of $\text{Crit}(f)$ such that

$$\hat{B}_{r\varepsilon/2}(c) \subset W_c \subset \hat{B}_{r\varepsilon}(c).$$

For each $c \in \text{Crit}(f)$, let V_c be the union of $\hat{B}_\varepsilon(c)$ and all return domains of W which intersects $\hat{B}_\varepsilon(c)$. Clearly, $V = \bigcup_c V_c$ is a puzzle neighborhood of $\text{Crit}(f)$ and moreover, for each $x \in \partial V$ and $k \geq 1$, $f^k(x) \notin W$. Provided r is large enough, each component of V is deep inside a component of W . It follows that each return domain of V is deep inside V , see for example Theorem B(2) in [SV]. \square

If I is an interval which contains a critical point and J (with $J \cap \text{Crit} \neq \emptyset$) is a unicritical pull back of I then we say that J is a *child* of I . So there exists $c \in \text{Crit}(f)$, $s \geq 1$ (called the *transition time from J to I*) and an interval $\tilde{J} \ni f(c)$ such that

- f^{s-1} maps \tilde{J} diffeomorphically onto I ;
- J is the component of $f^{-1}(\tilde{J})$ which contains c .

We shall use the following lemma in the next section:

Lemma 4. *Let $c, c' \in \text{Crit}(f)$, let $I \ni c$ be a λ -nice interval with $|I|$ small and let*

$$J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_m$$

be children of I which contain c' . Then

$$|f(J_i)| \leq \rho^{i-1} |f(J_1)|$$

holds for all i , where $\rho = \rho(\lambda) > 0$ is such that $\rho \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. By definition, for each $i \geq 1$, there exists a positive integer s_i and an interval $T_i \supset fJ_i$ such that f^{s_i-1} maps T_i diffeomorphically onto I and J_i is the component of $f^{-1}(T_i)$ containing c' . Note that $f^{s_i}(T_{i+1})$ is contained in a return domain of I , hence λ -well inside I . By the Koebe Principle, it follows that each T_{i+1} is λ_1 -well inside T_i , where $\lambda_1 \rightarrow \infty$ as $\lambda \rightarrow \infty$. The conclusion follows. \square

3. PULL BACK OF INTERVALS

The goal of this section is to prove the following:

Proposition 6. *For $\kappa \in (0, 1)$, $N \in \mathbb{N}$ and $\ell_{\max} > 1$, there exists $r > 1$ such that if $f \in \mathcal{A}(N, \ell_{\max})$ satisfies $BC(r)$, then there exists a neighborhood U of $\text{Crit}(f)$ such that for any interval $A \subset U$ and any $n \geq 0$, the following holds:*

$$(6) \quad |f^{-n}(A)| \leq M |f(A)|^{\kappa/\ell_{\max}},$$

where M is a constant depending on f .

Let r be a large constant, and assume that f satisfies $BC(r)$. By Lemma 3, there exist $\lambda = \lambda(r, \ell_{\max})$ and $\varepsilon_0 = \varepsilon_0(f) > 0$ small such that for any $\varepsilon < \varepsilon_0$ there exists a λ -nice puzzle neighborhood of $\text{Crit}(f)$ which lies in-between \hat{B}_ε and $\hat{B}_{2\varepsilon}$. Moreover, $\lambda \rightarrow \infty$ as $r \rightarrow \infty$.

Set

$$(7) \quad r_1 = r^{1/\ell_{\max}}.$$

For each $n \geq 0$ and $\delta > 0$, define

$$L_n(\delta) = \sup\{|f^{-m}(A)| : 0 \leq m \leq n, A \subset \hat{B}_{\varepsilon_0/r_1} \text{ is an interval, } |fA| \leq \delta\}.$$

Lemma 5. *Let I be a λ -nice interval such that $\hat{B}_\varepsilon(c) \subset I \subset \hat{B}_{2\varepsilon}(c)$, where $c \in \text{Crit}(f)$ and $\varepsilon < \varepsilon_0$, and let A be an interval such that*

$$A \subset \hat{B}_{\varepsilon/2}(c) \text{ and } \bar{A} \not\subset \hat{B}_{\varepsilon/r_1}(c).$$

Then for all $n \geq 1$,

$$(8) \quad |f^{-n}(A)| \leq C \frac{|A|}{|I|} |f^{-n}(I)| + 2N \sum_{i=1}^{\infty} L_{n-1}(\rho^i |fA|),$$

where $N = \#\text{Crit}(f)$ and $C > 0, \rho \in (0, 1)$ are constants depending only on r and ℓ_{\max} . Moreover, $\rho \rightarrow 0$ as $r \rightarrow \infty$ (for a fixed ℓ_{\max}).

Proof. Let \mathcal{J}_0 be the collection of all components J of $f^{-n}(I)$ such that $f^n : J \rightarrow I$ is a diffeomorphism. Let \mathcal{J}_1 be the collection of all other components of $f^{-n}(I)$.

