LECTURES ON INSTANTONS AND CONTACT STRUCTURES

STEVEN SIVEK

These notes are a somewhat expanded version of a 4-hour minicourse given at the summer school on "Gauge Theory and Applications," held 17–21 July 2018 at the University of Regensburg. The aim was to introduce instanton gauge theory, first in the form of Donaldson's polynomial invariants and then instanton Floer homology, and then to use this to explain recent joint work with John Baldwin [BS18] proving that Khovanov homology detects the trefoils.

1. Yang-Mills theory and Donaldson invariants

The goal of this lecture is to introduce Donaldson invariants, as defined in [Don90], and some of their basic properties. These are discussed in detail in [DK90] and [Mor98]; see also [FU91] for background.

Let $P \to X$ be a principal $SU(2)$-bundle over a closed smooth 4-manifold X; these are classified by their second Chern number $c_2(P) \in \mathbb{Z}$. Given a Riemannian metric on X, the Hodge star $* : \Omega^2(X) \to \Omega^2(X)$ extends to $\Omega^2(X; \mathrm{ad}(P))$, where it satisfies $*^2 = 1$ and defines an inner product

$$\langle \alpha, \beta \rangle = -\int_X \mathrm{tr}(\alpha \wedge *\beta).$$

We note that elements of $\mathfrak{su}(2)$ are skew-adjoint, and $g^* = -g$ implies that $-\mathrm{tr}(g^2) = |g|^2$, which explains the minus sign.

Since $*^2 = 1$, we can split $\Omega^2(X; \mathrm{ad}(P))$ into the ± 1-eigenspaces $\Omega^2_\pm(X; \mathrm{ad}(P))$ of $*$, and these are orthogonal with respect to $\langle \cdot, \cdot \rangle$. Chern-Weil theory tells us that

$$8\pi^2 c_2(P) = \int_X \mathrm{tr}(F_A \wedge F_A),$$

and if we split the curvature as $F_A = F_A^+ + F_A^-$ and let $\epsilon, \epsilon' \in \{+, -\}$ then we have

$$\int_X \mathrm{tr}(F_A^\epsilon \wedge F_A^{\epsilon'}) = \epsilon' \int_X \mathrm{tr}(F_A^\epsilon \wedge *F_A^{\epsilon'}) = -\epsilon' \langle F_A^\epsilon, F_A^{\epsilon'} \rangle,$$

so that

$$8\pi^2 c_2(P) = ||F_A^-||^2 - ||F_A^+||^2.$$

On the other hand, the *Yang-Mills functional* $YM : \mathcal{A} \to \mathbb{R}$ measures the L^2-norm of the curvature,

$$\begin{aligned} YM(A) = ||F_A||^2 &= ||F_A^+||^2 + ||F_A^-||^2 \\ &= 8\pi^2 c_2(P) + 2||F_A^+||^2. \end{aligned}$$

Thus YM achieves an absolute minimum at any connection with $F_A^+ = 0$.

Definition 1.1. *A connection A on $P \to X$ is* anti-self-dual*, or ASD, if $F_A^+ = 0$.*

We will also refer to ASD connections as *instantons*.

Remark 1.2. From the above formula for $8\pi^2 c_2(P)$, we see that there are no ASD connections if $c_2(P) < 0$, and that all ASD connections are flat if $c_2(P) = 0$. Thus the most interesting cases are those where $c_2(P) > 0$.

The space of all ASD connections is enormous whenever it is nonempty: for example, it is closed under symmetries of the bundle P, so we divide out by these.

Definition 1.3. *Let $P \xrightarrow{\pi} X$ be a principal G-bundle. A* gauge transformation *is a G-equivariant bundle automorphism $u : P \to P$, meaning an invertible map satisfying $\pi(u(p)) = \pi(p)$ and $u(p \cdot g) = u(p) \cdot g$ for all $p \in P$ and $g \in G$.*

Since a gauge transformation preserves the fibers of P, we can define $\tilde{u} : P \to G$ by the formula $u(p) = p \cdot \tilde{u}(p)$. These satisfy

$$(p \cdot g) \cdot \tilde{u}(p \cdot g) = u(p \cdot g) = u(p) \cdot g = p \cdot \tilde{u}(p)g = (p \cdot g) \cdot g^{-1}\tilde{u}(p)g$$

and so $\tilde{u}(p \cdot g) = g^{-1}\tilde{u}(p)g$. Thus the bijection $u \leftrightarrow \tilde{u}$ identifies the group $\mathcal{G}$ of gauge transformations with sections of $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} G$. (We use Ad and ad to refer to the actions of G by conjugation on G and $\mathfrak{g}$, respectively.)

The gauge group $\mathcal{G}$ acts on the space $\mathcal{A}$ of connections by $u^*A = u^{-1}Au + u^{-1}du$, and transforms the curvature by $F_{u^*A} = u^{-1}F_A u$. We will let $\mathcal{A}^* \subset \mathcal{A}$ denote the space of irreducible connections, and write $\mathcal{B} = \mathcal{A}/\mathcal{G}$ and $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$. Letting $\mathcal{M}_{P,g} \subset \mathcal{B}^*$ be the moduli space of irreducible ASD connections up to gauge equivalence, we wish to see that $\mathcal{M}$ is a smooth manifold. This will require us to work with appropriate Sobolev completions throughout, so that all spaces of interest are Banach and we can apply tools such as the implicit function theorem and the Sard-Smale theorem, but we will omit all of the details.

Proposition 1.4. *The tangent space to the orbit $\mathcal{O}_A = \mathcal{G} \cdot A$ at a connection A can be identified with*

$$\mathrm{Im}\left(d_A : \Omega^0(X; \mathrm{ad}(P)) \to \Omega^1(X; \mathrm{ad}(P))\right).$$

Proof. Identifying $\mathrm{Lie}(\mathcal{G})$ with the space of sections of $\mathrm{ad}(P) \to X$, we fix $v \in C^\infty(X, \mathrm{ad}(P))$ and compute the tangent vector to the path $(e^{tv})^*A$ at $t = 0$:

$$\begin{aligned} \tfrac{d}{dt}\left((e^{tv})^*A\right)\Big|_{t=0} &= \tfrac{d}{dt}\left(e^{-tv}Ae^{tv} + e^{-tv}d(e^{tv})\right)\Big|_{t=0} \\ &= dv + [A, v] = d_A v. \end{aligned}$$

□

We can then identify the tangent space to $\mathcal{B}^*$ at an irreducible $[A]$ with the L^2-orthogonal complement to $\mathrm{Im}(d_A)$: a Hodge decomposition theorem gives

$$T_{[A]}\mathcal{B}^* = \ker\left(d_A^* : \Omega^1(X; \mathrm{ad}(P)) \to \Omega^0(X; \mathrm{ad}(P))\right).$$

As for the moduli space $\mathcal{M}_{P,g} \subset \mathcal{B}^*$, we linearize the curvature operator at A by

$$F_{A+ta} = F_A + d_A(ta) + \tfrac{1}{2}[ta \wedge ta] = F_A + t d_A a + O(t^2).$$

Letting $F_A^+ = F_A + *F_A$ denote the self-dual part, we see that the linearization of $F_A^+ = 0$ at A is $d_A^+ = 0$, where $d_A^+ a$ is the projection of $d_A a \in \Omega^2(X; \mathrm{ad}(P))$ onto $\Omega^2_+(X; \mathrm{ad}(P))$. Thus the tangent space to the ASD moduli space near $[A]$ is governed by the complex

$$\Omega^0(X; \mathrm{ad}(P)) \xrightarrow{d_A} \Omega^1(X; \mathrm{ad}(P)) \xrightarrow{d_A^+} \Omega^2_+(X; \mathrm{ad}(P)),$$

which is indeed a complex because $d_A^+ \circ d_A = F_A^+ = 0$, and moreover it is elliptic.

More precisely, the cohomology group $H_A^0 = \ker(d_A)$ is the tangent space to the stabilizer of the $\mathcal{G}$-action on A, so it vanishes if and only if A is irreducible. If $H_A^2 = 0$, we say that

A is *regular*. The remaining group $H^1_A = \ker(d^+_A)/\operatorname{Im}(d_A)$ is identified with $T_{[A]}\mathcal{M}_{P,g}$. For $G = SU(2)$, we have:

Theorem 1.5. *The tangent space $T_{[A]}\mathcal{M}_{P,g}$ is isomorphic to the kernel of*

$$D_A = d^*_A \oplus d^+_A : \Omega^1(X; \operatorname{ad}(P)) \to \Omega^0(X; \operatorname{ad}(P)) \oplus \Omega^2_+(X; \operatorname{ad}(P)),$$

which is an elliptic operator of index $\operatorname{ind}(D_A) = 8c_2(P) - 3(1 - b_1(X) + b^+_2(X))$.

Then $\mathcal{M}_{P,g}$ has dimension $\operatorname{ind}(D_A)$ at A precisely when D_A has trivial cokernel, which happens when A is regular and irreducible.

Let $\mathcal{C}$ denote the space of C^r conformal structures on X for some fixed $r \geq 3$. If $c_2(P) > 0$, then Freed and Uhlenbeck's generic metrics theorem [FU91] says that $\mathcal{C}$ contains a countable intersection of open dense subsets of equivalence classes of metrics g for which all g-ASD connections are regular. Moreover, one can show that the subspace of all $[g] \in \mathbb{C}$ which admit reducible ASD connections is a countable union of codimension-$b^+_2(X)$ submanifolds. Thus if $b^+_2(X) > 1$, the subspace of $\mathcal{C}$ where all instantons are regular and irreducible, and hence $\mathcal{M}_{P,g}$ is a smooth manifold, is path-connected.

Ideally, we would also like the $\mathcal{M}_{P,g}$ to be compact. Then its bordism class would be a smooth invariant of X, assuming $b^+_2(X) > 1$: given any two such choices of metric g_0 and g_1, we pick a generic path g_t between them through the space of such metrics, and the union $\bigcup_t \left(\{t\} \times \mathcal{M}_{P,g_t}\right)$ would be a cobordism from $\mathcal{M}_{P,g_0}$ to $\mathcal{M}_{P,g_1}$. (Here genericity means that the cobordism is transversely cut out, rather than the individual $\mathcal{M}_{P,g_t}$.) Unfortunately, this does not hold in general, but these moduli spaces admit a nice compactification as follows.

Theorem 1.6 (Uhlenbeck compactness [Uhl82])**.** *Let $P \to X$ be a principal $SU(2)$ bundle with $c_2(P) > 0$, and let $A_1, A_2, \dots$ be a sequence of ASD connections on P. Then we can find a subsequence $A_{n_1}, A_{n_2}, \dots$, together with:*

- *a principal $SU(2)$ bundle $P' \to X$, with $0 \leq c_2(P') \leq c_2(P)$;*
- *finitely many points $x_1, \dots, x_t \in X$, not necessarily unique, with $t = c_2(P) - c_2(P')$;*
- *bundle isomorphisms $f_i : P'|_{X \smallsetminus \{x_1,\dots,x_t\}} \to P|_{X \smallsetminus \{x_1,\dots,x_t\}}$;*
- *an ASD connection A_∞ on P'*

such that the connections $f^*_i(A_{n_i}|_{X \smallsetminus \{x_1,\dots,x_t\}})$ converge to $A_\infty|_{X \smallsetminus \{x_1,\dots,x_t\}}$ on compact sets, and the functions $|F_{A_{n_i}}| : X \to \mathbb{R}$ converge to the measure $|F_{A_\infty}| + 8\pi^2 \sum_i \delta_{x_i}$.

Thus if we fix a metric on X and write $\mathcal{M}_k = \mathcal{M}_{P_k,g}$ where $c_2(P_k) = k \geq 0$, we can write

$$\overline{\mathcal{M}_k} \subset \mathcal{M}_k \cup \left(\mathcal{M}_{k-1} \times \operatorname{Sym}^1(X)\right) \cup \left(\mathcal{M}_{k-2} \times \operatorname{Sym}^2(X)\right) \cup \dots.$$

We note that generically, each stratum $\mathcal{M}_{k-j} \times \operatorname{Sym}^j(X)$ has dimension $\dim(\mathcal{M}_k) - 4j$ when $j < k$, and is empty for $j > k$. For $j = k$, we note that $\dim(\mathcal{M}_0 \times \operatorname{Sym}^k(X))$ is independent of the metric, since $\mathcal{M}_0$ consists of flat connections mod gauge and is thus identified with $\operatorname{Hom}(\pi_1(X), SU(2))$/conjugation. Supposing that $\pi_1(X) = 0$ for simplicity, there is only the trivial connection and so this stratum has codimension

$$\left(8k - 3(1 + b^+_2(X))\right) - 4k \geq 2$$

if we are in the *stable range* $k \geq \frac{1}{4}(3b^+_2(X) + 5)$; this implies that $\mathcal{M}_k$ has a fundamental class.

For example, if X is simply connected and $b^+_2(X) = 0$, then $\dim(\mathcal{M}_1) = 5$, and $\mathcal{M}_1$ can be compactified by adding $\mathcal{M}_0 \times X = \{\mathrm{pt}\} \times X$, so then $\overline{\mathcal{M}_1} = \mathcal{M}_1 \cup X$. There are reducible

ASD connections, one for every pair $\pm x \in H_2(X)$ with $x^2 = -1$ – a reduction amounts to a splitting $P_1 \times_{SU(2)} \mathbb{C}^2 = L \oplus L^{-1}$, so $1 = c_2(P_1) = -c_1(L)^2$ and thus each pair determines such a splitting – and neighborhoods of these look like cones on $\overline{\mathbb{CP}}^2$. Thus $\overline{\mathcal{M}_1}$ gives a cobordism from X to

$$n_X := \tfrac{1}{2}\#\{x \in H_2(X;\mathbb{Z}) \mid x^2 = -1\}$$

copies of $\overline{\mathbb{CP}}^2$.

