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THE CONTACT HOMOLOGY OF LEGENDRIAN KNOTS WITH MAXIMAL THURSTON–BENNEQUIN INVARIANT

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We show that there exists a Legendrian knot with maximal Thurston–Bennequin invariant whose contact homology is trivial. We also provide another Legendrian knot which has the same knot type and classical invariants but nonvanishing contact homology.

1. Introduction

The Chekanov–Eliashberg invariant [**2, 4**], which assigns to each Legendrian knot K a differential graded algebra $(Ch(K), \partial)$ over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, has been a powerful tool for classifying Legendrian knots in the standard contact S^3. The closely related characteristic algebra $\mathcal{C}(K)$ was defined by Ng [**13**] to be the quotient of $Ch(K)$ by the two-sided ideal $\langle \mathrm{Im}(\partial) \rangle$; if two knots K and K' are Legendrian isotopic, then we can add some free generators to $\mathcal{C}(K)$ and $\mathcal{C}(K')$ to make them isomorphic. Both of these invariants only provide information about nondestabilizable knots: if K is a stabilized knot, then both the Legendrian contact homology $H_*(Ch(K))$ and the characteristic algebra $\mathcal{C}(K)$ vanish. For an introduction to Legendrian knots, see [**6**].

Shonkwiler and Vela-Vick [**20**] gave the first examples of Legendrian knots with nonvanishing contact homology which do not have maximal Thurston–Bennequin invariant, representing the knot types $m(10_{161})$ and $m(10_{145})$. Conversely, there are conjecturally nondestabilizable knots of type $m(10_{139})$, 10_{161}, and $m(12n_{242})$ with nonmaximal tb and vanishing contact homology [**3, 20**]. On the other hand, it is an open question whether there is a Legendrian knot K for which $tb(K)$ is maximal but the contact homology of K vanishes. We will answer this question and show that it is not determined solely by the classical invariants tb and r of K:

Theorem. *There are distinct tb-maximizing Legendrian representatives K_1 and K_2 of $m(10_{132})$ with the same classical invariants such that K_1 has trivial contact homology, even with $\mathbb{Z}[t, t^{-1}]$ coefficients, while K_2 does not.*

These Legendrian knots, found in Chongchitmate and Ng's atlas of Legendrian knots [**3**], can be specified as plat diagrams by the following braid words:

$$K_1: \quad 6,7,4,3,7,5,3,6,4,2,5,1,3,2,5,2,4,6,2$$
$$K_2: \quad 4,5,3,5,3,2,4,1,3,2,4,2,5,1,3,2,4,4,3,5,4,2$$

Indeed, both knots have classical invariants $tb = -1$ and $r = 0$, and Ng [**16**] showed that $\overline{tb}(m(10_{132})) = -1$ by bounding $\overline{tb}$ for an appropriate cable of $m(10_{132})$. We will prove this theorem in Section 2.

Finally, the proof that K_2 has nonvanishing contact homology uses an action of $\mathcal{C}(K_2)$ on an infinite-dimensional vector space, just as the nonvanishing examples in [**20**] did. In Section 3, we will show that this is necessary in the sense that $\mathcal{C}(K_2)$ does not have any finite-dimensional representations. It is completely understood when a characteristic algebra $\mathcal{C}$ does not have any one-dimensional representations, but we will ask if such a $\mathcal{C}$ can admit maps $\mathcal{C} \to \mathrm{Mat}_n(\mathbb{F})$ for some finite $n \geq 2$. We will show that this is possible in general by constructing two-dimensional representations for specific Legendrian representatives of negative torus knots.

2. The $m(10_{132})$ examples

2.1. The vanishing example. Let K_1 be the Legendrian representative of $m(10_{132})$ whose front diagram is shown in Figure 1. Its Chekanov–Eliashberg algebra is generated freely over $\mathbb{Z}[t, t^{-1}]$ by crossings $x_1, \dots, x_{19}$ in order from left to right and right cusps $x_{20}, \dots, x_{23}$ in order from top to bottom. The differentials are specified in Appendix A.