By the Koebe Principle, for all $J \in \mathcal{J}_0$, we have

$$(9) \quad \frac{|f^{-n}(A) \cap J|}{|J|} \leq C \frac{|A|}{|I|},$$

where C is a constant depending only on ℓ_{\max} .

For each $J \in \mathcal{J}_1$, there exist $n_1 = n_1(J) \in \{0, 1, \dots, n-1\}$ and an interval $J' \supset f^{n-n_1}(J)$ such that J' has a common endpoint with $f^{n-n_1}(J)$ and such that J' is a child of I .

Claim. There exists a constant $C > 0$ depending only on ℓ_{\max} such that for any $J \in \mathcal{J}_1$, and each component A' of $f^{-n_1}(A) \cap J'$, we have

$$(10) \quad |f(A')| \leq Cr_1^{\ell_c-1} \frac{|fJ'|}{|fI|} |f(A)|.$$

In fact, by definition of a child, there exists $\tilde{J}' \supset f(J')$ such that f^{n_1-1} maps \tilde{J}' diffeomorphically onto I . By the One-sided Koebe Principle, there exists a constant C such that

$$\frac{|fA'|}{|fJ'|} \leq C \frac{|A|}{|f^{n_1}(J')|}.$$

Since $A \subset I \subset \hat{B}_{2\varepsilon}(c)$, and $A \not\subset \hat{B}_{\varepsilon/r_1}(c)$, we have

$$\frac{|fI|}{|I|} \frac{|A|}{|fA|} \leq (2r_1)^{\ell_c-1}.$$

Since $A \subset \hat{B}_{\varepsilon/2}(c)$, $I \supset \hat{B}_\varepsilon(c)$ and since $f^{n_1}(J')$ contains a component of $I \setminus A$, $|f^{n_1}(J')|/|I|$ is bounded away from zero. Inequality (10) follows by redefining the constant C .

For any child P of I , let $s(P)$ be the transition time from P to I and let $\mathcal{J}_1(P)$ be the collection of all elements $J \in \mathcal{J}_1$ with $J' = P$. Clearly,

$$\sum_{J \in \mathcal{J}_1(P)} |f^{-n}(A) \cap J| = |f^{-(n-s(P))}(f^{-s(P)}(A) \cap P)|.$$

Since $f^{-s(P)}(A) \cap P$ has at most two components, applying (10) we obtain that

$$(11) \quad \sum_{J \in \mathcal{J}_1(P)} |f^{-n}(A) \cap J| \leq 2L_{n-1} \left(Cr_1^{\ell_c-1} \frac{|fP|}{|fI|} |fA| \right).$$

For each $c' \in \text{Crit}(f)$, let

$$P_1(c') \supsetneq P_2(c') \supsetneq \dots$$

be all the children of I which contain c' . By the $BC(r)$ property,

$$|f(P_1(c'))| \leq 2\varepsilon/r.$$

By Lemma 4,

$$|f(P_i(c'))| \subset \rho_1^{i-1} |f(P_1(c'))|,$$

where ρ_1 is a constant depending on λ , and $\rho_1 \rightarrow 0$ as $\lambda \rightarrow \infty$. So

$$f(P_i(c')) \subset B_{2\rho_1^{i-1}\varepsilon/r}(c').$$

Note that since r is large, and $r_1 = r^{1/\ell_{\max}}$, for each c , $r_1^{\ell_c-1}/r \leq r_1^{\ell_{\max}-1}/r$ is small. Hence

$$\rho := \max(\rho_1, 2Cr_1^{\ell_{\max}-1}/r)$$

is close to zero. By (11),

$$\sum_{J \in \mathcal{J}_1(P_i(c'))} |f^{-n}(A) \cap J| \leq 2L_{n-1}(\rho^i |fA|).$$

Thus

$$\begin{aligned} \sum_{J \in \mathcal{J}_1} |f^{-n}(A) \cap J| &= \sum_{c'} \sum_{i=1}^{\infty} \sum_{J \in \mathcal{J}_1(P_i(c'))} |f^{-n}(A) \cap J| \\ &\leq 2N \sum_{i=1}^{\infty} L_{n-1}(\rho^i |fA|). \end{aligned}$$

Combining this with (9), we obtain inequality (8). \square

Proof of Proposition 6. Fix $\kappa \in (0, 1)$ and write $\alpha = \kappa/\ell_{\max}$. Let

$$M = \left(\frac{r_1}{\varepsilon_0} \right)^\alpha.$$

Assume that r is so large that

$$4Cr_1^{(\kappa-1)/\ell_{\max}} \leq \frac{1}{2},$$

where C is as in Lemma 5. We shall prove by induction on n that the following inequality (12) holds:

$$(12) \quad L_n(\delta) \leq M\delta^\alpha.$$

The case $n = 0$ is trivial. So assume that the inequality holds for all $n < n_0$. To show that inequality (12) is true for $n = n_0$, we shall prove by induction on m the following:

$$(13) \quad L_n \left(\frac{\varepsilon_0}{2^m} \right) \leq M \left(\frac{\varepsilon_0}{2^m} \right)^\alpha.$$

The choice of M clearly guaranteed that inequality (13) holds for $m = 0$. Assuming that the inequality holds for all m less than some positive integer m_0 , let us prove it for $m = m_0$.