Since signature is a cobordism invariant and X is definite, we now have

$$n_X = |\sigma(X)| = b_2(X).$$

At the same time, we can use any pair $\pm x$ of elements of square -1 to decompose the intersection form Q_X into orthogonal summands $\langle -1 \rangle \oplus x^\perp$, so by induction on the rank we see that $n_X \le b_2(X)$, and that if equality holds then Q_X must be the standard form with matrix $-I$. This is how Donaldson proved his celebrated theorem:

Theorem 1.7 ([Don83]). *Let X be a smooth, simply-connected 4-manifold with negative definite intersection form Q_X. Then Q_X can be diagonalized over the integers.*

Returning to more general $SU(2)$-bundles $P \to X$, we wish to construct smooth invariants of X out of the moduli spaces $\mathcal{M}_{P,g}$. These will take the form

$$q_{X,d} : H_2(X)^{\otimes d} \to \mathbb{Z},$$

which is obtained from the moduli space $\mathcal{M}_{P,g}$ of dimension $2d$, meaning that

$$d = 4c_2(P) - \frac{3}{2}(1 - b_1(X) + b_2^+(X)),$$

if such P exists (and we let $q_{X,d} = 0$ otherwise). We do so roughly by constructing a homomorphism

$$\mu : H_2(X;\mathbb{Z}) \to H^2(\mathcal{M}_{P,g};\mathbb{Z})$$

and evaluating the cup product $\mu(\alpha_1) \cup \cdots \cup \mu(\alpha_k)$ on the fundamental class $[\mathcal{M}_{P,g}]$, though the fact that $\mathcal{M}_{P,g}$ is not compact causes some amount of difficulty.

To construct the map, we first construct the principal $SO(3)$ bundle

$$\mathbb{P} = \mathcal{A}^*(P) \times_{\mathcal{G}(P)} P \to \mathcal{B}^*(P) \times X.$$

Then we let $\mu(\Sigma) = -\frac{1}{4}p_1(P)/[\Sigma]$, where / is the slant product

$$H^4(\mathcal{B}^*(P) \times X) \times H_2(X) \to H^2(\mathcal{B}^*(P)),$$

amounting to integration along fibers; then restriction to $\mathcal{M}_{P,g}$ gives the desired μ. (Similarly, given any $x \in H_i(X)$ we can define $\mu(x) \in H^{4-i}(\mathcal{M}_{P,g})$.)

Theorem 1.8. *The invariants $q_{X,d}(\alpha_1, \ldots, \alpha_d)$ are smooth invariants of X, assuming $b_2^+(X) > 1$.*

Proof. We can find codimension-2 cycles V_{α_i} Poincaré dual to the classes $\mu(\alpha_i)$ and in general position so that $q_{X,d}(\alpha_1, \ldots, \alpha_d)$ is a signed count of points in the finite intersection $\mathcal{M}_{P,g} \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_d}$. Now given two different metrics g_0, g_1 on X and a generic path g_t of metrics between them, the parametrized moduli space $\bigcup_t \mathcal{M}_{P,g_t}$ gives a cobordism between $\mathcal{M}_{P,g_0}$ and $\mathcal{M}_{P,g_1}$, and its intersection with $V_{\alpha_1} \cap \cdots \cap V_{\alpha_d}$ is a cobordism between the corresponding intersections at either end, so they have the same signed number of points. □

We state without proof some of the main properties of these invariants, proved by Donaldson in [Don90].

Theorem 1.9. *Let X be a simply connected Kähler surface, with $b_2^+(X) > 1$ odd, and let $H \in H_2(X)$ be the class of a hyperplane section coming from some embedding $X \hookrightarrow \mathbb{CP}^n$. Then $q_{X,d}(H, H, \ldots, H) > 0$ for all sufficiently large d in the appropriate residue class* (mod 4).

The proof uses prior work of Donaldson [Don85] relating the ASD moduli space for $P \to X$ to the moduli space of *H-stable* rank-2 vector bundles $E \to X$ with $c_1(E) = 0$ and $c_2(E) = c_2(P)$: these are bundles $E \to X$ satisfying the stability property $c_1(F) \cdot H < \frac{1}{2}c_1(E) \cdot H$ for every line bundle $F \subset E$. The analogous result for bundles over Riemann surfaces is a classical theorem of Narasimhan and Seshadri [NS65], and Atiyah-Bott [AB83] previously used it in the opposite direction: they calculated the cohomology of the moduli space of stable bundles on a Riemann surface via the Morse theory of the Yang-Mills functional.

Theorem 1.10. *If X is simply connected with $b_2^+(X)$ odd, and X is diffeomorphic to a connected sum $X_1 \# X_2$ where $b^+(X_i) > 0$ for each i, then the invariants $q_{X,d}$ are all identically zero.*

As a corollary, we can prove the following.

Corollary 1.11. *Let X be a K3 surface. If we have a diffeomorphism $X = X_1 \# X_2$, then one of the X_i is a homotopy sphere.*

Proof. Theorems 1.9 and 1.10 together tell us that some X_i is negative definite, say $b_2^+(X_2) = 0$ without loss of generality. Then the intersection form on $H_2(X_2)$ is diagonalizable over $\mathbb{Z}$ by Theorem 1.7, so if $b = b_2(X) = b_2^-(X)$ is positive then there is an integral basis $v_1, \ldots, v_b$ of $H_2(X_2)$ for which $Q_{X_2}(v_i, v_j) = -\delta_{ij}$. In particular $Q_{X_2}(v_1, v_1) = -1$, which is impossible since X has even intersection form. □

2. Instanton Floer homology

This lecture focuses on the construction of instanton Floer homology, which was first constructed by Floer in [Flo88]. Its construction and basic properties are outlined in [Sav02], and developed in much more detail in [Don02].

Previously we studied ASD connections on closed 4-manifolds, but now we turn our attention to 4-manifolds with cylindrical ends, or even better, to 4-manifolds of the form $\mathbb{R} \times Y$. We will restrict our attention to finite-energy connections, i.e. those A which satisfy $\int_{\mathbb{R}\times Y} |F_A|^2 < \infty$.

We first note that any principal $SU(2)$-bundle P on a 3-manifold Y is trivial, so we may take the product connection A_0 as our base point for the space $\mathcal{A}$ of connections on P and thus identify a connection with its 1-form $a \in \Omega^1(Y; \mathrm{ad}(P)) = \Omega^1(Y; \mathfrak{su}(2))$. The same holds for $\mathbb{R} \times Y$, which we give the product metric $dt^2 + g_Y$.

Suppose we have a connection A on $P \to Y$, and define a connection $\tilde{A}$ on $[0,1] \times P \to [0,1] \times Y$ with 1-form tA. Then we have

$$\begin{aligned} F_{\tilde{A}} &= d(tA) + \tfrac{1}{2}[tA \wedge tA] \\ &= dt \wedge A + t dA + \tfrac{1}{2}t^2[A \wedge A], \end{aligned}$$

and so we compute the integral which would determine $c_2(P)$ if $[0,1]\times Y$ were a closed manifold:

$$\frac{1}{8\pi^2}\int_{[0,1]\times Y}\mathrm{tr}(F_{\tilde{A}}\wedge F_{\tilde{A}}) = \frac{1}{8\pi^2}\int_0^1\int_Y dt\wedge\mathrm{tr}(A\wedge 2tdA + t^2 A\wedge[A\wedge A])$$
$$= \frac{1}{8\pi^2}\int_Y \mathrm{tr}\left(A\wedge dA + \tfrac{1}{3}A\wedge[A\wedge A]\right).$$

This is the *Chern-Simons functional* on the space $\mathcal{A}$ of connections on P. On the space $\mathcal{B}=\mathcal{A}/\mathcal{G}$ of connections up to gauge, it gives a functional

$$cs:\mathcal{B}\to\mathbb{R}/\mathbb{Z}$$

because one can take connections on $[0,1]\times Y$ corresponding to A and u^*A and glue the corresponding bundles together by id at one end and u at the other to get a principal $SU(2)$ bundle on Y-bundle over S^1, from which the difference $cs(u^*A)-cs(A)$ is c_2 of that bundle. (In fact, u changes the value of cs by $\deg(u:Y\to SU(2)\cong S^3)$.) This functional satisfies

$$cs(A+sa) = cs(A) + \frac{s}{4\pi^2}\int_Y \mathrm{tr}(F_A\wedge a) + O(s^2),$$

so its directional derivative in the direction $a\in\Omega^1(Y;\mathfrak{su}(2))$ is

$$\frac{1}{4\pi^2}\int_Y \mathrm{tr}(*(*F_A)\wedge a) = \langle *F_A, a\rangle$$

and thus it has L^2-gradient $\nabla cs(A) = *F_A$. In particular, the critical points of cs are flat connections on P.

We now attempt to understand ASD connections on $\mathbb{R}\times P\to\mathbb{R}\times Y$.

Lemma 2.1. *Let A be a finite-energy connection on a principal $SU(2)$-bundle $\mathbb{R}\times P\to\mathbb{R}\times Y$. There is a gauge transformation u such that u^*A is in temporal gauge, meaning that u^*A has no dt-term.*

Proof. Write $A=\beta_t+\eta_t\,dt$, where $\beta_t\in\Omega^1(Y;\mathfrak{su}(2))$ and $\eta_t\in\Omega^0(Y;\mathfrak{su}(2))$ may depend on t. The dt-coefficient of u^*A is then

$$u^{-1}\eta_t u + u^{-1}\frac{\partial u}{\partial t},$$

which vanishes if and only if u solves the first order differential equation $\dot{u}+\eta_t u = 0$; there is a unique solution for any choice of $u|_{\{0\}\times Y}$. □

Given a connection $\tilde{A}=A_t$ in temporal gauge on $\mathbb{R}\times Y$, we have

$$F_{\tilde{A}} = d\tilde{A} + \tfrac{1}{2}[\tilde{A},\tilde{A}] = dt\wedge\frac{\partial A_t}{\partial t} + F_{A_t}$$

and from this we compute that

$$*F_{\tilde{A}} = *_Y\left(\frac{\partial A_t}{\partial t}\right) + dt\wedge *_Y F_{A_t},$$

which means that $\tilde{A}$ is ASD if and only if $\frac{\partial A_t}{\partial t} = -*F_{A_t} = -\nabla cs(A_t)$. In other words, the ASD connections on $\mathbb{R}\times Y$ up to gauge equivalence can be identified with downward gradient flow lines of the Chern-Simons functional.

All of the above suggests that we should attempt to construct equivariant Morse homology for the functional $cs:\mathcal{B}\to\mathbb{R}/\mathbb{Z}$. This would be a complex $CI_*(Y)$ whose generators are flat connections on Y up to gauge equivalence, which are the same as conjugacy classes of representations $\pi_1(Y) \to SU(2)$ (via the bijection $A \leftrightarrow \mathrm{hol}_A$), and whose differential counts rigid ASD connections on $\mathbb{R} \times Y$ which limit to given flat connections as $t \to \pm\infty$. For homology 3-spheres this construction was carried out by Floer [Flo88].

In this setting we can again try to study the moduli space of ASD connections on $P \to X = \mathbb{R} \times Y$ near a connection A with 1-form a_t in temporal gauge. Then $d_A = d + a_t$, and the operator

$$d_A^* \oplus d_A^+ : \Omega^1(X; \mathrm{ad}(P)) \to \Omega^0(X; \mathrm{ad}(P)) \oplus \Omega^2_+(X; \mathrm{ad}(P))$$

can be expressed in simpler terms. We write 1-forms as $\beta_t + \eta_t dt \in \Omega^0 \oplus \Omega^1$, and identify $\Omega^2_+ \cong \Omega^1$ via

$$dt \wedge \omega_t + *_Y \omega_t \quad \longleftrightarrow \quad \omega_t,$$

and then $D_A := d_A^* \oplus d_A^+ : \Omega^0 \oplus \Omega^1 \to \Omega^0 \oplus \Omega^1$ has the form

$$(d_A^* \oplus d_A^+) \begin{pmatrix} \eta_t \\ \beta_t \end{pmatrix} = \left(\frac{\partial}{\partial t} + \begin{pmatrix} 0 & -d_{A_t}^* \\ -d_{A_t} & *d_{A_t} \end{pmatrix} \right) \begin{pmatrix} \eta_t \\ \beta_t \end{pmatrix}.$$

In other words, it can be written $\frac{\partial}{\partial t} + L_A$ where L_A is self-adjoint and elliptic. If $\alpha = \lim_{t\to-\infty} A|_{\{t\}\times Y}$ and $\beta = \lim_{t\to\infty} A|_{\{t\}\times Y}$ are flat connections on Y, then L_A is Fredholm if we work in an appropriately weighted Sobolev space. It has index

$$\mathrm{ind}(D_A(\alpha, \beta)) = \mathrm{sf}(B_t),$$

where B_t is a path of operators from L_α to L_β and sf denotes spectral flow [APS76]. However, if α' and α are gauge equivalent and A' has limiting flat connections α' and β, then we only have $\mathrm{ind}(D_{A'}(\alpha', \beta)) \equiv \mathrm{ind}(D_A(\alpha, \beta)) \pmod 8$; this will lead to a relative $\mathbb{Z}/8\mathbb{Z}$ grading on gauge equivalence classes of flat connections.

One serious problem, however, is that some of the flat connections which supposedly generate $CI_*(Y)$ are necessarily reducible: up to conjugation, they are identified via their holonomy with $\hom(H_1(Y; \mathbb{Z}), U(1))$. If Y is an integral homology sphere, there is only the trivial connection, and Floer omitted it from CI_* but used it to define an absolute grading. We will instead work with nontrivial $SO(3)$ bundles or $U(2)$ bundles from now on, as developed by Floer in [Flo90]; see also [Don02, §5.6].