To show that K_1 has vanishing contact homology, we need to find a relation $\partial x = 1$ in $Ch(K_1)$. Recall that $Ch(K_1)$ uses a signed Leibniz rule $\partial(vw) = (\partial v)w + (-1)^{|v|}v(\partial w)$, where $|v|$ is the grading of the homogeneous

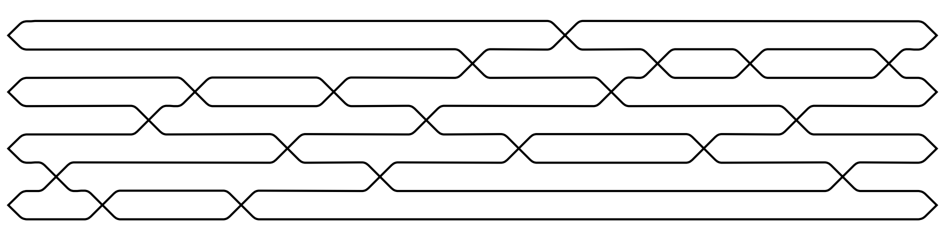


Figure 1. A front diagram of the representative K_1 of $m(10_{132})$, defined as the plat closure of the braid word $6,7,4,3,7,5,3,6,4,2,5,1,3,2,5,2,4,6,2$ in the notation of [**11**]. The numbers label the crossings from left to right, where each k indicates the kth strand crossing over the $(k+1)$st strand as numbered from 1 at the top to 8 at the bottom.

element v, and note that the generators with odd grading are

$$x_2, x_3, x_5, x_9, x_{11}, x_{12}, x_{13}, x_{15}, x_{20}, x_{21}, x_{22}, x_{23}.$$

Let

$$\begin{aligned} a &= x_{12}(x_4(1 + x_2x_5) - x_8) + x_{14}x_5 \\ b &= x_{22} + x_{12} - ax_{18}; \end{aligned}$$

then $\partial a = x_{10}x_4(1 + x_2x_5) - x_{10}x_8 + x_{13}x_5$, and so

$$\begin{aligned} \partial b &= 1 + x_{17}x_7 + (\partial a)x_{18} - ((\partial a)x_{18} + ax_{15}x_7) \\ &= 1 + (x_{17} - ax_{15})x_7. \end{aligned}$$

Then if $c = b(x_6 - x_4x_1) + (x_{17} - ax_{15})(x_9 + x_2)$, we can compute $\partial c = x_6 - x_4x_1$ and so

$$\partial(x_{20} - c(1 + x_{16}x_{19})) = 1.$$

Thus K_1 has trivial contact homology over $\mathbb{Z}[t, t^{-1}]$, as desired.

2.2. The nonvanishing example. Let K_2 be the Legendrian representative of $m(10_{132})$ in Figure 2. The algebra $Ch(K_2)$ is generated freely over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ by crossings $x_1, \dots, x_{22}$ from left to right and right cusps x_{23}, x_{24}, x_{25} from top to bottom with differentials specified in Appendix B. In order to show that K_2 has nontrivial contact homology, it will suffice to show that the characteristic algebra $\mathcal{C}_2 = \mathcal{C}(K_2)$ is nonvanishing [**20**]. We remark that the abelianization of $\mathcal{C}_2$ does vanish, however, so that both K_1 and K_2 have the same abelianized characteristic algebra.

The differential in $\mathcal{C}_2$ immediately gives us $x_1 = x_6 = 0$, and

$$x_{12} = \partial(x_{12}x_{23} + x_{15}x_{22} + x_{17}x_{18})$$

gives $x_{12} = 0$, hence $\partial x_{24} = 0$ becomes $(1 + x_5(x_2 + x_3))x_{20} = 1$. Then we can use $(\partial x_{13})x_{20} = 0$ and $(\partial x_{17})x_{20} = 0$ to get $x_{11} = 0$ and $x_{15} = 0$, so

$$x_1 = x_6 = x_{11} = x_{12} = x_{15} = 0.$$

Furthermore, $\partial x_{21} = 0$ becomes $x_{14} = cx_{20}$, so $\partial x_{25} = 0$ gives us $x_{14} = x_{20}$.