Let $A \subset \hat{B}_{\varepsilon_0/r_1}$ be an open interval with $|fA| \leq \varepsilon_0/2^{m_0}$. We need to show that

$$(14) \quad |f^{-n}(A)| \leq M|fA|^\alpha.$$

Let $\varepsilon \in (0, \varepsilon_0)$ be minimal such that $A \subset \hat{B}_{\varepsilon/r_1}(c)$ for some $c \in \text{Crit}(f)$. Let $I \ni c$ be a λ -nice interval such that

$$\hat{B}_\varepsilon \subset I \subset \hat{B}_{2\varepsilon}.$$

Then by Lemma 5, we have

$$|f^{-n}(A)| \leq C \frac{|A|}{|I|} |f^{-n}(I)| + 2N \sum_{i \geq 1} L_{n-1}(\rho^i |fA|).$$

The second term in the right hand side is bounded from above by

$$2NM \sum_{i \geq 1} (\rho^i |fA|)^\alpha = 2NM |fA|^\alpha \frac{\rho^\alpha}{1 - \rho^\alpha} \leq |fA|^\alpha M/2,$$

provided that ρ is small enough. If $I \subset \hat{B}_{\varepsilon_0/r_1}$, then by the induction hypothesis,

$$|f^{-n}(I)| \leq M|fI|^\alpha,$$

since $|fI| \geq \varepsilon_0/2^{m_0-1}$. The same estimate holds in the case $I \supset \hat{B}_{\varepsilon_0/r_1}$ by the choice of M since $|f^{-n}(I)| \leq 1$. Thus

$$|f^{-n}(A)| \leq CM \frac{|A|}{|I|} |fI|^\alpha + \frac{M}{2} |fA|^\alpha.$$

Since

$$\frac{|A|}{|I|} \leq \frac{|fA|}{|fI|} \cdot 4r_1^{1-1/\ell_c},$$

it follows that

$$|f^{-n}(A)| \leq M \left(4Cr_1^{(\kappa-1)/\ell_{\max}} + \frac{1}{2} \right) |fA|^\alpha \leq M|fA|^\alpha.$$

This completes the proof of inequality (13), hence the induction step in (12) and the proposition. \square

4. INVARIANT MEASURE AND THE DENSITY

In this section we prove Proposition 2. So let f be a multimodal interval map as in the proposition.

The strategy of the proof of (3) is to use the Minimum Principle (Proposition 3 (i)) to relate the size of the preimage of a general set to the size of preimages of intervals “at the end of branches”, see Lemma 6 below. This corresponds to the “sliding argument” used in [NS2].

Take $\kappa_1 \in (\kappa, 1)$. Assume that f satisfies $BC(r)$ for a large r . Then by Proposition 6, there exists $\varepsilon_0 > 0$ (small) such that for any interval $Q \subset \hat{B}_{\varepsilon_0}$ and any $n \geq 0$,

$$(15) \quad |f^{-n}(Q)| \leq M_0 |fQ|^{\kappa_1/\ell_{\max}},$$

where $M_0 > 0$ is a constant.

We say that a sequence of open intervals $\{G_i\}_{i=0}^s$ is a *quasi-chain* if for each $0 \leq i < s$, G_i contains a component of $f^{-1}(G_{i+1})$. The *order* of the quasi-chain is defined to be the number of $i \in \{0, 1, \dots, s-1\}$ such that G_i contains a critical point. We say that such a quasi-chain is λ -*admissible* if for each $0 \leq i \leq s$, either $G_i \cap \text{Crit}(f) = \emptyset$, or G_i is λ -nice.

By the argument in Sect. 2, choosing ε_0 smaller if necessary, we may assume that for each $j \geq 0$, there exists a λ -nice interval $V_{j,c}$ with the following property:

$$\hat{B}_{\varepsilon_0/2^{j+1}}(c) \subset V_{j,c} \subset \hat{B}_{\varepsilon_0/2^j}(c),$$

where $\lambda \rightarrow \infty$ as $r \rightarrow \infty$. For each interval \tilde{G} which contains a critical point c and with $\tilde{G} \subset V_{0,c}$, let j be maximal such that $\tilde{G} \subset V_{j,c}$. We call that $V_{j,c}$ is the *enlargement* of \tilde{G} . Clearly, the enlargement of any critical interval is uniformly comparable to itself in size.

Let $V = \bigcup_{c \in \text{Crit}(f)} V_{0,c}$. Fix a positive integer n throughout the rest of this section. For each component J of $f^{-n}(V)$, we define a λ -admissible quasi-chain $\{G_i\}_{i=0}^n$ as follows:

- $G_s = V$;
- Assume that $G_{i+1} \supset f^{i+1}(J)$ is defined, and let \tilde{G}_i be the component of $f^{-1}(G_{i+1})$ which contains $f^i(J)$. Then
 - if \tilde{G}_i is not critical, then $G_i := \tilde{G}_i$;
 - if \tilde{G}_i is critical, then G_i is the enlargement of \tilde{G}_i defined as above.