Definition 2.2. *A rank-2 unitary bundle $E \to Y$ is* admissible *if there is a closed, embedded surface $\Sigma \subset Y$ such that the pairing $\langle c_1(E), \Sigma \rangle$ is odd.*

Since $SO(3) \cong U(2)/Z(U(2))$, we can associate to E an $\mathfrak{so}(3)$ bundle $\mathrm{ad}(P)$, consisting of the traceless, skew-adjoint automorphisms of E; and a complex line bundle $\Lambda^2 E$. A connection on E induces connections on $\mathrm{ad}(E)$ and $\Lambda^2 E$, and vice versa. We will let $\mathcal{A}(E)$ denote the space of connections on E which induce a fixed connection on $\Lambda^2 E$; the ones which give a flat connection on $\mathrm{ad}(E)$ are called *projectively flat*, though we will drop the word "projective" for convenience. We also use a modified gauge group $\mathcal{G}(E)$ of *determinant-1* gauge transformations on E, which are the gauge transformations which fix $\Lambda^2 E$. Morally these behave like $SO(3)$ connections up to gauge, but the reducibles are easier to understand.

Lemma 2.3. *If $E \to Y$ is an admissible $U(2)$-bundle then it does not have any reducible flat connections.*

Proof. Let Σ be a surface with $\langle c_1(E), \Sigma \rangle$ odd. A reducible connection on E induces a splitting $E = L \oplus L'$ with connections A and A' on the summands, where the curvatures of

A and A' are equal. (These need not be zero, because the original connection on E is only *projectively* flat.) But then $\langle c_1(E), \Sigma\rangle$ is equal to

$$\langle c_1(L), \Sigma\rangle + \langle c_1(L'), \Sigma\rangle = \frac{i}{2\pi}\int_\Sigma \mathrm{tr}(F_A) + \frac{i}{2\pi}\int_\Sigma \mathrm{tr}(F_{A'}),$$

and since both terms on the right are integers their sum cannot be odd. □

We can now define instanton Floer homology for a 3-manifold with an admissible bundle.

Definition 2.4. *Let $w \to Y$ be a Hermitian line bundle such that $c_1(w)$ has odd pairing with some class in $H_2(Y)$. Take a $U(2)$-bundle $E \to Y$ together with a fixed identification $\Lambda^2 E \cong w$. Then we define a chain complex*

$$CI_*(Y)_w = \bigoplus_{[\alpha]} \mathbb{C}\langle[\alpha]\rangle,$$

where $[\alpha]$ ranges over equivalence classes of projectively flat connections in $\mathcal{A}(E)/\mathcal{G}(E)$. It has a relative $\mathbb{Z}/8\mathbb{Z}$ grading $\mathrm{gr}(\alpha,\beta)$ given by the spectral flow of a path from L_α to L_β.

Given any two generators α and β, and the product bundle $\mathbb{R}\times E \to \mathbb{R}\times Y$, we define a moduli space

$$\mathcal{M}(\alpha,\beta) = \left\{ A \in \mathcal{A}(\mathbb{R}\times E) \,\middle|\, A \text{ is projectively ASD}, ||F_A||^2 < \infty, \right.$$
$$\left. \lim_{t\to-\infty} A|_{\{t\}\times E} = \alpha, \lim_{t\to\infty} A_{\{t\}\times E} = \beta \right\} / \mathcal{G}(\mathbb{R}\times E).$$

If $\alpha \neq \beta$ then this has a free $\mathbb{R}$-action given by $(x\cdot A)|_{\{t\}\times E} = A|_{\{t+x\}\times E}$, and we write $\hat{\mathcal{M}}(\alpha,\beta) = \mathcal{M}(\alpha,\beta)/\mathbb{R}$. Then we define a differential on $CI(Y)_w$ by

$$\partial\alpha = \sum_\beta \#\hat{\mathcal{M}}(\alpha,\beta)\cdot\beta,$$

where $\#\hat{\mathcal{M}}$ is a signed count of isolated points.

Implicit in this definition is a choice of data $\mathcal{D}$, consisting of a metric on Y and a perturbation of the ASD equation; we can arrange for the Chern-Simons functional to be Morse-Smale by defining a suitable family of perturbations and using the Sard-Smale theorem. We thus write $CI_*(Y,\mathcal{D})_w$ for now to indicate the dependence on this data. For generic $\mathcal{D}$ it follows from the above discussion that $\langle\partial\alpha,\beta\rangle \neq 0$ only if $\mathrm{gr}(\alpha,\beta) = 1$, and so ∂ lowers the degree by 1.

The fact that $\partial^2 = 0$ follows just as in Morse theory: a gluing theorem asserts that each coefficient

$$\langle\partial^2\alpha,\gamma\rangle = \sum_\beta \#\hat{\mathcal{M}}(\alpha,\beta)\cdot\#\hat{\mathcal{M}}(\beta,\gamma)$$

counts points in the boundary of the 1-manifold built by compactifying the 1-dimensional components of $\hat{\mathcal{M}}(\alpha,\gamma)$, and this signed count is zero. Thus $CI_*(Y,\mathcal{D})_w$ is indeed a chain complex, and we write $I_*(Y,\mathcal{D})_w$ for its homology.

Example 2.5. Consider the Hermitian line bundle $w \to T^3$ with $c_1(w)$ Poincaré dual to an S^1 factor. Flat connections on the corresponding $SO(3)$ bundle do not lift to $SU(2)$ connections: this is obstructed by the second Stiefel-Whitney class, which is $c_1(w) \pmod 2$. However, we can view them as flat $SU(2)$ connections on $T^3 \smallsetminus S^1 = (T^2 \smallsetminus \{\mathrm{pt}\})\times S^1$, with holonomy -1 around a meridian of the S^1. By taking holonomy, these are in bijection with representations

$$\rho : \langle x, y, z \mid [x, z] = [y, z] = 1\rangle \to SU(2), \qquad \rho(xyx^{-1}y^{-1}) = -I$$

where x, y lie in the punctured torus and $xyx^{-1}y^{-1}$ thus represents the meridian.

Write $X = \rho(x) \in SU(2)$ and likewise Y, Z. Then taking traces of both $YXY^{-1} = -X^{-1}$ and $XYX^{-1} = -Y^{-1}$ says that $\operatorname{tr}(X) = -\operatorname{tr}(X^{-1})$ and $\operatorname{tr}(Y) = -\operatorname{tr}(Y^{-1})$, but elements of $SU(2)$ are conjugate to and thus have the same trace as their inverses, so $\operatorname{tr}(X) = \operatorname{tr}(Y) = 0$. There is then a unique ρ in each conjugacy class such that

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and Z has to commute with both of these, so it must be $\pm I$. Thus the chain complex $CI_*(T^3)_w$ has two generators, and their relative grading is 4 (mod 8), so the differential vanishes and thus $I_*(T^3)_w \cong \mathbb{C}^2$.

Proposition 2.6. *Let X be a cobordism from Y_1 to Y_2, and let $P \to W$ be a $U(2)$-bundle whose restrictions to Y_1 and Y_2 are both admissible. We write w for the associated line bundle over X. Then any choice $\mathcal{D}$ of metric and perturbation on X gives a well-defined homomorphism*

$$I(X)_w : I_*(Y_1, \mathcal{D}|_{Y_1})_{w|_{Y_1}} \to I_*(Y_2, \mathcal{D}|_{Y_2})_{w|_{Y_2}}$$

which does not depend on $\mathcal{D}$. If $(X_1, w_1, \mathcal{D}_1) : Y_1 \to Y_2$ and $(X_2, w_2, \mathcal{D}_2) : Y_2 \to Y_3$ are cobordisms for which the bundles w_i and data $\mathcal{D}_i$ agree over Y_2, then

$$I(X_1 \cup_{Y_2} X_2)_{w_1 \cup w_2} = I(X_2)_{w_2} \circ I(X_1)_{w_1}$$

as maps $I_*(Y_1, \mathcal{D}_1|_{Y_1})_{w_1|_{Y_1}} \to I_*(Y_3, \mathcal{D}_2|_{Y_3})_{w_2|_{Y_3}}$.

Proof (sketch). The map $I(X)_w$ is defined at the chain level by the formula

$$f(\alpha) = \sum_\beta \#\mathcal{M}_X(\alpha, \beta) \cdot \beta,$$

where $\#\mathcal{M}_X$ is a count of isolated (perturbed) instantons on the completed manifold

$$\big((-\infty, 0] \times Y_1\big) \cup X \cup \big([0, \infty) \times Y_2\big)$$

which limit to α at $\{-\infty\} \times Y_1$ and to β at $\{\infty\} \times Y_2$. (Note that there is no longer an $\mathbb{R}$-action on these moduli spaces.) The proof that $f \circ \partial_1 = \partial_2 \circ f$ again follows as in Morse theory: we wish to show that for all generators $\alpha \in CI_*(Y_1, \mathcal{D}|_{Y_1})$ and $\gamma \in CI_*(Y_2, \mathcal{D}|_{Y_2})$, the expression

$$\sum_{\beta_1} \#\hat{\mathcal{M}}_{Y_1}(\alpha, \beta_1) \cdot \#\mathcal{M}_X(\beta_1, \gamma) - \sum_{\beta_2} \#\mathcal{M}_X(\alpha, \beta_2) \cdot \#\hat{\mathcal{M}}_{Y_2}(\beta_2, \gamma)$$

is zero. Again, this is zero because it counts points in the boundary of the compactification of $\mathcal{M}_X(\alpha, \gamma)$.

Now suppose we have two different choices $\mathcal{D}_0$ and $\mathcal{D}_1$ of metrics and perturbations on X with the same restriction to $\partial X = -Y_1 \sqcup Y_2$, and let f_0 and f_1 be the corresponding chain maps. Then we choose a generic path $\mathcal{D}_t$ between them, and define a chain homotopy from f_0 to f_1 by counting instantons of index -1 over the 1-parameter family $\mathcal{D}_t$; this shows that f_0 and f_1 induce the same maps on homology. ∎

Theorem 2.7. *The instanton homology groups* $I_*(Y)_w$ *do not depend on the metric and perturbation.*

Proof. Let $X = [0,1] \times Y$, and write w for both the bundle on Y and its pullback to X. For any metric and perturbation $\mathcal{D}$ on Y, we take the corresponding $\mathbb{R}$-invariant metric and perturbation on $\mathbb{R} \times Y$ to see that the chain map

$$f : CI_*(Y, \mathcal{D})_w \to CI_*(Y, \mathcal{D})_w$$

is the identity: the translation invariance says that any nonempty moduli space $\mathcal{M}_X(\alpha, \beta)$ is at least 1-dimensional unless $\alpha = \beta$, in which case the only isolated point in $\mathcal{M}_X(\alpha, \alpha)$ is the product connection. Thus $I(X)_w : I_*(Y, \mathcal{D})_w \to I_*(Y, \mathcal{D})_w$ is the identity for all $\mathcal{D}$.

Pick two choices $\mathcal{D}_1$ and $\mathcal{D}_2$, and let $X = [0,1] \times Y$. Then we glue two copies of X together, with perturbation data interpolating between $\mathcal{D}_1$ and $\mathcal{D}_2$, to see that the composition of the cobordism maps

$$I_*(Y, \mathcal{D}_1)_w \xrightarrow{I(X)_w} I_*(Y, \mathcal{D}_2)_w \xrightarrow{I(X)_w} I_*(Y, \mathcal{D}_1)$$

is the identity, and likewise if we compose in the other direction, so each $I_*(X)_w$ must be an isomorphism. □

We can specialize the cobordism maps to the case where one end is empty: if X is a smooth 4-manifold with $\partial X = Y$ and a Hermitian line bundle $w \to X$ such that $w|_Y$ is admissible, then we have a relative invariant

$$\psi_{X,w} : \mathrm{Sym}(H_0(X) \oplus H_2(X)) \to I_*(Y)_w.$$

The invariant $\psi_{X,w}(1)$ is just a weighted sum of flat connections α on Y, where α is weighted by a signed count of isolated, finite-energy instantons which are asymptotic to α along the boundary. More generally, given $z \in \mathrm{Sym}(H_0(X) \oplus H_2(X))$, we can define the cohomology class $\mu(z)$ just as for closed Donaldson invariants and use it to obtain invariants $\psi_{X,w}(z)$ from higher-dimensional moduli spaces.

This leads to some interesting operators on instanton homology, as follows. Let $\Sigma \subset Y$ be a closed, oriented surface such that $\langle c_1(w), [\Sigma] \rangle$ is odd. Then there is a degree-2 operator

$$\mu(\Sigma) : I_*(Y)_w \to I_*(Y)_w$$

which we can define as follows. We take the cobordism $[0,1] \times Y$ and remove $\{\frac{1}{2}\} \times N(\Sigma)$ to get a new cobordism X with associated map

$$I(X)_w : I_*(Y)_w \otimes I_*(S^1 \times \Sigma) \to I_*(Y)_w.$$

Then we let $\mu(\Sigma) = I(X)_w\big(\cdot \otimes \psi_{D^2 \times \Sigma, w}(\Sigma)\big)$. If we write $X = X_1 \cup_Y X_2$ with $b_2^+(X_i) > 0$, then there is a natural pairing

$$I_*(Y)_w \otimes I_*(-Y)_w \to \mathbb{C}$$

for which the relative invariants of X_1 and X_2 recover the Donaldson invariants on X.