Consider the quotient of $\mathcal{C}_2$ by the two-sided ideal

$$\mathcal{I} = \langle x_3, x_7, x_8, x_9, x_{10}, x_{13} + 1 + x_2x_5, x_{17}, x_{19}, x_{21}, \dots, x_{25}\rangle.$$

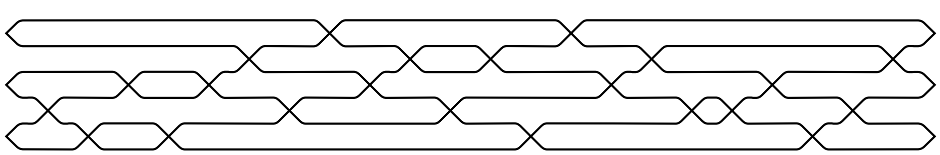


Figure 2. A front diagram of the representative K_2 of $m(10_{132})$, defined as the plat closure of the braid word 4, 5, 3, 5, 3, 2, 4, 1, 3, 2, 4, 2, 5, 1, 3, 2, 4, 4, 3, 5, 4, 2.

The quotient $\mathcal{C}_2/\mathcal{I}$ is generated by $x_2, x_4, x_5, x_{14}, x_{16}, x_{18}$, and its nontrivial relations are $c = x_2 + x_{14}(1 + x_2x_5) + x_{16}(1 + x_5x_2) = 1$ and

$$\begin{aligned} x_4 &= x_5(1 + x_2x_4), \\ x_{18} &= 1 + x_2x_4, \\ 0 &= (1 + x_5x_2)x_{18}, \\ 1 &= (1 + x_2x_5)x_{18}, \\ 1 &= (1 + x_5x_2)x_{14}. \end{aligned}$$

Note that the pair of relations $x_4 = x_5(1 + x_2x_4)$ and $x_{18} = 1 + x_2x_4$ are equivalent to $x_4 = x_5x_{18}$ and $(1 + x_2x_5)x_{18} = 1$, the latter of which is already known, so we can replace the pair with $x_4 = x_5x_{18}$. Furthermore, multiplying the $c = 1$ equation on the right by x_{18} gives $x_{14} = (1 + x_2)x_{18}$, hence the last relation becomes $(1 + x_5x_2)x_2x_{18} = 1$. Then, the $c = 1$ equation becomes

$$x_{16}(1 + x_5x_2) = (1 + x_2)(1 + x_{18}(1 + x_2x_5)),$$

so we multiply on the right by x_2x_{18} and get

$$x_{16} = (1 + x_2)(x_2x_{18} + x_{18}x_2(1 + x_5x_2)x_{18}) = (1 + x_2)x_2x_{18}.$$

Thus, we see that x_4, x_{14}, and x_{16} can be expressed in terms of x_2, x_5, and x_{18}, and $c = 1$ can be rewritten as

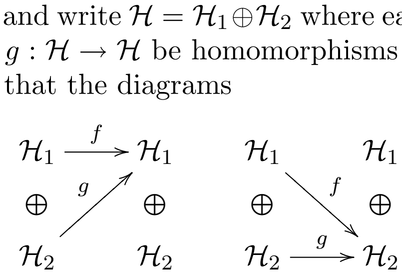
$$0 = (1 + x_2)\left(1 + x_{18}(1 + x_2x_5) + x_2x_{18}(1 + x_5x_2)\right).$$

Relabeling x_2, x_5, x_{18} as a, b, c, respectively, we have a homomorphism from $\mathcal{C}_2/\mathcal{I}$ to the quotient R of the free algebra $\mathbb{F}\langle a, b, c\rangle$ by the two-sided ideal generated by the relations

$$\begin{aligned} 0 &= 1 + c(1 + ab) + ac(1 + ba), \\ 0 &= (1 + ba)c, \\ 1 &= (1 + ab)c, \\ 1 &= (1 + ba)ac. \end{aligned}$$

Proposition 2.1. *The algebra R is nontrivial.*

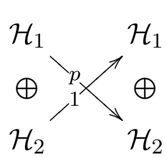
Proof. We will construct an infinite-dimensional representation of R, following ideas from [**20**]. Let $\mathcal{H}$ be a countable-dimensional $\mathbb{F}$-vector space, with basis $\{v_0, v_1, v_2, \dots\}$, and write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where each $\mathcal{H}_i$ summand is isomorphic to $\mathcal{H}$. Let $f, g : \mathcal{H} \to \mathcal{H}$ be homomorphisms defined by $f(v_i) = v_{2i}$ and $g(v_i) = v_{2i+1}$, so that the diagrams



