Ribbon concordance and fibered predecessors

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(joint work with John A. Baldwin, Jonathan Hanselman)

We say two knots $K_0, K_1 \subset S^3$ are *concordant* if there is a smooth, properly embedded annulus $A \subset S^3 \times [0,1]$ with boundary components $K_0 \times \{0\}$ and $K_1 \times \{1\}$. If the $[0,1]$-coordinate is a Morse function on A with no local maxima, then we say that K_0 is *ribbon concordant* to K_1, or $K_0 \leq K_1$.

Ribbon concordance was introduced by Gordon [6], who observed that some notions of knot complexity seem to behave monotonically with respect to it. For example, if $K_0 \leq K_1$ then Gilmer proved that the Alexander polynomial of K_0 divides that of K_1; Silver and Kochloukova proved that if K_1 is fibered then so is K_0; and Zemke [9] proved that the knot Floer homology $\widehat{HFK}(K_0)$ injects into $\widehat{HFK}(K_1)$, and so deduced an inequality $g(K_0) \leq g(K_1)$ between their Seifert genera. This culminated in work of Agol [1], who proved Gordon's conjecture that ribbon concordance is in fact a partial order.

Gordon also conjectured that ribbon concordance satisfies a descending chain condition, meaning that every infinite sequence

$$\cdots \leq K_3 \leq K_2 \leq K_1 \leq K_0$$

must eventually stabilize. In light of Agol's theorem, this would be implied by the following stronger conjecture, which was the subject of this talk.

Conjecture 1. *For any fixed knot $K \subset S^3$, there are only finitely many knots $J \subset S^3$ such that J is ribbon concordant to K.*

Conjecture 1 remains open, but we did manage to prove the following.

Theorem 2 ([2, 3])**.** *For any fixed knot $K \subset S^3$, there are only finitely many* fibered *knots ribbon concordant to K.*

Remark 3. Given the work of Silver and Kochloukova mentioned above, Conjecture 1 follows immediately whenever the knot K is fibered.

To prove Theorem 2, we suppose that $J \leq K$ for some fibered knot J, and we use K to bound the complexity (in some appropriate sense) of the *monodromy* of J. We first outline the proof when J is hyperbolic, in which case its monodromy is freely isotopic to a pseudo-Anosov map whose dilatation will be the right notion of complexity.

To start, Zemke [9] provides us with an inequality

$$\operatorname{rank} \widehat{HFK}(J) \leq \operatorname{rank} \widehat{HFK}(K), \tag{1}$$

but this is not enough because there can be infinitely many J with the same knot Floer homology. Perhaps more usefully, we can apply a deep relationship between the knot Floer homology of a fibered knot and its monodromy, due to Ni [8] and Ghiggini–Spano [5]: the summand $\widehat{HFK}(J, g(J)-1)$ is determined by the symplectic Floer homology of the monodromy of J. When J is hyperbolic and its monodromy is a pseudo-Anosov map $\phi : S \to S$, then this relationship is simply

$$\#\mathrm{Fix}(\phi) = \operatorname{rank} \widehat{HFK}(J, g(J) - 1) - 1, \tag{2}$$

and combining this with (1) gives us

$$\#\mathrm{Fix}(\phi) < \operatorname{rank} \widehat{HFK}(K).$$

It turns out that this is not the right bound to use either, though, because this might still leave us with infinitely many knots to consider.

On the other hand, if we can bound the fixed point counts of sufficiently many iterates ϕ^n, then we get a bound on the dilatation $\lambda(\phi)$, via the relation

$$\lambda(\phi) = \lim_{n\to\infty} \left(\#\mathrm{Fix}(\phi^n)\right)^{1/n}.$$

This is a more reasonable goal, because for any constant C and any fixed genus there are only finitely many conjugacy classes of monodromy with dilatation at most C; there are also only finitely many possible genera, because we know that $g(J) \leq g(K)$, so this will show that there are only finitely many corresponding fibered knots J.

If $J \subset S^3$ has monodromy ϕ, then ϕ^n arises naturally as the monodromy of a knot J_n in the n-fold cyclic branched cover $\Sigma_n(J)$, where J_n is the lift of the branch locus J. In this case we restate (2) with ϕ^n in place of ϕ to get

$$\#\mathrm{Fix}(\phi^n) < \dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(J), J_n). \tag{3}$$

We can also take n-fold branched covers of the entire ribbon concordance from J to K, and for suitable n — any large enough prime will do — we get what Daemi, Lidman, Vela–Vick, and Wong [4] call a *ribbon $\mathbb{Z}/2$-homology cobordism* from $J_n \subset \Sigma_n(J)$ to $K_n \subset \Sigma_n(K)$. They prove that these satisfy the analogue of Zemke's rank inequality (1):

$$\dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(J), J_n) \leq \dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(K), K_n). \tag{4}$$

Levine [7] constructed an explicit chain complex for $\widehat{HFK}(\Sigma_n(K), K_n)$ with at most $\big(\delta(K)!\big)^n$ generators, where $\delta(K)$ is the arc index of K, so we combine this with (3) and (4) to get

$$\#\mathrm{Fix}(\phi^n) < \big(\delta(K)!\big)^n$$

for any large enough prime n. In the limit as $n \to \infty$ this means that $\lambda(\phi) \leq \delta(K)!$, and so there are finitely many monodromies up to conjugacy and hence finitely many hyperbolic J as explained above.

Having understood the case where J is hyperbolic, we can try to repeat this proof for more general J. In this case we put the monodromy $h : S \to S$ of J in Nielsen–Thurston form, and the argument only tells us about the outermost (i.e., boundary-adjacent) component S_0 of this decomposition. If the restriction $h|_{S_0}$ is pseudo-Anosov, then we learn that its dilatation is again at most $\delta(K)!$; if it is periodic, then this corresponds either to a cabling operation of bounded genus or a connected sum. Either way, there are only finitely many options up to conjugacy for the monodromy on S_0. But what about the non-outermost components of this Nielsen–Thurston decomposition?

These other components tell us how J can be expressed as a satellite knot, because the reducing set for h is a collection of curves that suspend to satellite tori in the mapping torus $S^3 \setminus J$ of h. If we express J as such a satellite, with companion $C \subset S^3$ and pattern $P \subset S^1 \times D^2$, then the monodromy on S_0 tells us nothing about the companion C. But we get around this by proving a new inequality for knot Floer homology of satellite knots:

Theorem 4. *Let J be a satellite knot with companion C and pattern P of nonzero winding number. Then*

$$\dim_{\mathbb{Z}/2} \widehat{HFK}(C) \leq \dim_{\mathbb{Z}/2} \widehat{HFK}(J).$$

Theorem 4 holds not just for satellite knots in S^3, where it is already new, but for a generalized notion of satellites in other 3-manifolds: in particular

$$\dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(C), C_n) \leq \dim_{\mathbb{Z}/2} \widehat{HFK}(\Sigma_n(J), J_n)$$

for many values of n, including again all large enough primes. So we combine this with (4) as before to deduce that $\lambda(\psi) \leq \delta(K)!$, where ψ is any pseudo-Anosov component of the Nielsen–Thurston decomposition. Now there are only finitely many possible monodromies for each component, just as before, and only finitely many ways to glue these components together to produce the monodromy of our original knot J, establishing Theorem 2.

References

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