Muñoz [Muñ99] computed the instanton homology groups $I_*(S^1 \times \Sigma_g)_w$ where $g \geq 1$ and $c_1(w) = PD(S^1 \times \{\mathrm{pt}\})$. In fact, he determined their ring structure, where the multiplication maps are cobordism maps corresponding to Σ_g times a pair of pants. One can deduce from his presentation that the commuting operators $\mu(\Sigma_g)$ and $\mu(\mathrm{pt})$ have simultaneous eigenvalues

$$(2k, 2) \text{ and } (2k\sqrt{-1}, -2), \qquad |k| \leq g - 1,$$

and that the $(2g-2,2)$-eigenspace is 1-dimensional. It follows by construction that the operators $\mu(\Sigma)$ and $\mu(\mathrm{pt})$ on $I_*(Y)_w$ also have eigenvalues belonging to this set. In fact, since $\mu(\Sigma)$ depends only on the homology class of Σ, its eigenvalues are at most $2g-2$ in magnitude whenever there is a genus-g surface homologous to Σ.

3. Sutured manifolds and contact invariants

Our goal for this lecture is to define invariants of contact 3-manifolds with boundary, as elements of instanton Floer homology for sutured 3-manifolds. Sutured manifolds were originally defined by Gabai [Gab83], who used them to prove the existence of taut foliations on many 3-manifolds and then deduce in [Gab87] that 0-surgery on a nontrivial knot K in S^3 is prime and not $S^1 \times S^2$. (See also the foliation-free approach of [Sch89].) A version of Heegaard Floer homology for sutured manifolds was developed by Juhász [Juh06, Juh08], and Honda–Kazez–Matić [HKM09] constructed invariants of contact structures within sutured Floer homology. We will study the analogous instanton Floer homology theory due to Kronheimer and Mrowka [KM10b], and the corresponding contact invariant defined in joint work with Baldwin [BS16a].

Definition 3.1. *A (balanced)* sutured manifold (M,γ) *consists of a compact 3-manifold M with nonempty boundary, and an embedded, oriented 1-manifold $\gamma \subset \partial M$ which separates ∂M into two pieces $R_+(\gamma)$ and $R_-(\gamma)$ satisfying:*

- *Every component of ∂M contains a component of γ.*
- *$R_+(\gamma)$ and $R_-(\gamma)$ admit orientations such that $\gamma = \partial R_+(\gamma) = -\partial R_-(\gamma)$.*
- *$R_+(\gamma)$ and $R_-(\gamma)$ have the same Euler characteristic.*

Example 3.2. Let F be a connected, oriented surface with nonempty boundary. Then $(F \times [-1,1], \partial F \times \{0\})$ is a sutured manifold.

Example 3.3. If Y is a closed 3-manifold, then we let $Y(1) = (Y \smallsetminus \mathrm{int}(B^3), S^1)$.

Example 3.4. If $K \subset Y$ is a knot, then $Y(K) = (Y \smallsetminus \mathrm{int}(N(K)), \mu \cup -\mu)$ where μ is an oriented meridian.

Example 3.5. Let (M,ξ) be a contact 3-manifold with boundary. We say that ∂M is *convex* if it is transverse to some ξ-preserving vector field v on a neighborhood of ∂M. Giroux [Gir91] proved that this is a C^∞-generic condition, and that the *dividing curves*

$$\Gamma = \{x \in \partial M \mid v(x) \in \xi_x\}$$

determine ξ up to isotopy on a neighborhood of ∂M. Then (M,Γ) is a sutured manifold.

Kronheimer and Mrowka [KM10b] define invariants of (M,γ) by the following procedure. They embed (M,γ) in a closed 3-manifold Y with an admissible bundle w and a distinguished surface $R \subset Y \smallsetminus M$, so that the topology of $Y \smallsetminus M$ is as simple as possible. Then they define $\mathit{SHI}(M,\gamma)$ as the "top" eigenspace of the operator $\mu(R)$, acting on $I_*(Y)_w$, and prove its invariance by an excision theorem relating the top eigenspaces of such operators before and after cutting along such a surface and regluing. This excision theorem generalizes Floer's excision theorem for tori [BD95], and the *instanton knot homology*

$$\mathit{KHI}(Y,K) := \mathit{SHI}(Y(K))$$

was already defined by Floer [Flo90], though *SHI* is defined for a much larger class of 3-manifolds.

Definition 3.6. *Let (M,γ) be a sutured manifold. We construct a* closure (Y,R,η,α) *of (M,γ) by the following procedure.*

(1) Fix a tubular neighborhood $A(\gamma)\subset\partial M$ of γ and a surface with boundary T, and take an orientation-reversing homeomorphism $h:\partial T\times[-1,1]\to A(\gamma)$ for which $h(\partial T\times\{\pm 1\})\subset R_\pm(\gamma)\smallsetminus A(\gamma)$.

(2) Let $M' = M\cup_h T\times[-1,1]$, and identify

$$\partial M' = \partial_+ M' \sqcup \partial_- M'$$

where $R_\pm(\gamma)$ intersects $\partial_\pm M'$ nontrivially. We observe that ∂_+M' is homeomorphic to $\partial_- M'$, since they are connected surfaces of the same Euler characteristic.

(3) Choose a homeomorphism $\phi:\partial_+M'\to\partial_-M'$, and form

$$Y = M'\cup\partial_+M'\times[-1,1]$$

by the identifications

$$\partial_+M'\times\{-1\}\xrightarrow{\mathrm{id}}\partial_+M', \qquad \partial_+M'\times\{1\}\xrightarrow{\phi}\partial_-M'.$$

We then take $R=\partial_+M'\times\{0\}\subset Y$ and write $R\times[-1,1]$ for $\partial_+M'\times[-1,1]$.

(4) Let $\eta\subset R$ be an oriented, homologically essential curve.

(5) Let $\alpha\subset Y\smallsetminus M$ be a closed curve which intersects $R\times[-1,1]$ in an arc of the form $\{p\}\times[-1,1]$.

We require that $g(R)\geq 1$. Then the sutured instanton homology

$$SHI(M,\gamma) := I_*(Y|R)_{\alpha\sqcup\eta}$$

is defined as the generalized $(2-2g(R),2)$-eigenspace of $(\mu(R),\mu(\mathrm{pt}))$ acting on $I_*(Y)_{\alpha\sqcup\eta}$. Here the subscript means that we have chosen a Hermitian line bundle w on Y where $c_1(w)$ is Poincaré dual to $\alpha\sqcup\eta$.

Example 3.7. Fix a surface F with boundary and let

$$(M,\gamma) = (F\times[-1,1], \partial F\times\{0\}).$$

As illustrated in Figure 1, we can pick a surface T and gluing map h so that the resulting $M' = M\cup_h[-1,1]$ has the form

$$M' = \Sigma\times[-1,1], \qquad \Sigma := F\cup_\partial T.$$

Then $\partial_\pm M' = \Sigma\times\{\pm1\}$, and we let $\phi:\partial_+M'\to\partial_-M'$ be the identity, so that $Y=\Sigma\times S^1$ and $R=\Sigma\times\{\mathrm{pt}\}$. We let $\alpha=\{\mathrm{pt}\}\times S^1$ and pick η arbitrarily, and then a slight generalization of Muñoz's computation from [Muñ99] (using Kronheimer and Mrowka's excision theorem [KM10b]) yields

$$SHI(F\times[-1,1],\partial F\times\{0\}) = I_*(Y|R)_{\alpha\sqcup\eta}\cong\mathbb{C}.$$

Theorem 3.8 ([KM10b, BS15])**.** *The $\mathbb{C}$-vector space $SHI(M,\gamma)$ is an invariant of (M,γ) up to isomorphism. More precisely, for any two choices $\mathcal{D},\mathcal{D}'$ made in the construction, there is an isomorphism*

$$\Psi_{\mathcal{D},\mathcal{D}'}: SHI_{\mathcal{D}}(M,\gamma)\xrightarrow{\sim} SHI_{\mathcal{D}'}(M,\gamma)$$

which is canonical up to multiplication by an element of $\mathbb{C}^\times$, and which also satisfies $\Psi_{\mathcal{D},\mathcal{D}''} = \Psi_{\mathcal{D}',\mathcal{D}''}\circ\Psi_{\mathcal{D},\mathcal{D}'}$ up to a scalar in $\mathbb{C}^\times$ for any $\mathcal{D},\mathcal{D}',\mathcal{D}''$.

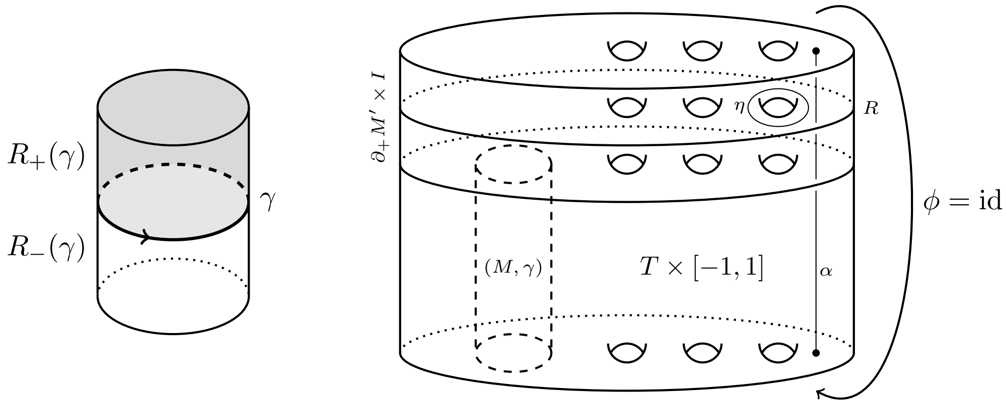


FIGURE 1. The sutured manifold $(M, \gamma) = (D^2 \times [-1, 1], \partial D^2 \times \{0\})$, with $R_+(\gamma)$ shaded (left), and a closure $(Y, R) \cong (\Sigma_3 \times S^1, \Sigma_3 \times \{\mathrm{pt}\})$ of (M, γ) built by taking T to be a genus-3 surface with one boundary component (right).

The proof in [KM10b] only establishes the invariance of *SHI* up to isomorphism, but for some applications we need more. Namely, if (M, Γ) carries a contact structure ξ, then we wish to define an invariant

$$\theta(\xi) \in SHI(-M, -\Gamma)$$

as an *element* of the sutured instanton homology. By promoting *SHI* to an invariant up to (nearly) canonical isomorphism in [BS15], we are able to make sense of elements: these are tuples consisting of one element $\theta_{\mathcal{D}}$ in each $SHI_{\mathcal{D}}$, such that the $\theta_{\mathcal{D}}$ are preserved by the canonical isomorphisms. Since these isomorphisms are well-defined up to scalars, so is the resulting θ.

We outline the proof of Theorem 3.8 here. We need to show invariance under several choices, but we focus on the gluing map $\phi : \partial_+ M' \to \partial_- M'$. In [KM10b], this is done in a single step using a cobordism map, as illustrated in Figure 2. The map induces an isomorphism on the top eigenspace of R,

$$f_\psi : SHI_{\mathcal{D}}(M, \gamma) \otimes I_*(R \times_\psi S^1 | R)_w \to SHI_{\mathcal{D}'}(M, \gamma) \tag{3.1}$$

for some w, which gives an isomorphism $SHI_{\mathcal{D}} \cong SHI_{\mathcal{D}'}$ since $I_*(R \times_\psi S^1 | R)_w \cong \mathbb{C}$.

Unfortunately, this argument does not show that the composition of the resulting isomorphisms $SHI_{\mathcal{D}} \to SHI_{\mathcal{D}'}$ and $SHI_{\mathcal{D}'} \to SHI_{\mathcal{D}''}$ is the same as our isomorphism $SHI_{\mathcal{D}} \to SHI_{\mathcal{D}''}$, which is needed for naturality. Thus instead of using the excision isomorphism, if we wish to change our gluing map from ϕ to $\phi \circ \psi$, we write

$$\psi = D_{\alpha_1} \circ \cdots \circ D_{\alpha_k}$$

as a product of right-handed Dehn twists along curves $\alpha_i \subset R$, and realize the change of gluing map by performing -1-framed surgery on curves $\alpha_i \times \{t_i\} \subset R \times [-1, 1] \subset Y$. Each surgery cobordism gives an isomorphism $SHI(Y|R) \to SHI(Y'|R)$, by Floer's exact triangle [Flo90, BD95]: the third term in the triangle has the form $SHI(Y''|R)$ where the surgery

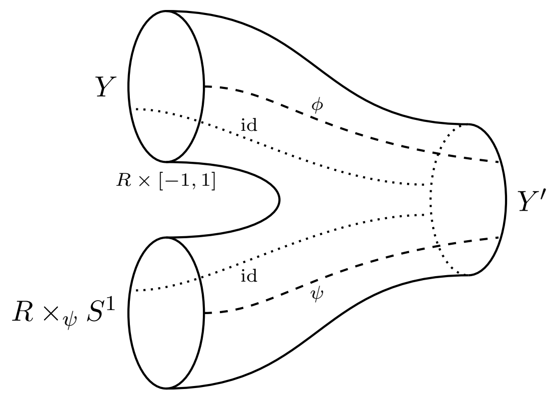


FIGURE 2. A cobordism which induces an isomorphism on *SHI*, changing the gluing isomorphism from ϕ to $\phi \circ \psi$. The middle region in between the dotted lines represents R times an octagon, glued to $Y \smallsetminus R \times [-1,1]$ above and to $R \times [1,1]$ below by the indicated maps.

producing Y'' has compressed R, and since R is homologous to a surface R' of strictly lower genus, the $(2-2g(R))$-eigenspace of $\mu(R) = \mu(R')$ is empty. Thus we can take the isomorphism

$$\Psi_{\mathcal{D},\mathcal{D}'} : \mathit{SHI}_{\mathcal{D}}(M,\gamma) \to \mathit{SHI}_{\mathcal{D}'}(M,\gamma)$$

to be the composition of the corresponding cobordism maps.