represent isomorphisms $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_1$ and $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_2$, respectively. We also define homomorphisms $p, s : \mathcal{H} \to \mathcal{H}$ by $p(v_i) = v_{i-1}$ for $i \geq 1$, $p(v_0) = 0$ and $s(v_i) = v_{i+1} + v_{2(i+1)}$. It is straightforward to check the identities

$$s \circ p = f + 1, \quad p \circ g = f, \quad p \circ s = g + 1.$$

We define a right action of a and b on $\mathcal{H} \cong \mathcal{H}_1 \oplus \mathcal{H}_2$ by the diagrams



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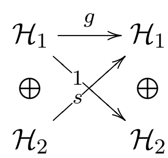
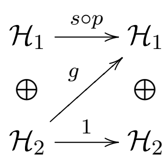


respectively. Then we can compute the action of ab and ba by concatenating the a and b diagrams to get



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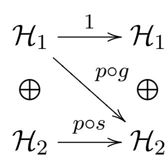
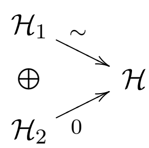
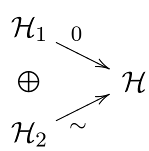


respectively, hence by the above identities $1 + ab$ and $1 + ba$ are exactly the specified isomorphisms $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_1$ and $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_2$. Finally, let c act on $\mathcal{H}$ as the map



where the indicated isomorphism is the inverse of $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_1$. Then the composition ac is the homomorphism



where the isomorphism is inverse to $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_2$. It is now easy to check that $(1 + ab)c = 1$, $(1 + ba)c = 0$, and $(1 + ba)ac = 1$. Finally, we note that $c(1 + ab)$ is the projection of $\mathcal{H}$ onto $\mathcal{H}_1 \subset \mathcal{H}$ and likewise $ac(1 + ba)$ is the projection onto $\mathcal{H}_2$, hence

$$1 = c(1 + ab) + ac(1 + ba).$$

Therefore, the action that we have constructed satisfies all of the defining relations of R. $\square$

Since R is nonvanishing and we have a homomorphism $\mathcal{C}_2 \to \mathcal{C}_2/\mathcal{I} \to R$, we conclude that $\mathcal{C}_2$ (and hence the contact homology of K_2) is nonvanishing as well.

3. Finite-dimensional representations of $\mathcal{C}(K)$

Although the Legendrian knot K_2 of Section 2.2 is now known to have nontrivial contact homology and characteristic algebra, one can ask for a simpler proof of this fact; in particular, one can ask if $\mathcal{C}_2$ has any finite-dimensional representations. The answer in this case is no.

Lemma 3.1. *Suppose that an $\mathbb{F}$-algebra $\mathcal{A}$ has a relation of the form $ab = 1$. If the quotient of $\mathcal{A}$ by the two-sided ideal $\langle ba - 1\rangle$ is trivial, i.e. if $0 = 1$ in $\mathcal{A}/\langle ba - 1\rangle$, then there is no representation $\mathcal{A} \to \mathrm{Mat}_n(\mathbb{F})$ for any n.*

Proof. Suppose there is a homomorphism $\varphi : \mathcal{A} \to \mathrm{Mat}_n(\mathbb{F})$, so in particular $\varphi(1) = 1$. The equation $\varphi(ab-1) = 0$ implies that $\varphi(a)$ and $\varphi(b)$ are inverse matrices, so they commute and $\varphi(ba-1) = 0$ as well. Then, φ factors through the quotient $\mathcal{A}/\langle ba - 1\rangle$ in which $0 = 1$, hence $\varphi(1) = \varphi(0) = 0$, which is a contradiction. ☐

Now in $\mathcal{C}_2$, we showed in Section 2.2 that $x_{11} = x_{12} = 0$ and $(1 + x_5(x_2 + x_3))x_{20} = 1$. If we impose the relation $x_{20}(1 + x_5(x_2 + x_3)) = 1$, then $x_{18} = x_{20}(\partial x_{22}) = 0$ as well and so $0 = \partial x_{23} = 1$, hence $\mathcal{C}_2$ has no finite-dimensional representations by Lemma 3.1. This argument also proves the claim made in Section 2.2 that the abelianization of $\mathcal{C}_2$ is trivial.