Let \mathcal{J}_m be the collection of all components of $f^{-n}(V)$ such that the order of the corresponding quasi-chain $\{G_i\}_{i=0}^n$ is equal to m . Each $J \in \mathcal{J}_m$ is associated with a sequence of positive integers k_1, k_2, \dots, k_m and critical points c_0, c_1, \dots, c_m as follows.

- $s_0 = s > s_1 > \dots > s_m \geq 0$ are all the integers such that G_{s_j} is critical, and c_j is the critical point contained in G_{s_j} ;
- \tilde{G}_{s_j} is the k_j -th child of $G_{s_{j-1}}$ which contains c_j .

For any $\mathbf{c} = c_0 c_1 \dots c_m \in \text{Crit}(f)^{m+1}$ and $\mathbf{k} = k_1 k_2 \dots k_m \in \mathbb{Z}_+^m$, we use $\mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ to denote the collection of all components J of $f^{-n}(V)$ with the parameters \mathbf{c} and \mathbf{k} . Note that if $J' \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ and $\{G'_j\}_{j=0}^n$ is the corresponding quasi-chain, then $s_j = s'_j$ and $G_{s_j} = G'_{s'_j}$ for all $0 \leq j \leq m$. Define

$$H_{\mathbf{c}}^{\mathbf{k}} = G_{s_m},$$

and

$$CV_{\mathbf{c}}^{\mathbf{k}} = \{f^{s_0 - s_j}(c_j) : 0 \leq j \leq m\}$$

which has $m + 1$ elements. (Recall that f has no critical relation.) Notice that for each $J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$, the critical values of $f^n|_J$ are contained in $CV_{\mathbf{c}}^{\mathbf{k}}$.

Lemma 6. *For any \mathbf{c}, \mathbf{k} , and any Borel set $A \subset V$, the following holds:*

$$(16) \quad \sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A) \cap J| \leq 3M_0(m+2)|fA|^{\kappa_1/\ell_{\max}},$$

where M_0 is as in (15).

Proof. Let us label the set $CV_{\mathbf{c}}^{\mathbf{k}}$ as $v_1 < v_2 < \dots < v_m$, and let v_0 and v_{m+2} be the left and right endpoints of V_{c_0} . These points divide V_{c_0} into subintervals $Q_k = (v_k, v_{k+1})$, $0 \leq k \leq m+1$. Let $A_k = A \cap Q_k$. It suffices to prove that for each k , the following holds:

$$(17) \quad \sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k) \cap J| \leq 2M_0\rho^{\kappa_1/\ell_{\max}},$$

where $\rho = |fA|$. If $|fQ_k| \leq \rho$, then by Proposition 6,

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k) \cap J| \leq |f^{-n}Q_k| \leq M_0\rho^{\kappa_1/\ell_{\max}}.$$

So assume that $|fQ_k| > \rho$. To show inequality (17), choose $a_k, b_k \in Q_k$ such that

$$L_k = (v_k, a_k), \text{ and } R_k = (b_k, v_{k+1})$$

satisfies

$$|fL_k| = |fR_k| = |fA_k|.$$

For each $J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}$ and each component J' of $f^{-n}Q_k \cap J$, $f^{n+1} : J' \rightarrow fQ_k$ is a diffeomorphism. By the Minimum Principle it follows that

$$\begin{aligned} |(f^n|_{J'})^{-1}(A_k)| &= |(f^{n+1}|_{J'})^{-1}(fA_k)| \\ &\leq 0.9^{-1} \left(|(f^{n+1}|_{J'})^{-1}(fL_k)| + |(f^{n+1}|_{J'})^{-1}(fR_k)| \right) \\ &= 0.9^{-1} \left(|f^{-n}L_k \cap J'| + |f^{-n}R_k \cap J'| \right). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |f^{-n}(A_k) \cap J| &\leq 0.9^{-1} \sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} \left(|f^{-n}L_k \cap J| + |f^{-n}R_k \cap J| \right) \\ &\leq 0.9^{-1} \left(|f^{-n}L_k| + |f^{-n}R_k| \right) \\ &\leq 3M_0\rho^{\kappa_1/\ell_{\max}}, \end{aligned}$$

where in the last step we used (15). This proves (17), and hence completes the proof of the lemma. \square

Applied to the case $\mathbf{c} = \{c\}$, $c \in \text{Crit}(f)$, the lemma gives

$$(18) \quad \sum_{J \in \mathcal{J}_0} |f^{-n}(A) \cap J| \leq 6NM_0 |fA|^{\kappa_1/\ell_{\max}}.$$

Lemma 7. *There is $\sigma > 0$ depending only on r , such that $\sigma \rightarrow 0$ as $r \rightarrow +\infty$, and such that for each $\mathbf{c} = c_0 c_1 \dots c_m$ and $\mathbf{k} = k_0 k_1 \dots k_m$ the following hold:*

(i)

$$\bigcup_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} J \subset f^{-(n-s_m)}(H_{\mathbf{c}}^{\mathbf{k}}),$$

(ii)

$$|f(H_{\mathbf{c}}^{\mathbf{k}})| \leq \sigma^{k_1+k_2+\dots+k_m} |fV|,$$

(iii) *if we put $\sigma_1 = \sigma^{\kappa_1/\ell_{\max}}$, then*

$$\sum_{J \in \mathcal{J}_{\mathbf{c}}^{\mathbf{k}}} |J| \leq M_1 \sigma_1^{k_1+k_2+\dots+k_m},$$

where $M_1 = M_0 |fV|^{\kappa_1/\ell_{\max}}$, and M_0 is as in (15).