The fact that the maps $\Psi_{\mathcal{D},\mathcal{D}'}$ compose as expected is immediate from their description in terms of 2-handle cobordisms corresponding to Dehn twists. The fact that they do not depend on the factorization into Dehn twists is illustrated in Figure 3: the cobordism is the same as the excision cobordism, after we fill in the $R \times_\psi S^1$ boundary component using a Lefschetz fibration X_ψ over D^2 with fiber D^2 and vanishing cycles the α_i. In other words, we have

$$\Psi_{\mathcal{D},\mathcal{D}'} = f_\psi(\cdot, z) : \mathit{SHI}_{\mathcal{D}}(M,\gamma) \to \mathit{SHI}_{\mathcal{D}}(M,\gamma)$$

where f_ψ is the map (3.1) and $z \in I_*(R \times_\psi S^1|R)_w$ is the relative invariant of X_ψ, projected into the $(2-2g(R))$-eigenspace of $\mu(R)$. This relative invariant is both nonzero (which is equivalent to the nonvanishing of Donaldson invariants of symplectic manifolds, see [KM10b, Siv15]) and an element of $I_*(R \times_\psi S^1|R)_w \cong \mathbb{C}$, so the elements corresponding to two different choices differ by a nonzero factor and hence so do the corresponding maps $\Psi_{\mathcal{D},\mathcal{D}'}$. This explains why these maps are well-defined up to scalars.

We now explain the construction of attaching maps on *SHI* associated to contact 1- and 2-handles, as done in [BS16a]. A contact 1-handle is a tight contact $D^1 \times D^2$, attached to (M,γ) along a $\partial D^1 \times D^2$ neighborhood of two points on γ as in Figure 4. We remark that (M,γ) need not have a contact structure of its own; the contact structure on the 1-handle simply tells us where to place the sutures, which are identified with dividing curves.

Proposition 3.9. *Let (M',γ') be obtained from (M,γ) by attaching a contact 1-handle H. Then there is a natural isomorphism*

$$F_H : \mathit{SHI}(-M,-\gamma) \to \mathit{SHI}(-M',-\gamma'),$$

induced by and depending only on H.

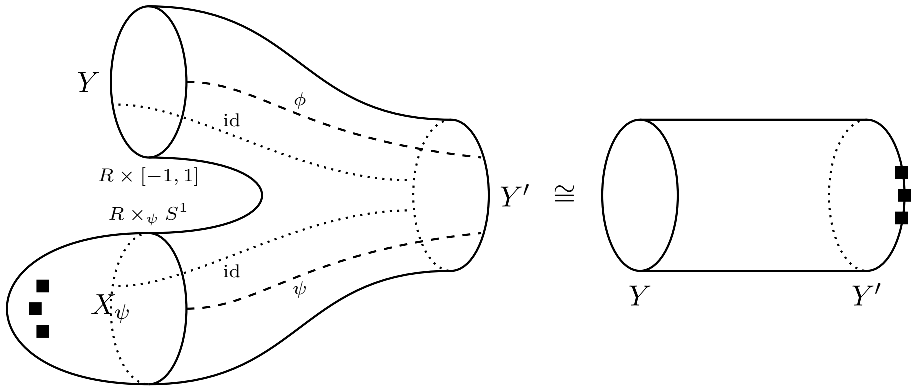


FIGURE 3. Attaching 2-handles to $Y \times [0,1]$ to perform Dehn twists (right) produces the same cobordism as if we had used the 2-handles to build a Lefschetz fibration $X_\psi \to D^2$ and glued this to the $R \times_\psi S^1$ boundary component of the excision cobordism (left).

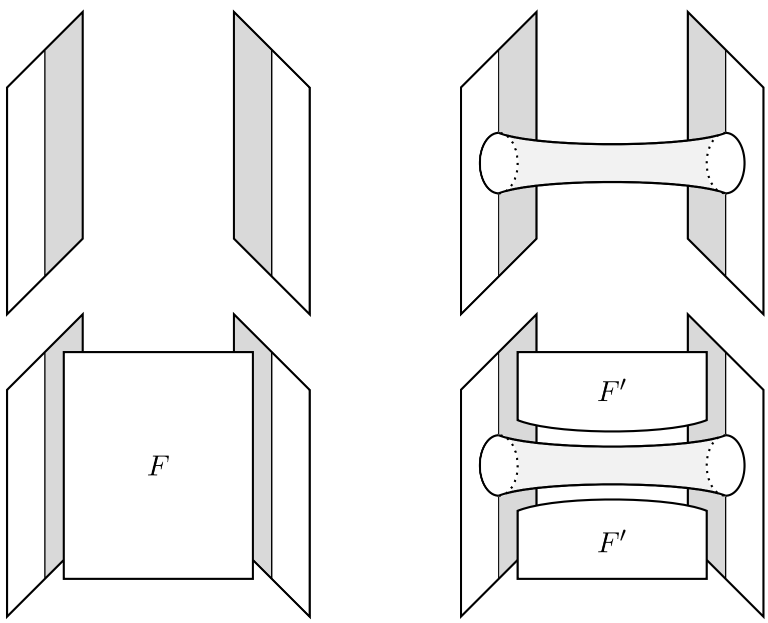


FIGURE 4. Attaching a contact 1-handle to (M,γ) along ∂M (top left) to get (M',γ') (top right). Here we have again shaded $R_+(\gamma)$. The corresponding auxiliary surfaces F and F' are shown at bottom left and bottom right.

Proof. We take a closure of (M',γ'), built by gluing an auxiliary surface F' to γ', and absorb the 1-handle H into F' to get an auxiliary surface F from which we can build a closure of (M,γ). This is pictured at the bottom of Figure 4. The end result is that the corresponding closures of (M,γ) and (M',γ') are identical, so for *SHI* defined from these closures we can declare F_H to be the identity map. For the "natural" claim, we then check that if we define F_H using a different closure of (M',γ'), then the two definitions of F_H commute with the canonical isomorphisms relating the different $SHI(M,\gamma)$ and $SHI(M',\gamma')$ invariants. $\square$

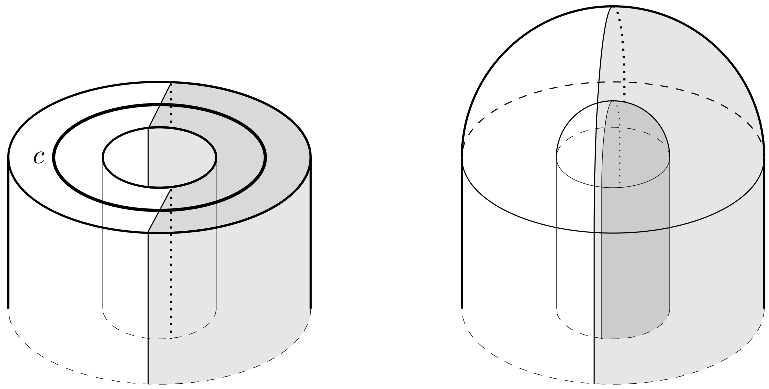


FIGURE 5. Attaching a contact 2-handle to (M, γ) along $c \subset \partial M$ (left) to get (M', γ') (right).

Similarly, a contact 2-handle is a tight contact $D^2 \times D^1$ attached to (M, γ) along a neighborhood $\partial D^2 \times D^1$ of a curve $c \subset \partial M$, as in Figure 5. We require that each of $c \cap R_+(\gamma)$ and $c \cap R_-(\gamma)$ is an arc.

Proposition 3.10. *Let (M', γ') be obtained from (M, γ) by attaching a contact 2-handle H. Then there is a natural homomorphism*

$$F_H : SHI(-M, -\gamma) \to SHI(-M', -\gamma')$$

induced by and depending only on H.

Proof. Given a closure (Y, R, η, α) of (M, γ), we construct a closure (Y', R, η, α) of (M', γ') by performing ∂M-framed surgery on the attaching curve $c \subset \partial M \subset Y$. Then F_H is defined to be the corresponding 2-handle cobordism map, and again we can check that it is natural in the same sense as in Proposition 3.9. We remark that the effect on the auxiliary surface F used to close up (M, γ) is to attach a neighborhood of the arc $c \cap R_+(\gamma)$ as a 1-handle to get the surface F' which leads to a closure of (M', γ'). □

These contact handle maps suggest a way to define a contact invariant. Namely, if ξ_{std} is the standard tight contact structure on B^3 with convex boundary, then since

$$SHI(-B^3, -S^1) \cong \mathbb{C},$$

we declare $\theta(\xi_{\text{std}})$ to be any nonzero element of $SHI(-B^3, -S^1)$ up to rescaling. Then for any contact 3-manifold (M, ξ) with dividing curves Γ on the convex boundary ∂M, we construct (M, ξ) by attaching contact 1- and 2-handles to (B^3, ξ_{std}) and let $\theta(\xi)$ be the image of $\theta(\xi_{\text{std}})$ under the corresponding handle attaching maps.

Unfortunately, this definition is too general to allow for a proof of invariance, so we restrict to a particularly nice class of handle decompositions. Closed contact 3-manifolds are supported by many open book decompositions, which according to the Giroux correspondence [Gir02] are all related by a notion of stabilization. For contact 3-manifolds with convex boundary we have the following *relative* Giroux correspondence [HKM09].

Definition 3.11. *A* partial open book *for a sutured contact manifold (M, Γ, ξ) is a tuple $(S, P, h, \mathbf{c})$, where*

- *S is a surface with boundary and $P \subset S$ is a surface;*
- *$h : P \hookrightarrow S$ is an embedding which restricts to the identity on $\partial P \cap \partial S$;*

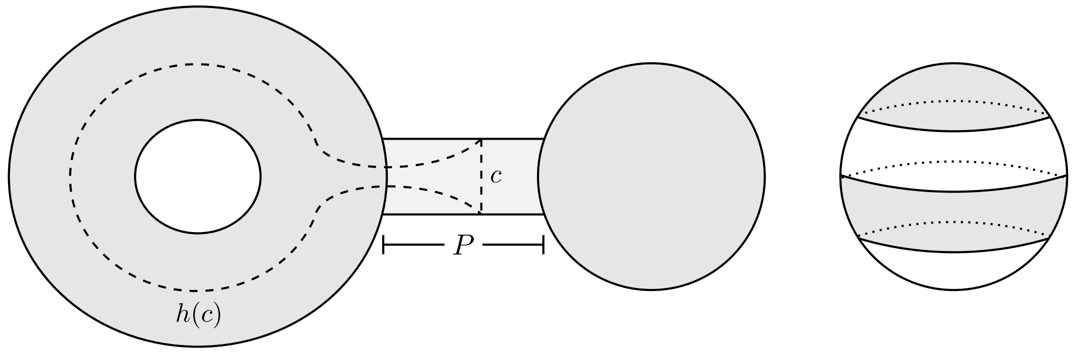


FIGURE 6. A page S of a partial open book for a neighborhood of an overtwisted disk (left), and the convex boundary of this neighborhood (right).

- $\mathbf{c} = \{c_1, \ldots, c_n\}$ *is a set of disjoint, properly embedded arcs in* P *such that* $S \setminus \{\mathbf{c}\}$ *deformation retracts onto* $S \setminus P$*.*

We form a contact manifold from a partial open book by the following procedure. We take a handlebody $H = H(S)$ by taking the $[-1,1]$-invariant contact structure ξ_S on $S \times [-1, 1[$ whose dividing curves Γ_S on $S \times \{1\}$ consist of a boundary-parallel arc on each component of ∂S, oriented the same way as ∂S, and then rounding corners. Its boundary ∂H is the double of S, with dividing set Γ isotopic to ∂S and $R_\pm(\Gamma)$ identified with $S \times \{\pm 1\}$. Next, we identify disjoint closed curves

$$\gamma_i = \big(c_i \times \{1\}\big) \cup \big(\partial c_i \times [-1,1]\big) \cup \big(h(c_i) \times \{-1\}\big) \tag{3.2}$$

in ∂H, and attach contact 2-handles along each γ_i to get a contact manifold $M(S, P, h, \mathbf{c})$. A *partial open book decomposition* of (M, γ, ξ) is then a tuple

$$(S, P, h, \mathbf{c}, f : M(S, P, h, \mathbf{c}) \to (M, \Gamma, \xi)),$$

where $(S, P, h, \mathbf{c})$ is a partial open book and f is a contactomorphism.

Example 3.12. A standard neighborhood (B^3, ξ_{ot}) of an overtwisted disk has the partial open book decomposition shown in Figure 6, as described in [HKM09, Example 1]. In the resulting sutured contact manifold (M, Γ, ξ), we can identify $R_+(\Gamma)$ with $S \setminus P$ and $R_-(\Gamma)$ with $S \setminus h(P)$; both are the disjoint union of an annulus and a disk.

When we form a closure (Y, R, η, α) of $(-M, -\Gamma)$, the surface R is compressible because we can identify $R_+(\Gamma)$ with a subsurface of R; the core of the annulus in $R_+(\Gamma)$ is essential in R and compressible in Y since it bounds a disk in M. Since R is compressible, the operator $\mu(R)$ does not have $2 - 2g(R)$ as an eigenvalue and so

$$SHI(-M, -\Gamma) = I_*(Y|R)_{\alpha \sqcup \eta}$$

is zero.