Lemma 3.1 can also be used to prove that the characteristic algebra of the Legendrian $m(10_{161})$ studied in [**20**] has no finite-dimensional representations, by adding $x_{28}x_{13} = 1$ to the relations $\partial x_i = 0$ in [**20,** Appendix A] and showing that $0 = 1$ as a consequence, and similarly for the $m(10_{145})$ representative mentioned in the same article. Neither one of these knots has maximal Thurston–Bennequin invariant.

On the other hand, it is interesting to ask when the characteristic algebra $\mathcal{C}$ of a Legendrian knot K has n-dimensional representations. For $n = 1$ the answer depends only on tb and the topological knot type:

Proposition 3.2. *There is a homomorphism $\mathcal{C} \to \mathrm{Mat}_1(\mathbb{F}) \cong \mathbb{F}$ if and only if the Kauffman bound*

$$tb(K) \leq \text{min-deg}_a F_K(a, x) - 1$$

*(see [**10**]) is sharp.*

Proof. The Kauffman bound for K is achieved if and only if a front diagram for K admits an ungraded normal ruling [**18**], which happens if and only if $Ch(K)$ admits an ungraded augmentation [**8, 9, 19**]. An augmentation is an algebra homomorphism $Ch(K) \xrightarrow{\epsilon} \mathbb{F}$ which satisfies $\epsilon \circ \partial = 0$, and these correspond bijectively to algebra homomorphisms $\mathcal{C} \to \mathbb{F}$, so the latter exists if and only if the Kauffman bound is sharp. □

In particular, the Kauffman bound is known to be sharp for all knots with at most 9 crossings except for $m(8_{19})$ and $m(9_{42})$ (see [**15**]); for all 10-crossing knots except $m(10_{124})$, $m(10_{128})$, $m(10_{132})$, and $m(10_{136})$ [**1, 14**]; and for all alternating knots [**18**]. Thus, the characteristic algebra of a Legendrian representative of any alternating knot or knot with at most ten crossings other than the six exceptions above has a one-dimensional representation if and only if it is *tb*-maximizing.

We will now demonstrate the existence of infinitely many Legendrian knots whose characteristic algebras have n-dimensional representations for $n = 2$ but not for $n = 1$. For convenience, we will use the following presentation of $\mathrm{Mat}_2(\mathbb{F})$.

Lemma 3.3. *The ring* $\mathrm{Mat}_2(\mathbb{F})$ *can be presented as*

$$\frac{\mathbb{F}\langle a, b\rangle}{\langle a^2 = b^2 = 0, ab + ba = 1\rangle}.$$

Proof. Let R be the $\mathbb{F}$-algebra with the given presentation, and consider a map $\varphi : R \to \mathrm{Mat}_2(\mathbb{F})$ of the form

$$a \mapsto A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$b \mapsto B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that $A^2 = B^2 = 0$ and $AB + BA = I$, so φ is a valid homomorphism, and since A, B, AB, BA form an additive basis of $\mathrm{Mat}_2(\mathbb{F})$ it is surjective. To check that φ is also injective, we note that any nonzero monomial in R is equal to one of $1, a, b, ab$, or $ba = 1 + ab$, and so $1, a, b, ab$ span R as an $\mathbb{F}$-vector space; since the image of φ has order $|\mathrm{Mat}_2(\mathbb{F})| = 16 \geq |R|$ it follows that φ is injective. □

Let $T_{p,-q}$ be the Legendrian representative of the $(p, -q)$-torus knot as in Figure 3, where $q > p \geq 3$; there are p numbered left cusps at the leftmost edge of the diagram, $q - p$ left cusps in the innermost region of the diagram, and q right cusps. The algebra $Ch(T_{p,-q})$ can be computed following [**13**]: the front projection is simple, so $Ch(T_{p,-q})$ is generated by crossings and right cusps and the differential counts admissible embedded disks in the diagram.

We label the generators of $Ch(T_{p,-q})$ as follows. On the left half of the diagram, x_{ij} is the intersection of the strands through the numbered left cusps i and j for $1 \leq i < j \leq p$. On the right half, y_{ij} denotes the intersection of strands through the numbered right cusps i and j for $1 \leq i < j \leq \min(q, i + p - 1)$, and z_i is the ith right cusp.