Proof. The first item is clear. The second follows from Lemma 4, and $|\tilde{G}_{s_i}| \asymp |G_{s_i}|$. For the third item, notice that by item (i), the left hand side of the inequality does not exceed $|f^{-(n-s_m)}(H_{\mathbf{c}}^{\mathbf{k}})|$. So the inequality follows from the item (ii) of this lemma by (15). \square

Lemma 8. *Provided that r is large enough, the following holds: for any Borel set $A \subset V$ and any $n \geq 1$,*

$$|f^{-n}(A)| \leq M |fA|^{\kappa/\ell_{\max}},$$

where M is a constant.

Proof. Pick D_0 to be the maximal integer such that

$$(19) \quad 6(1+N)^{D_0+1} D_0 \leq \left(\frac{1}{\rho}\right)^{(\kappa_1-\kappa)/\ell_{\max}},$$

where $\rho = |fA|$ and $N = \#\text{Crit}(f)$. For any $m \geq 1$ and $D \geq 1$, let

$$\mathcal{J}_m(D) = \{J \in \mathcal{J}_m : k_1(J) + k_2(J) + \dots + k_m(J) = D\},$$

$$\mathcal{J}(D) = \bigcup_{m=1}^{\infty} \mathcal{J}_m(D).$$

For each $1 \leq m \leq D$, the number of tuples $\mathbf{k} = k_1 k_2 \dots k_m$ satisfying

$$k_1 + k_2 + \dots + k_m = D$$

is given by the binomial coefficient $C_{D-1}^{m-1} := \binom{D-1}{m-1}$. Associated to such a $\mathbf{k} = k_1 k_2 \dots k_m$, there could be at most N^{m+1} corresponding choices of $\mathbf{c} = c_0 c_1 \dots c_m$. From this and Lemma 7 it follows that

$$\sum_{J \in \mathcal{J}(D)} |J| \leq \sum_{m=1}^D \sum_{J \in \mathcal{J}_m(D)} |J| \leq M_1 \sum_{m=1}^D N^{m+1} C_{D-1}^{m-1} \sigma_1^D \leq M_1 N \sigma_2^D,$$

where $\sigma_2 = (N + 1)\sigma_1$ and M_1 is as in Lemma 7. Thus

$$\sum_{D=D_0+1}^{\infty} \sum_{J \in \mathcal{J}(D)} |f^{-n}(A) \cap J| \leq \sum_{D=D_0+1}^{\infty} \sum_{J \in \mathcal{J}(D)} |J| \leq M_1 \frac{N\sigma_2}{1 - \sigma_2} \sigma_2^{D_0}.$$

Provided that r is large enough, σ_2 is close to 0 so that

$$\sigma_2^{\hat{D}} \leq [6(1 + N)^{\hat{D}+2}(\hat{D} + 1)]^{-\tau}$$

for any $\hat{D} \geq 1$ and where τ is so that $\tau(\kappa_1 - \kappa) = \kappa$. This and (19) imply

$$\sigma_2^{D_0+1} \leq \rho^{\tau(\kappa_1 - \kappa)/\ell_{\max}} = \rho^{\kappa/\ell_{\max}}.$$

Since $\rho = |fA|$ we get therefore

$$\sum_{D=D_0+1}^{\infty} \sum_{J \in \mathcal{J}(D)} |f^{-n}(A) \cap J| \leq M_2 |fA|^{\kappa/\ell_{\max}},$$

where M_2 is a constant.

On the other hand, by (16), we have

$$\begin{aligned} \sum_{D=1}^{D_0} \sum_{J \in \mathcal{J}(D)} |f^{-n}(A) \cap J| &\leq \sum_{m=1}^{D_0} \sum_{J \in \mathcal{J}_m} |f^{-n}(A) \cap J| \\ &\leq \sum_{m=1}^{D_0} C_{D_0-1}^{m-1} N^{m+1} 3M_0(m+2) |fA|^{\kappa_1/\ell_{\max}} \\ &\leq 6M_0(1+N)^{D_0+1} D_0 |fA|^{\kappa_1/\ell_{\max}} \\ &\leq M_0 |fA|^{\kappa/\ell_{\max}}, \end{aligned}$$

where in the last inequality we used (19) again. Combining with (18), these estimates imply the lemma. \square

Proof of Proposition 2. Fix $\kappa \in (0, 1)$. Assume that f satisfies $BC(r)$ for a large r . We want to show that there exists $M > 0$ such that inequality (3) holds for any Borel set A and any $n \geq 0$.