Definition 3.13. *A* stabilization *of the partial open book* $(S, P, h, \mathbf{c})$ *is a partial open book* $(S', P', h', \mathbf{c}')$*, where*

- S' *and* P' *are obtained by attaching a 1-handle* H_0 *to* S *and* P*;*
- $h' = D_\beta \circ h$*, where* D_β *is a right-handed Dehn twist along a curve* $\beta \subset S'$ *having a single transverse intersection with a cocore* c_0 *of* H_0*;*
- $\mathbf{c}' = \mathbf{c} \cup \{c_0\}$*.*

Theorem 3.14 ([HKM09])**.** *Every sutured contact manifold (M, Γ, ξ) is supported by a contact open book. Moreover, any two supporting open books for (M, Γ, ξ) are related by a sequence of stabilizations.*

With the above definitions in hand, we now use partial open book decompositions to construct our contact invariant. Recall for any surface S with boundary that the product handlebody $(H(S), \Gamma_S)$ has sutured instanton homology $SHI(-H(S), -\Gamma_S) \cong \mathbb{C}$, by Example 3.7.

Definition 3.15. *Let $(S, P, h, \mathbf{c}, f)$ be a partial open book decomposition of (M, Γ, ξ), and let $1 \in SHI(-M, -\Gamma)$ be any nonzero element. Form (M, γ, ξ) from $(H(S), \xi_S)$ by attaching contact 2-handles $H_1, \dots, H_n$ along curves $\gamma_1, \dots, \gamma_n$ as defined in* (3.2)*. Then we use the handle attaching maps*

$$F_{H_n} \circ F_{H_{n-1}} \circ \cdots \circ F_{H_1} : SHI(-H(S), -\Gamma_S) \to SHI(-M, -\Gamma)$$

of Proposition 3.10 to define

$$\theta(\xi) = (F_{H_n} \circ F_{H_{n-1}} \circ \cdots \circ F_{H_1})(1) \in SHI(-M, -\Gamma).$$

Theorem 3.16. *Let (M, Γ, ξ) be a sutured contact manifold. Then the element $\theta(\xi) \in SHI(-M, -\Gamma)$ associated to a supporting open book is invariant under stabilization, and hence an invariant of the contact structure. Moreover, if we build a new contact manifold (M', Γ', ξ') by attaching a contact 1-handle or a contact 2-handle, which we call H, then the map*

$$F_H : SHI(-M, -\Gamma) \to SHI(-M', -\Gamma')$$

of Proposition 3.9 or 3.10 satisfies $F_H(\theta(\xi)) = \theta(\xi')$.

Proof. We explain the proof of invariance here. Let $(S, P, h, \mathbf{c} = \{c_1, \dots, c_n\})$ be an open book supporting (M, Γ, ξ), and let

$$(S', P', h', \mathbf{c}' = \mathbf{c} \cup \{c_0\})$$

be a stabilization. Then $M(S', P', h', \{c_0\})$ is formed by attaching a Darboux ball (a cancelling pair of contact 1- and 2-handles) to $H(S)$, so we have a natural identification

$$SHI(-M(S', P', h', \{c_0\})) \cong SHI(-H(S)) \cong \mathbb{C}$$

and we claim that the 2-handle attachment map

$$F_{H_0} : SHI(-H(S')) \to SHI(-M(S', P', h', \{c_0\})),$$

induced by a handle H_0 attached along γ_0, is nonzero as a map $\mathbb{C} \to \mathbb{C}$. Then the elements 1 and $F_{H_0}(1)$ of $SHI(-M(S', P', h', \{c_0\})) \cong \mathbb{C}$ agree up to a scalar, hence so do their images under the analogous map

$$F_{H_n} \circ \cdots \circ F_{H_1}.$$

But these images are by definition $\theta(M(S, P, h, \mathbf{c}))$ and $\theta(M(S', P', h', \mathbf{c}'))$ respectively, so the two must be equal.

To prove the claim, we consider γ_0 as shown in Figure 7. As a $\partial H(S')$-framed curve, it is isotopic to the $(\partial H(S') + 1)$-framed curve $\beta' \subset S' \times \{1\} \subset R_+(\Gamma_{S'})$, which is a parallel copy of the curve β for which $h' = D_\beta \circ h$. We can then further isotope γ_0 into the surface R' which is part of the closure data for $-H(S')$, and thus we realize the cobordism which defines F_{H_0} as $(+1)$-surgery on an essential curve in R'. But this is exactly the sort of cobordism which realizes the canonical isomorphisms between different closures of $-H(S')$

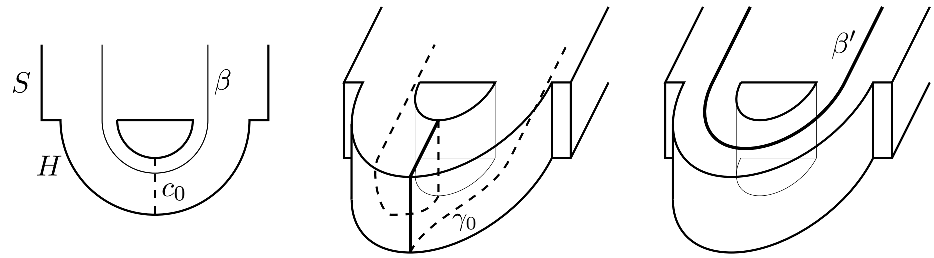


FIGURE 7. The arc $c_0 \subset S'$ (left), and the arc γ_0 both before (middle) and after (right) an isotopy.

– here it matters that we have reversed orientation, so that the framing with respect to $-H(S')$ is -1 – and so F_{H_0} is indeed an isomorphism. □

Proposition 3.17. *If (M, Γ, ξ) is a sutured contact manifold and ξ is overtwisted, then* $\theta(\xi) = 0$.

Proof. We build (M, Γ, ξ) by attaching contact 1- and 2-handles to a small neighborhood $(B^3, \Gamma_{\text{ot}}, \xi_{\text{ot}})$ of an overtwisted disk. Letting F be the composition of the corresponding handle maps, the map

$$F : SHI(-B^3, -\Gamma_{\text{ot}}) \to SHI(-M, -\Gamma)$$

sends $\theta(\xi_{\text{ot}})$ to $\theta(\xi)$ by Theorem 3.16. But we saw in Example 3.12 that $SHI(-B^3, -\Gamma_{\text{ot}}) = 0$, so $\theta(\xi_{\text{ot}}) = 0$ and hence the same is true of its image. □

We can see from the definition that θ is not uniformly zero: it is nonzero for all product sutured contact manifolds $(H(S), \xi_S)$, since these are supported by partial open books with $P = \mathbf{c} = \emptyset$. In fact, if (Y, ξ) is a closed contact 3-manifold which is Stein fillable, then $\theta(\xi|_{Y(1)}) \neq 0$, and it follows that if a sutured contact manifold (M, Γ, ξ) embeds in a Stein fillable closed contact 3-manifold then $\theta(\xi) \neq 0$. See [BS16a], or [BS16b] for a stronger version with a different proof.

4. Khovanov homology and the trefoils

In this lecture we build on Kronheimer and Mrowka's proof that Khovanov homology detects the unknot [KM11] to prove that it also detects the left- and right-handed trefoils [BS18]. For further reading, we recommend [BN02] as an introduction to Khovanov homology. This lecture also makes use of quite a few notions from 3-dimensional contact geometry, and all of the necessary background (and more) can be found in Etnyre's lecture notes on Legendrian and transverse knots [Etn05] and on open book decompositions [Etn06].

The first categorification of a knot polynomial was given by Khovanov [Kho00], who defined for any knot $K \subset S^3$ a bigraded abelian group

$$Kh^{*,*}(K)$$

whose graded Euler characteristic recovers the Jones polynomial,

$$V_K(q) = \sum_{i,j} (-1)^i q^j \cdot \text{rank}(Kh^{i,j}(K)).$$

(Strictly speaking, we are using Kh to denote *reduced* Khovanov homology.)

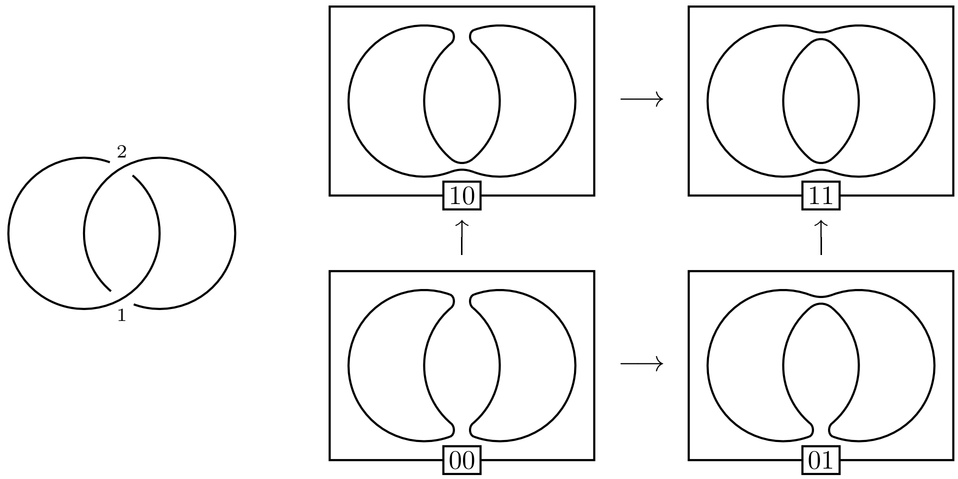
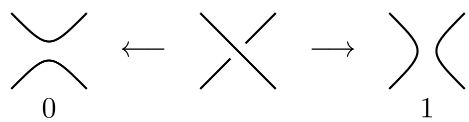


FIGURE 8. The cube of resolutions for the Hopf link. We have $V_{00} \cong V_{11} \cong \mathbb{Z}^2$ and $V_{10} \cong V_{01} \cong \mathbb{Z}$.

Khovanov homology is defined using a "cube of resolutions", in which we take a diagram D of K whose crossings are labeled $1, \dots, n$ and then associate to each element $a = (a_1, \dots, a_n)$ of $\{0,1\}^n$ the unlink diagram D_a in which crossing i is replaced with its a_i-resolution for all i, according to the following convention:



(See Figure 8 for an example.) We then place $V_a = (\mathbb{Z}^2)^{\otimes(|D_a|-1)}$ at each vertex of an n-dimensional cube, where $|D_a|$ is the number of components of D_a, and label the edges from a to b (where b is obtained from a by changing one $a_j = 0$ to 1) with certain maps $V_a \to V_b$ determined by whether the change in resolution merges two components of D_a or splits one component into two. All of this is combinatorial and forms a chain complex whose homology is $Kh(K)$. See [BN02] for details of the original "unreduced" construction, which is essentially the same but slightly larger.

It is still unknown whether the Jones polynomial detects the unknot $U \subset S^3$: in other words, if $V_K(q) = V_U(q)$, must K be the unknot? The categorified version of this question was answered affirmatively by Kronheimer and Mrowka.

Theorem 4.1 ([KM11])**.** *Let K be a knot in S^3. Then $Kh(K)$ has rank 1 if and only if K is the unknot.*

The proof of Theorem 4.1 proceeds in several steps. First, Kronheimer and Mrowka define an invariant $I^\natural(K)$, the *singular instanton knot homology* of K, which associates abelian groups to knots in 3-manifolds and homomorphisms to cobordisms between them. They then construct a spectral sequence with E_2 page $Kh(K)$ and converging to $I^\natural(\overline{K})$, based on the observation that the groups and maps in Khovanov's cube of resolutions are the same as those assigned by $I^\natural$ to the various D_a and saddle cobordisms between them; this was originally due to Ozsváth and Szabó [OS05], whose spectral sequence converged instead to $\widehat{HF}(\Sigma(\overline{K}))$. Finally, they prove an isomorphism $I^\natural(K) \otimes \mathbb{C} \cong KHI(K)$, so all of this gives a rank inequality

$$\operatorname{rank}(Kh(K)) \geq \operatorname{rank}(KHI(K)), \tag{4.1}$$

and it suffices to show that $\operatorname{rank}(KHI(K)) \geq 1$ with equality if and only if K is unknotted.

This last claim was proven in [KM10b] as follows. There is an Alexander grading

$$KHI(K) = \bigoplus_{i=-g(K)}^{g(K)} KHI(K,i)$$

which is symmetric and detects the Seifert genus, fiberedness, and Alexander polynomial of K, meaning the following:

- $KHI(K,i) \cong KHI(K,-i)$ for all i;
- $\operatorname{rank} KHI(K,g(K)) \geq 1$, with equality if and only if K is fibered.
- Each $KHI(K,i)$ has a canonical $\mathbb{Z}/2\mathbb{Z}$ grading, and these satisfy

$$\Delta_K(t) = -\sum_{i=-g(K)}^{g(K)} \chi(KHI(K,i))t^i. \tag{4.2}$$

(The last property is not needed for unknot detection; it was proved separately in [KM10a].) So if K is not the unknot then it has Seifert genus $g(K) \geq 1$, and the summand

$$KHI(K,g(K)) \oplus KHI(K,-g(K)) \subset KHI(K)$$

has rank at least 2 by the above properties.

The goal of this lecture, building on Kronheimer and Mrowka's work, is to show that Khovanov homology also detects the trefoils. We remark that it clearly distinguishes the left and right handed trefoils from each other, since they have different Jones polynomials.

Theorem 4.2 ([BS18])**.** *Let K be a knot other than the unknot. Then* $\operatorname{rank} Kh(K) \geq 3$*, with equality if and only if K is a trefoil.*

Proof. By (4.1), it suffices to show that if K is not the unknot, then $\operatorname{rank}(KHI(K)) \geq 3$ with equality if and only if K is a trefoil. For K which are not fibered, the rank is at least 4 since it must be at least 2 in each of the Alexander gradings $g(K)$ and (by symmetry) $-g(K)$, so we can assume from now on that K is fibered. We also assume for now the following proposition.