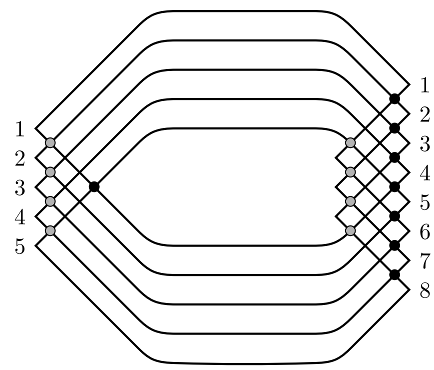


Figure 3. A Legendrian representative $T_{5,-8}$ of the $(5,-8)$-torus knot.

We define an algebra homomorphism $f : Ch(T_{p,-q}) \to \mathrm{Mat}_2(\mathbb{F})$ by sending all generators to 0 except

$$x_{i,i+1}, y_{j,j+p-1} \mapsto a,$$
$$x_{1,p}, y_{j,j+1} \mapsto b.$$

In Figure 3, f is equal to a on the crossings marked with gray dots, b on the crossings marked with black dots, and 0 on all other crossings and right cusps. If we can show that $f(\partial v) = 0$ for all generators v, then f is a morphism of DGAs (where $\mathrm{Mat}_2(\mathbb{F})$ has trivial differential) and it induces a representation $\mathcal{C}(T_{p,-q}) \to \mathrm{Mat}_2(\mathbb{F})$.

Proposition 3.4. *The homomorphism* $f : Ch(T_{p,-q}) \to \mathrm{Mat}_2(\mathbb{F})$ *satisfies* $f(\partial v) = 0$ *for all* v.

Proof. Call an admissible disk *nontrivial* if none of its corners are in $\ker(f)$. Then it is easy to see that any nontrivial disk has exactly two corners, and if both corners have the same color (in the sense of Figure 3, i.e., if they are sent to the same element of $\mathrm{Mat}_2(\mathbb{F})$) then the contribution of this disk to $f(\partial v)$ is either $a^2 = 0$ or $b^2 = 0$. Thus we can determine $f(\partial v)$ by only counting disks with initial vertex at v and having exactly one gray corner and one black corner.

If v is the right cusp z_i, then there are two nontrivial disks contributing ab and ba to the differential, so $f(\partial z_i) = 1 + ab + ba = 0$. For all crossings v, however, the only possible black corner for a nontrivial disk is $x_{1,p}$. Such a disk must include either the first or the pth numbered left cusp on its boundary depending on whether the interior of the disk is immediately above or below $x_{1,p}$, but then the boundary of the disk must pass through either z_1 or z_q, which in particular is to the right of v, and so it cannot contribute to $f(\partial v)$. We conclude that $f(\partial v) = 0$ for all generators v of $Ch(T_{p,-q})$, as desired. □

We can compute $tb(T_{p,-q}) = -pq$ for all p and q, hence $T_{p,-q}$ is tb-maximizing by the classification of Legendrian torus knots [**7**], but for odd p the Kauffman bound is $tb(K) \leq -pq + q - p$ [**5**]. Using Proposition 3.2, we conclude:

Corollary 3.5. *Let $p \geq 3$ be odd and $q > p$. Then the characteristic algebra $\mathcal{C}(T_{p,-q})$ admits an n-dimensional representation for $n = 2$ but not for $n = 1$.*

Remark 3.6. The knots $T_{3,-4}$ and $T_{3,-5}$ are the unique tb-maximizing representatives of $m(8_{19})$ and $m(10_{124})$ up to change of orientation [**7**], so if any tb-maximizing Legendrian representative of a knot with at most 10 crossings has vanishing contact homology or characteristic algebra (such as the $m(10_{132})$ of Section 2.1) then it must represent one of $m(9_{42})$, $m(10_{128})$, $m(10_{132})$, or $m(10_{136})$. The characteristic algebras of the known tb-maximizing Legendrian $m(9_{42})$, $m(10_{128})$, and $m(10_{136})$ knots, which have plat diagrams with braid words

$$\begin{aligned} m(9_{42}):&\quad 2,1,1,4,5,3,5,3,2,4,3,3,2,4 \\ m(10_{128}):&\quad 6,5,5,4,3,3,2,1,5,4,3,2,2,4,1,3,5,7,1,2,3,4,5,6 \\ m(10_{136}):&\quad 6,5,4,3,7,5,3,3,2,1,4,3,2,4,5,2,3,1,1,2,3,4,5,6 \end{aligned}$$

respectively, can also be shown to have two-dimensional representations, so they do not vanish. The $m(9_{42})$ knot is given in the table of [**12**], and the others both appear in [**3**].