By Lemma 8, there exists a neighborhood V of $\text{Crit}(f)$ such that the inequality holds when $A \subset V$.

The general case follows by Mănă's theorem, which asserts that $f|(X \setminus V)$ is uniformly expanding, i.e., there exists $C > 0$ and $\gamma \in (0, 1)$ such that for any $x \in X$ and $k \geq 1$, if $x, f(x), \dots, f^{k-1}(x) \notin V$ then $|(f^k)'(x)| \geq C\gamma^{-k}$. It follows that for any Borel set A and $m \geq 1$,

$$A_m := \{x \in X : x, f(x), \dots, f^{m-1}(x) \notin V, f^m(x) \in A\}$$

has length $C_1 \gamma_1^m |A|$, where $C_1 > 0$ and $\gamma_1 \in (0, 1)$ are constants.

Set $Q_n = A_n$ and for $0 \leq m < n$, set

$$Q_m := \{x : f^m(x) \in V \text{ and } f^{m+1}(x) \in A_{n-m-1}\}.$$

Clearly,

$$f^{-n}(A) = \bigcup_{m=0}^n Q_m.$$

By the argument in the previous paragraph,

$$|Q_n| \leq |A_n| \leq C_1 \gamma_1^n |A|,$$

and for $0 \leq m < n$, since $f^m(Q_m) \subset V \cap f^{-1}A_{n-m-1}$, we have

$$\begin{aligned} |Q_m| &\leq |f^{-m}(f^m(Q_m))| \leq |f^{-m}(V \cap f^{-1}A_{n-m-1})| \\ &\leq M|A_{n-m-1}|^\alpha \leq M' \gamma_2^{n-m} |A|^\alpha, \end{aligned}$$

where the third inequality follows by Lemma 8, and $M' > 0$ and $\gamma_2 \in (0, 1)$ are constants. Inequality (3) follows by redefining M . \square

APPENDIX: GROWTH OF DERIVATIVES AND BACKWARD CONTRACTION

Let $f : X \rightarrow X$ be a map in the class \mathcal{A} , i.e., f is C^3 with non-flat critical points and all periodic points hyperbolic repelling. Let $CV = CV(f) = f(\text{Crit}(f))$. Given $\delta' > \delta > 0$ we will say that f is (δ, δ') -backward contracting if for every critical point c of f , every $n \geq 1$, and every connected component W of $f^{-n}(\hat{B}_{\delta'}(c))$ we have that

$$\text{dist}(W, CV) < \delta \text{ implies } |W| < \delta.$$

For a given constant $r > 1$, the map f satisfies the backward contracting property with constant r , as defined in the introduction, if for every $\delta > 0$ sufficiently small the map f is $(\delta, \delta r)$ -backward contracting. Given $\delta_0 > 0$ and a function $r : (0, \delta_0) \rightarrow (1, +\infty)$, we will say that f is *backward contracting with growth function r* , if for every $\delta \in (0, \delta_0)$ sufficiently small the map f is $(\delta, \delta r(\delta))$ -backward contracting.

The purpose of this appendix is to prove the following result.

Theorem 1. *For $f \in \mathcal{A}$, the following properties hold.*

1. *For every $r > 1$ there is a constant $K > 0$ that depends only on ℓ_{\max} , such that if $f \in \bigcup_N \mathcal{A}(N, \ell_{\max}, K)$, then f satisfies the backward contracting property with constant r .*
2. *If f satisfies the summability condition with exponent $\alpha > 0$, then there exists $\delta_0 > 0$, and a function $r : (0, \delta_0) \rightarrow (1, +\infty)$ such that for every $\theta \in (0, 1)$ we have*

$$\sum_{n \gg 1} r(\theta^n)^{-\alpha} < +\infty,$$

and such that f is backward contracting with growth function r .

3. *If f satisfies the Collet-Eckmann condition, then there are constants $\alpha \in (0, 1]$ and $C > 0$, such that f is backward contracting with growth function $r(\delta) = C\delta^{-\alpha}$.*

Parts 2 and 3 were proved for rational maps in [R, Theorem A].

For the proof of the theorem, let $f \in \mathcal{A}$. Given $v \in CV$ and $c \in \text{Crit}$ let

$$0 \leq k_1(v, c) < k_2(v, c) < \dots$$

be all integers $k \geq 0$ satisfying the following property: If $r > 0$ the smallest number so that the closure of $\hat{B}_r(c)$ contains $f^k(v)$, then the pull-back of

the closure of $\hat{B}_r(c)$ by f^k to v is diffeomorphic. Observe that for certain v and c this sequence might be finite or non-existent. We denote by $\xi_i(v, c)$ the corresponding preimage of c by $f^{k_i(v, c)}$. Notice that when $\xi_i(v, c)$ is close to v , the integer $k_i(v, c)$ is large.