Proposition 4.3. *Let K be a fibered knot in S^3. Then* $\operatorname{rank}(KHI(K,g(K)-1)) \geq 1$*.*

Supposing that K is fibered with Seifert genus $g \geq 2$, it follows from this proposition that

$$\operatorname{rank}(KHI(K,i)) > 0 \quad \text{for} \quad i \in \{-g,-g+1,g-1,g\},$$

and these four gradings are all distinct, so $KHI(K)$ has total rank at least four.

The only remaining cases are those where K is a fibered knot with genus 1, in which case it is either a trefoil or the figure eight. But by (4.2), the rank of $KHI(K)$ is at least $\det(K) = |\Delta_K(-1)|$; applying this together with the upper bound (4.1), it follows that KHI has rank 5 for the figure eight knot and 3 for each of the trefoils, so the proof is complete. □

We now focus on the proof of Proposition 4.3. The analogous statement in knot Floer homology was discovered first [BVV18], but its proof makes use of the fact that $\widehat{HFK}(-Y,-K)$

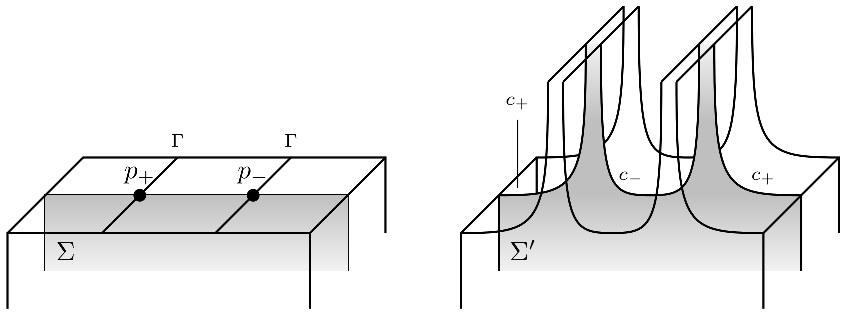


FIGURE 9. Building a closure of (M, Γ) (left) adapted to the surface Σ by attaching a handle to Σ inside $T \times [-1, 1]$ to form Σ' (right) and then gluing c_+ to c_-.

comes from a filtration on the Heegaard Floer chain complex $\widehat{CF}(-Y)$, and this has no analogue in instanton Floer homology. However, the relevant claim can be rephrased in terms of the $\mathbb{Z}/2\mathbb{Z}[U]$-module structure of $HFK^-(-Y, -K)$, which Etnyre, Vela-Vick and Zarev [EVVZ17] reinterpreted in terms of contact structures and sutured Heegaard Floer homology. This reinterpretation can be translated into *SHI*, and doing so leads us to the seemingly ad hoc proof of Proposition 4.3, which we develop over the next few subsections.

4.1. **The Alexander grading.** The first ingredient in the proof of Proposition 4.3 is an Alexander grading on *SHI*, which was defined by Kronheimer and Mrowka for *KHI* in [KM10b] and then slightly generalized in [BS18]. We refer back to Definition 3.6 for the definition of a closure of a sutured manifold.

Definition 4.4. *Let (M, Γ) be a sutured manifold, and $\Sigma \subset M$ a properly embedded, oriented surface with connected boundary which intersects Γ transversely in two points, $p_+ \in R_+(\Gamma)$ and $p_- \in R_-(\Gamma)$. We build a closure (Y, R) of (M, Γ) which is* adapted *to Σ by the following steps:*

- *Fix a properly embedded, nonseparating arc τ in the auxiliary surface T, and arrange that the homeomorphism*
$$h : \partial T \times [-1, 1] \to A(\Gamma) = \Gamma \times [-1, 1]$$
sends $\partial\tau \times [-1, 1]$ to $\{p_+, p_-\} \times [-1, 1]$. Define $\Sigma' \subset M'$ as the union of Σ and $\tau \times [-1, 1]$.
- *Define closed curves in the boundary of M' by*
$$c_\pm = \big(\partial\Sigma \cap (R_\pm(\Gamma) \smallsetminus A(\Gamma))\big) \cup \big(\tau \times \{\pm 1\}\big) \subset \partial_\pm M'$$
and let $\phi : \partial_+ M' \to \partial_- M'$ be a homeomorphism taking c_+ to c_-.

In the closed manifold $Y = M' \cup \partial_+ M' \times [-1, 1]$, we define $\hat{\Sigma} = \Sigma' \cup c_+ \times [-1, 1]$.

Given an adapted closure (Y, R), the associated surface $\hat{\Sigma}$ is a closed, oriented surface obtained by gluing a punctured torus to Σ. Thus $g(\hat{\Sigma}) = g(\Sigma) + 1$, and so the eigenvalues of $\mu(\hat{\Sigma})$ acting on $I_*(Y|R)_w$ are a subset of

$$\{2 - 2g(\hat{\Sigma}), 4 - 2g(\hat{\Sigma}), \ldots, 2g(\hat{\Sigma}) - 2\} = \{-2g(\Sigma), 2 - 2g(\Sigma), \ldots, 2g(\Sigma)\}.$$

Definition 4.5. *Let $\Sigma \subset (M,\Gamma)$ be a properly embedded, oriented surface with connected boundary which intersects Γ transversely in two points. Then there is a decomposition*

$$SHI(M,\Gamma) = \bigoplus_{i=-g(\Sigma)}^{g(\Sigma)} SHI(M,\Gamma,[\Sigma],i),$$

where $SHI(M,\Gamma,[\Sigma],i)$ is the generalized $2i$-eigenspace of $\mu(\hat{\Sigma})$ acting on $SHI(M,\Gamma)$ as constructed from an adapted closure.

Theorem 4.6 ([BS18]). *The decomposition of Definition 4.5 is independent of the choice of adapted closure, and depends only on the relative homology class $[\Sigma] \in H_2(M,\partial\Sigma)$.*

We refer to this decomposition as the *Alexander grading* on $SHI(M,\Gamma)$ relative to Σ. In the case of a sutured knot complement with Seifert surface Σ, this recovers Kronheimer and Mrowka's original Alexander grading on *KHI*; if the ambient manifold is a homology 3-sphere then the homology class of Σ is uniquely determined and so we can omit it from the notation.

4.2. **Bypasses and the Legendrian invariant.** The second ingredient in the proof of Proposition 4.3 is a Legendrian knot invariant built out of the contact class θ, using a construction of Stipsicz and Vértesi [SV09] in sutured Heegaard Floer homology which recovers an Legendrian invariant originally due to Lisca–Ozsváth–Stipsicz–Szabó [LOSS09]. In order to define it, we must first review the notion of a *bypass* in contact geometry.

A bypass in a contact manifold (Y,ξ) is a particular type of half-disk D, originally defined by Honda [Hon00, §3.4], which is attached to a convex surface Σ along an arc c. The arc c intersects the dividing curves $\Gamma \subset \Sigma$ transversely in exactly three points, two of which are the endpoints of c. A neighborhood of $\Sigma \cup D$ is then smoothly isotopic to a product $\Sigma \times [0,1]$, but not necessarily as a contact manifold: the dividing curves on $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ are generally different.

In more familiar (and precise) terms, we construct this neighborhood by attaching a contact 1-handle to Σ along ∂c, and then a contact 2-handle along a closed curve consisting of c and an arc on the boundary of the 1-handle, as shown in Figure 10. If we attach a bypass D to a sutured contact manifold (M,Γ,ξ) along its boundary, producing a new sutured contact manifold (M,Γ',ξ'), then the composition of the 1-handle and 2-handle maps of Propositions 3.9 and 3.10 is a map

$$F_D : SHI(-M,-\Gamma) \to SHI(-M,-\Gamma')$$

which takes $\theta(\xi)$ to $\theta(\xi')$ by Theorem 3.16.

Now a knot Λ in a contact manifold (Y,ξ) is called *Legendrian* if it is tangent everywhere to the contact planes, meaning that $T\Lambda \subset \xi|_\Lambda$. A Legendrian knot Λ has a standard contact neighborhood $N(\Lambda)$ and a natural framing, the *Thurston–Bennequin framing* $tb(\Lambda)$, given by the orthogonal complement of $T\Lambda$ inside $\xi|_\Lambda$. The boundary of a standard neighborhood is then a convex surface with dividing set Γ_{tb} a pair of parallel, oppositely oriented curves of slope $tb(\Lambda)$, and in the complement $Y \setminus N(\Lambda)$ there are several interesting ways to attach a bypass. Two of these are illustrated in Figure 11: one of them, along an arc labeled c_{SV}, turns these dividing curves into meridians and thus induces a bypass attachment map

$$F_{SV} : SHI(-Y \setminus N(\Lambda),\Gamma_{tb}) \to SHI(-Y \setminus N(\Lambda), \mu \cup -\mu) = KHI(-Y,-K).$$

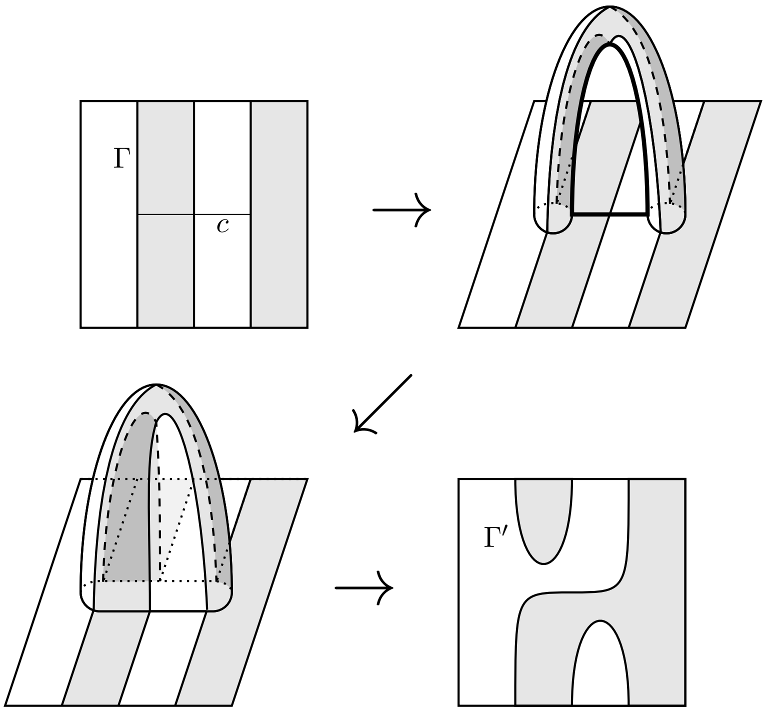


FIGURE 10. Attaching a bypass to Σ along an arc c (top left) by first attaching a contact 1-handle (top right), then attaching a contact 2-handle along the very thick curve to get a convex surface isotopic to Σ (bottom left) with new dividing curves Γ' (bottom right).

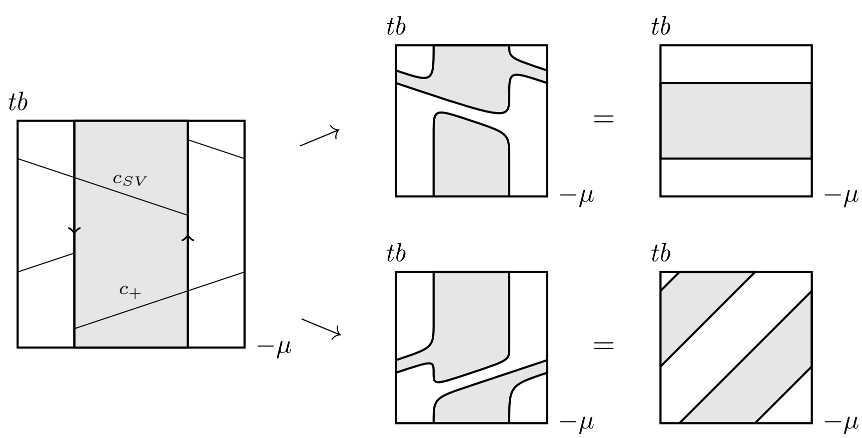


FIGURE 11. Attaching bypasses to the complement of a Legendrian knot along the arcs c_{SV} (top) and c_+ (bottom), as viewed from outside the complement.

The other, along the arc labeled c_+, turns the complement of Λ into the complement of its *positive stabilization* Λ_+, which satisfies $tb(\Lambda_+) = tb(\Lambda) - 1$. (An identical bypass with endpoints on the other dividing curve produces the negative stabilization Λ_- instead.)

Definition 4.7 (cf. [SV09]). *If* Λ *is a Legendrian knot in* (Y, ξ)*, we define*

$$\mathcal{L}(\Lambda) = F_{SV}(\theta(\xi|_{Y \smallsetminus N(K)})) \in KHI(-Y, -K)$$

as the class $\theta(\xi_\Lambda)$*, where* ξ_Λ *is the contact structure obtained by attaching a bypass to the complement of* Λ *along the curve* c_{SV}*.*

Evidently $\mathcal{L}(\Lambda)$ is an invariant of Λ up to Legendrian isotopy. If $\Lambda_\pm$ are the positive and negative stabilizations of Λ, then Stipsicz and Vértesi [SV09] proved that $\xi_{\Lambda_-} \cong \xi_\Lambda$ and that ξ_{Λ_+} is overtwisted, from which we have

$$\mathcal{L}(\Lambda_-) = \mathcal{L}(\Lambda), \qquad \mathcal{L}(\Lambda_+) = 0.$$

The first identity implies that $\mathcal{L}$ defines an invariant of *transverse* knots $\mathcal{K} \subset (Y, \xi)$, by declaring $\mathcal{T}(\mathcal{K}) := \mathcal{L}(\Lambda)$ where Λ is any Legendrian approximation of $\mathcal{K}$.