Let $\eta_0 = \eta_0(f) > 0$ be sufficiently small. Then for any $\eta \in (0, \eta_0)$, we have

$$|\hat{B}_\eta(c)| \approx 2 \left(\frac{\eta}{A_c} \right)^{1/\ell_c}$$

and for $x \in \hat{B}_{\eta_0}(c)$,

$$|f'(x)| \approx A_c \ell_c |x - c|^{\ell_c - 1},$$

where $A_c = \lim_{y \rightarrow c} |f(y) - f(c)|/|x - c|^{\ell_c}$. Here by writing $C_1 \approx C_2$ we mean that $C_2/2 \leq C_1 \leq 2C_2$.

Lemma 9. *There is a constant $C > 0$ only depending on ℓ_{\max} such that the following property holds. For $\delta > 0$ small put*

$$\rho(\delta) = \min \left\{ C \cdot A(\delta), \frac{\eta_0}{\delta} \right\},$$

where $A(\delta)$ is equal to

$$\inf_{\substack{\text{dist}(\xi_i(v, c), v) \geq \delta \\ f^{k_i(v, c)+1}(v) \in B_{\eta_0}(CV)}} \left(\frac{\text{dist}(\xi_i(v, c), v)}{\delta} |(f^{k_i(v, c)+1})'(v)| \right).$$

Then for every $c \in \text{Crit}$, every $n \geq 0$ and every $z \in f^{-n}(c)$ such that for every $i = 0, \dots, n-1$ we have $f^i(z) \notin B_\delta(CV)$, the pull-back of $\hat{B}_{\delta\rho(\delta)}(c)$ to z by f^n is diffeomorphic and disjoint from CV .

Proof. Consider a critical point $c \in \text{Crit}$, an integer $n \geq 0$ and $z \in f^{-n}(c)$. Given $\delta > 0$ small and $0 < r \leq \eta_0/\delta$, put $U_0 = \hat{B}_{\delta r}(c)$ and consider the successive pull-backs U_0, U_1, \dots, U_n , such that U_k contains $f^{n-k}(z)$. Let us suppose that for some $k = 0, \dots, n$ the set U_k contains a critical value $v \in CV$, and let k be minimal with this property. Note that the restriction of f^k to U_k is a diffeomorphism onto its image. Moreover, $k = k_i(v, c)$ for some $i \geq 1$ and $\xi = \xi_i(v, c) \in U_k$ is equal to $f^{n-k}(z)$. Then for a constant $C_1 > 0$ only depending on f , we have by the One-sided Koebe Principle,

$$\frac{\text{dist}(\xi, v)}{|\hat{B}_{\delta r}(c)|} \leq \frac{\text{dist}(\xi, v)}{\text{dist}(c, f^k(v))} \leq \left(\frac{\text{dist}(\xi, v)}{\text{dist}(c, f^k(v))} \right)^{\frac{\ell_c - 1}{\ell_c}} |(f^k)'(v)|^{-\frac{1}{\ell_c}} C_1 \left(\frac{1}{2} \right)^{\frac{1}{\ell_c}}.$$

There is thus a constant $C_1 > 0$ depending only on ℓ_c such that, if δ is such that δr is sufficiently small, then

$$\delta r > C_1 \text{dist}(\xi, v) |(f^{k+1})'(v)|.$$

By hypothesis $\text{dist}(\xi, v) \geq \delta$, so if we take the constant C in the definition of ρ equal to C_1 , then we have $\hat{B}_{\delta\rho(\delta)}(c) \subset U_0$ and for each $k = 0, \dots, n$ the set U_k is disjoint from CV . \square

Lemma 10. *There is a constant C_0 that depends only on ℓ_{\max} such that the following property holds. For $\delta > 0$ put*

$$\begin{aligned} r_0(\delta) &= \\ &= \min \left(31^{-\ell_{\max}} \rho(\delta), C_0 \inf_{\text{dist}(\xi_i(v,c),v) < \delta} \left(\left(\frac{\delta}{\text{dist}(\xi_i(v,c),v)} \right)^{\ell_c-1} |(f^{k_i(v,c)+1})'(v)| \right) \right). \end{aligned}$$

If for every $\delta > 0$ small we have $r_0(\delta) \geq 2$, then f is backward contracting with growth function r_0 .

Proof. We will choose the constant $C_0 > 0$ below. Given $c \in \text{Crit}$ and $\delta > 0$ small, consider successive pull-backs $U_0 = \hat{B}_{\delta r_0(\delta)}(c)$, U_1, \dots, U_k , such that $U_k \cap B_\delta(CV) \neq \emptyset$. Since for every $\delta > 0$ small we have $r_0(\delta) \geq 2$, arguing by induction, it is enough to consider the case when for every $i = 0, \dots, k-1$ we have $U_i \cap B_\delta(CV) = \emptyset$.

For each $i = 0, \dots, k$ let U'_i and U''_i be the corresponding pull-backs of $\hat{B}_{10^{\ell_{\max}} \delta r_0(\delta)}(c)$ and $\hat{B}_{31^{\ell_{\max}} \delta r_0(\delta)}(c)$ respectively, so that $U_i \subset U'_i \subset U''_i$. Since by definition $31^{\ell_{\max}} r_0(\delta) \leq \rho(\delta)$, the previous lemma implies that $f^k : U''_k \rightarrow U''_0$ is a diffeomorphism. Observe that U''_0 contains a 1-scaled neighborhood of U'_0 and that U'_0 contains a 4-scaled neighborhood of U_0 . In particular, the Koebe Principle implies that the distortion of f^k on U'_k is bounded by 5.

Case 1. U'_k is disjoint from CV . Since U'_0 contains a 4-scaled neighborhood of U_0 , it follows from the Macroscopic Koebe Principle (Proposition 3 (iv)) that U'_k contains a 1-scaled neighborhood of U_k . Since $U_k \cap B_\delta(CV) \neq \emptyset$ and $U'_k \cap CV = \emptyset$, it follows that $|U_k| < \delta$.

Case 2. There is $v \in U'_k \cap CV$. Then there is $i \geq 1$ such that $k = k_i(v, c)$ and $\xi = \xi_i(v, c) \in U'_k$ is the unique k -th preimage of c in U'_k . Then we must have that $\xi \in B(v, \delta)$; otherwise Lemma 9 would imply that $U''_k \cap CV = \emptyset$.

Suppose by contradiction that $|U'_k| \geq \delta$. Then, if $\delta > 0$ is such that $\delta r_0(\delta)$ is small, then we have

$$\begin{aligned} \frac{\delta}{|U_0|} &\leq 11 \frac{|U'_k|}{|U'_0|} \leq 55 \frac{\text{dist}(\xi, v)}{\text{dist}(c, f^k(v))} \\ &\leq 55 \cdot 5^{\frac{1}{\ell_c}} \left(\frac{\text{dist}(\xi, v)}{\text{dist}(c, f^k(v))} \right)^{\frac{\ell_c-1}{\ell_c}} |(f^k)'(v)|^{-\frac{1}{\ell_c}}. \end{aligned}$$

So, for a constant $C_2 > 0$ depending only on ℓ_{\max} , we have

$$r_0(\delta) > C_2 \left(\frac{\delta}{\text{dist}(\xi, v)} \right)^{\ell_c-1} |(f^{k+1})'(v)|.$$

Since $\text{dist}(\xi, v) < \delta$, letting $C_0 = C_2$ we obtain a contradiction. \square

Proof of Theorem 1. Part 1 is a direct consequence of Lemmas 9 and 10.

2. Given $\theta \in (0, 1)$ we have

$$\begin{aligned} & \sum_{n \gg 1} r(\theta^n)^{-\alpha} \leq \\ & \leq C' \sum_{i,v,c} \left(\sum_{n \geq 0, \theta^n < \text{dist}(\xi_i(v,c),v)} \left(\frac{\text{dist}(\xi_i(v,c),v)}{\theta^n} \right)^{-\alpha} |(f^{k_i(v,c)+1})'(v)|^{-\alpha} + \right. \\ & \quad \left. + \sum_{n \geq 0, \theta^n > \text{dist}(\xi_i(v,c),v)} \left(\frac{\theta^n}{\text{dist}(x_i(v,c),v)} \right)^{-\alpha(\ell_c-1)} |(f^{k_i(v,c)+1})'(v)|^{-\alpha} \right) \leq \\ & \leq C'' \sum_{i,v,c} |(f^{k_i(v,c)+1})'(v)|^{-\alpha} < +\infty. \end{aligned}$$

3. Let $C_0 > 0$ and $\lambda > 1$ be such that for every $v \in CV$ we have $|(f^k)'(v)| \geq C_0 \lambda^k$. By [NS1, Theorem B] there are constants $C_1 > 0$ and $\theta \in (0, 1)$ such that $\text{dist}(\xi_i(v,c),v) \geq C_1 \theta^{k_i(v,c)}$. Therefore there is $C_2 > 0$ and $\gamma \in (0, 1)$ such that,

$$|(f^{k_i(v,c)+1})'(v)| \geq C_0 \lambda^{k_i(v,c)} \geq C_2 (\text{dist}(\xi_i(v,c),v))^{-\gamma}.$$

Choose $\mu \in (0, 1)$ and note that, if $\frac{\text{dist}(\xi_i(v,c),v)}{\delta} \leq \delta^{-\mu}$, we have

$$|(f^{k_i(v,c)+1})'(v)| \geq C_2 \text{dist}(\xi_i(v,c),v)^{-\gamma} \geq C_2 \delta^{-\gamma(1-\mu)}.$$

Thus there is a constant $C_4 > 0$ such that $r_0(\delta) \geq C_4 \delta^{-\alpha}$, where $\alpha = \min(\mu, \gamma(1-\mu))$. \square

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Henk Bruin

Department of Mathematics,
University of Surrey,
UK

email: h.bruin@surrey.ac.uk

Juan Rivera-Letelier

Departamento de Matemáticas,
Universidad Católica del Norte,
CHILE

email: rivera-letelier@ucn.cl

Weixiao Shen

Department of Mathematics,
University of Science and Technology of China,
CHINA P.R.

email: wxshen@ustc.edu.cn

Sebastian van Strien

Department of Mathematics,
University of Warwick,
UK

email: strien@maths.warwick.ac.uk