If $K \subset Y$ is a fibered knot of genus g, with fibration $\pi : Y \smallsetminus K \to S^1$, then the open book (K, π) determines a contact structure on Y by a construction of Thurston and Winkelnkemper [TW75]. This contact structure, which is said to be *supported* by (K, π), is nearly tangent to the pages of the open book (i.e., the fibers of π), while K is transverse to it. It may be overtwisted, but we still have the following:

Theorem 4.8 ([BS18]). *Let* $K \subset Y$ *be a fibered knot, with fibration* $\pi : Y \smallsetminus K \to S^1$ *and Seifert surface* Σ *which is the closure of a fiber* $\pi^{-1}(\mathrm{pt})$*. If we view* K *as a transverse knot with respect to the contact structure supported by* (K, π)*, then the transverse invariant* $\mathcal{T}(K)$ *is a nonzero element of*

$$KHI(-Y, K, [\Sigma], g(\Sigma)) \cong \mathbb{C}.$$

Remark 4.9. There is no contradiction with Proposition 3.17 here, because while the contact structure on Y may be overtwisted, its restriction to $Y \smallsetminus N(K)$ is not. In other words, every overtwisted disk in Y has to pass through the binding of the open book.

In particular, Theorem 4.8 says that the sutured manifold $Y(K)$ carries a contact structure $\xi_{Y,K,\pi}$ – this is the ξ_Λ of Definition 4.7, where Λ is a Legendrian approximation to K – whose contact invariant $\theta(\xi_{Y,K,\pi}) = \mathcal{T}(K)$ generates $KHI(-Y, K, [\Sigma], g(\Sigma))$.

4.3. **The bypass exact triangle.** The third ingredient in the proof of Proposition 4.3 is the *bypass exact triangle*, which was originally described in sutured Heegaard Floer homology in unpublished work of Honda (see [EVVZ17, §6] for a proof). When we attach a bypass, there is an obvious candidate for a new attaching arc in a neighborhood of the old one, and if we attach another bypass and repeat we eventually get a 3-periodic sequence $\Gamma, \Gamma', \Gamma'', \dots$ of dividing curves, as shown in Figure 12. Their sutured instanton homologies are related as follows.

Theorem 4.10 ([BS18]). *Let* M *be a sutured manifold, and let* $\Gamma, \Gamma', \Gamma'' \subset \partial M$ *be sutures which differ inside a disk as in Figure 12 and agree outside that disk. Then there is an exact triangle*

$$\cdots \to SHI(-M, -\Gamma) \to SHI(-M, -\Gamma') \to SHI(-M, -\Gamma'') \to \dots,$$

where the homomorphisms are bypass attachment maps. In particular, if $\Gamma, \Gamma', \Gamma''$ *are the dividing sets of contact structures* ξ, ξ', ξ'' *on* M *which differ by attaching bypasses as shown, then these maps preserve the contact invariants* $\theta(\xi), \theta(\xi'), \theta(\xi'')$*.*

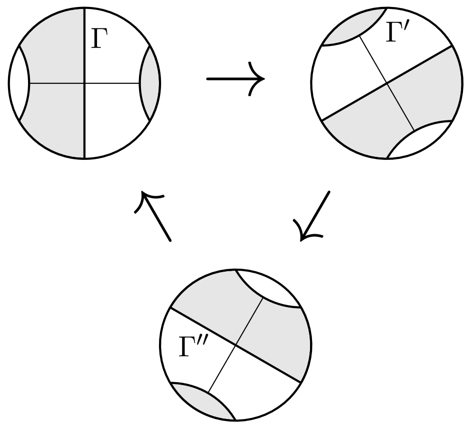


FIGURE 12. A 3-periodic sequence of bypasses.

Proof (sketch). We recall that the 1-handle attachment maps come from identifying appropriately chosen closures before and after attaching the handle, so we choose to attach all three 1-handles first. Then the bypass maps can be identified with the 2-handle attachment maps, which come from cobordisms induced by surgeries on the respective closures. Some handle-sliding eventually shows that we can arrange for these surgeries to be precisely the ones described in Floer's surgery exact triangle [Flo90, BD95]. The bundles on the bypass cobordisms do not quite match the bundles on the cobordisms which appear in the exact triangle, but we can arrange for any two out of three to match and thus verify exactness at their common vertex of the triangle. □

4.4. **Proof of Proposition 4.3.** Let $K \subset Y$ be a fibered knot, with fiber Σ of genus $g \geq 1$, and suppose that $Y \not\cong \#^{2g}(S^1 \times S^2)$. This assumption ensures that the monodromy of the fibration $\pi : Y \smallsetminus K \to S^1$ is not the identity, so up to replacing K with its mirror image we may assume that the monodromy is *non-right-veering*. This notion, due to Honda–Kazez–Matić [HKM07], implies that the contact structure ξ supported by (K, π) is overtwisted. More importantly, if Λ is a Legendrian approximation of K in (Y, ξ), it allows us to show that the complement $Y \smallsetminus N(\Lambda_+)$ of a positive stabilization of Λ is overtwisted [BS18, Lemma 1.14].

We consider a particular case of the bypass exact triangle shown in Figure 13, beginning with the sutured manifold $(Y \smallsetminus N(\Lambda), \Gamma_{tb(\Lambda)})$. Labeling the three bypasses c_+, c_{SV}, and c' in order and their bypass attachment maps F_+, F^1_{SV}, and G, we have a commutative diagram

(4.3)
$$\begin{array}{ccccc}
 & & KHI(-Y,K) & & \\
 & \overset{F^0_{SV}}{\nearrow} & & & \\
SHI(-Y \smallsetminus N(K), \Gamma_{tb(\Lambda)}) & & \xrightarrow{\quad F_+ \quad} & & SHI(-Y \smallsetminus N(K), \Gamma_{tb(\Lambda_+)}) \\
 & \underset{G}{\nwarrow} & & \underset{F^1_{SV}}{\swarrow} & \\
 & & KHI(-Y,K) & &
\end{array}$$

in which the triangle is the bypass exact triangle. The arrow labeled F^0_{SV} is the bypass attaching map for the bypass c_{SV} of Figure 11, which is attached to the convex surface with

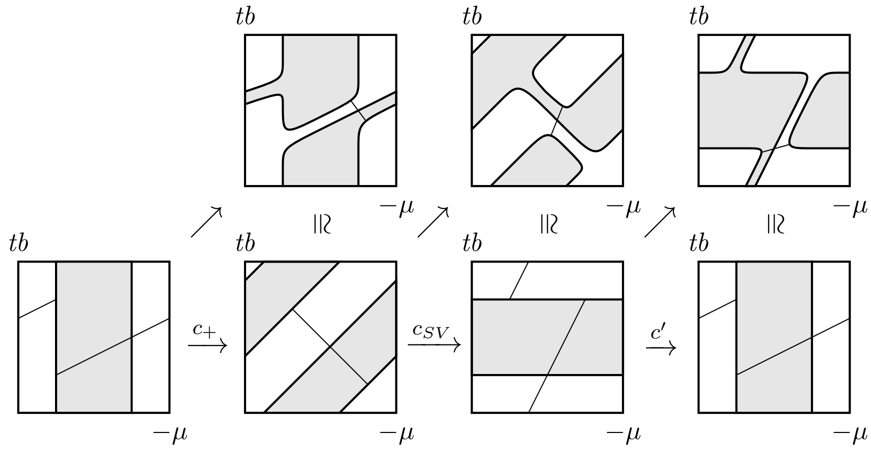


FIGURE 13. A 3-periodic sequence of bypasses on the boundary of $Y \smallsetminus N(K)$.

dividing set $\Gamma_{tb(\Lambda)}$ whereas F^1_{SV} corresponds to a bypass attached to a surface with dividing set $\Gamma_{tb(\Lambda_+)}$.

In (4.3), we now examine the element

$$x = \theta(\xi|_{Y\smallsetminus N(\Lambda)}) \in SHI(-Y \smallsetminus N(K), \Gamma_{tb(\Lambda)}).$$

Since $\xi|_{Y\smallsetminus N(\Lambda_+)}$ is overtwisted, we have

$$F_+(x) = \theta(\xi|_{Y\smallsetminus N(\Lambda_+)}) = 0,$$

and so by exactness we can find an element

$$y \in KHI(-Y, K) \quad \text{such that} \quad G(y) = x.$$

But we have $F^0_{SV}(x) = \mathcal{L}(\Lambda) = \mathcal{T}(K)$ by definition, so by Theorem 4.8 the map

$$\Psi := F^0_{SV} \circ G : KHI(-Y, K) \to KHI(-Y, K)$$

sends y to a generator of $KHI(-Y, K, \Sigma, g) \cong \mathbb{C}$.

Claim 4.11. The map $\Psi : KHI(-Y, K) \to KHI(-Y, K)$ is zero on $KHI(-Y, K, \Sigma, g)$.

Proof. Since $KHI(-Y, K, g)$ is generated by $\mathcal{L}(\Lambda)$, it suffices to show that $\Psi(\mathcal{L}(\Lambda)) = 0$. We use the fact that

$$\mathcal{L}(\Lambda) = \mathcal{L}(\Lambda_-) = F^1_{SV}(\theta(\xi|_{Y\smallsetminus N(\Lambda_-)})),$$

where $\theta(\xi|_{Y\smallsetminus N(\Lambda_-)}) \in SHI(-Y \smallsetminus N(K), \Gamma_{tb(\Lambda_+)})$ since Λ_- and Λ_+ have the same Thurston–Bennequin framing. Then

$$\Psi(\mathcal{L}(\Lambda)) = (F^0_{SV} \circ G \circ F^1_{SV})\big(\theta(\xi|_{Y\smallsetminus N(\Lambda_-)})\big),$$

which is zero since $G \circ F^1_{SV} = 0$ by exactness. □

Claim 4.12. The map $\Psi : KHI(-Y, K) \to KHI(-Y, K)$ changes grading by at most 1, so that if $z \in KHI(-Y, K, \Sigma, i)$ then

$$\Psi(z) \in \bigoplus_{|j-i|\le 1} KHI(-Y, K, \Sigma, j).$$

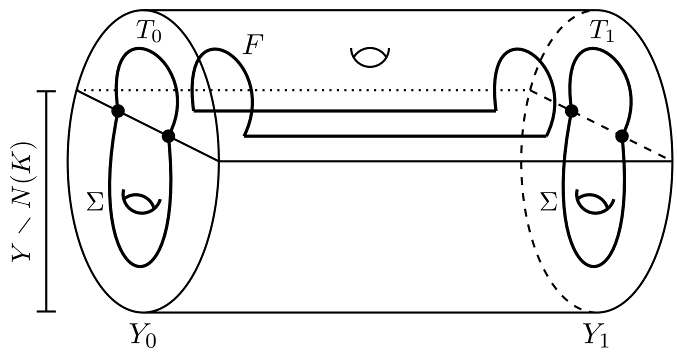


FIGURE 14. A schematic of the cobordism underlying the map Ψ. The bottom half is a product $[0,1] \times (Y \smallsetminus N(K))$, while the top contains a pair of 2-handles.

Proof. Since Ψ is a composition of two bypass attachment maps, we can arrange for it to be realized by a cobordism $W : Y_0 \to Y_1$, consisting of a pair of 4-dimensional 2-handles, between two closures (Y_0, R_0) and (Y_1, R_1); we can also arrange for both closures to be adapted to Σ. The handles are attached to the complement of $Y \smallsetminus N(K)$, so we have the schematic picture shown in Figure 14. Namely, the Alexander grading on the source $KHI(-Y, K)$ is the generalized eigenspace decomposition of $\mu(\hat{\Sigma}_0)$, where $\hat{\Sigma}_0$ is obtained by gluing a punctured torus T_0 to the Seifert surface Σ inside Y_0; and the grading on the target $KHI(-Y, K)$ likewise comes from the operator $\mu(\hat{\Sigma}_1)$, where $\hat{\Sigma}_1 = \Sigma \cup T_1$.

Within the cobordism W, there is a closed surface F of genus 2 and self-intersection 0 such that

$$[\hat{\Sigma}_0] + [F] = [\hat{\Sigma}_1]$$

in $H_2(W)$: it consists of the punctured tori $-T_0$ and T_1, together with an annulus of the form $[0,1] \times \partial\Sigma \subset [0,1] \times (Y \smallsetminus N(K))$ connecting them. One can therefore show (see [BS18, Proposition 2.8]) that if $z \in I_*(Y_0|R_0)_{w_0}$ lies in the generalized $2i$-eigenspace of $\mu(\hat{\Sigma}_0)$, then its image $\Psi(z) \in I_*(Y_1|R_1)_{w_1}$ lies in the direct sum of generalized eigenspaces of $\mu(\hat{\Sigma}_1)$ with eigenvalues

$$2i + (2 - 2g(F)), 2i + (4 - 2g(F)), \ldots, 2i + (2g(F) - 4), 2i + (2g(F) - 2),$$

which in this case is simply $2i - 2, 2i, 2i + 2$. These are the summands $KHI(-Y, K, \Sigma, j)$ where $j = i - 1, i, i + 1$, as claimed. □

We now examine the element y with $\Psi(y) = \mathcal{T}(K)$. It may not be homogeneous, but if we write it as

$$y = y_{-g} + y_{-g+1} + \cdots + y_{g-1} + y_g, \qquad y_j \in KHI(-Y, K, \Sigma, j) \text{ for all } j,$$

then Claim 4.12 says that $\Psi(y) = \mathcal{T}(K) \neq 0$ is in fact the component of $\Psi(y_{g-1}) + \Psi(y_g)$ in Alexander grading g, and Claim 4.11 says that $\Psi(y_g) = 0$, so this is only possible if $y_{g-1} \neq 0$. We conclude that $KHI(-Y, K, \Sigma, g - 1) \neq 0$, completing the proof of Proposition 4.3 and hence of Theorem 4.2.

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Department of Mathematics, Imperial College London

E-mail address: `s.sivek@imperial.ac.uk`