It is not known whether there are Legendrian knots whose characteristic algebras have representations of minimal dimension $n \geq 3$, or whether this minimal dimension can be used to distingush any Legendrian knots with nontrivial characteristic algebras and the same classical invariants. We leave open the question of which Legendrian knots K admit representations $\mathcal{C}(K) \to \mathrm{Mat}_n(\mathbb{F})$ for fixed $n \geq 2$ or even for any finite n.[1]

4. Appendix A: The differential of the vanishing $m(10_{132})$

Let K_1 be the representative of $m(10_{132})$ with braid word

$$6,7,4,3,7,5,3,6,4,2,5,1,3,2,5,2,4,6,2.$$

Then $Ch(K_1)$ has generators $x_1, \dots, x_{23}$ over $\mathbb{Z}[t,t^{-1}]$ with the following nonzero differentials [**11**]:

$\partial x_2 = -x_1,$

$\partial x_4 = x_3,$

$\partial x_6 = x_3 x_1,$

[1] Added in press: Ng and Rutherford [**17**] have shown that the question of whether $\mathcal{C}(K)$ as defined over $\mathbb{F}[t,t^{-1}]$ admits an n-dimensional representation depends only on $tb(K)$ and the smooth knot type of K, by proving that this occurs if and only if a particular Legendrian satellite of K admits a normal ruling.

$$\begin{aligned}
\partial x_8 &= x_3 + x_3x_2x_5 - x_6x_5,\\
\partial x_9 &= x_1 + x_7x_4x_1 - x_7x_6,\\
\partial x_{11} &= 1 + x_2x_5 + x_7x_4 + x_7x_4x_2x_5 - x_7x_8 + x_9x_5,\\
\partial x_{12} &= x_{10},\\
\partial x_{13} &= x_{10}x_4x_1 - x_{10}x_6,\\
\partial x_{14} &= -x_{12}x_4x_1 + x_{12}x_6 + x_{13},\\
\partial x_{17} &= x_{10}x_4x_{15} + x_{10}x_4x_2x_5x_{15} - x_{10}x_8x_{15} + x_{13}x_5x_{15},\\
\partial x_{18} &= -x_{15}x_7,\\
\partial x_{20} &= 1 - x_4x_1 + x_6 - x_4x_1x_{16}x_{19} + x_6x_{16}x_{19},\\
\partial x_{21} &= 1 - x_{12}x_4x_{15} - x_{12}x_4x_2x_5x_{15} + x_{12}x_8x_{15} - x_{14}x_5x_{15} + x_{17}\\
&\quad - x_{19}x_5x_{15} - x_{19}x_{16}x_{12}x_4x_{15} - x_{19}x_{16}x_{12}x_4x_2x_5x_{15}\\
&\quad + x_{19}x_{16}x_{12}x_8x_{15} - x_{19}x_{16}x_{14}x_5x_{15} + x_{19}x_{16}x_{17},\\
\partial x_{22} &= 1 - x_{10} + x_{17}x_7 + x_{10}x_4x_{18} + x_{10}x_4x_2x_5x_{18} - x_{10}x_8x_{18} + x_{13}x_5x_{18},\\
\partial x_{23} &= t^{-1} + x_{15}x_2 + x_{15}x_7x_4x_2 + x_{15}x_9 - x_{18}x_3x_2 + x_{18}x_6.
\end{aligned}$$

5. Appendix B: The differential of the nonvanishing $m(10_{132})$

Let K_2 be the representative of $m(10_{132})$ with braid word

$$4, 5, 3, 5, 3, 2, 4, 1, 3, 2, 4, 2, 5, 1, 3, 2, 4, 4, 3, 5, 4, 2.$$

Then $Ch(K_2)$ has generators $x_1, \ldots, x_{25}$ over $\mathbb{Z}/2\mathbb{Z}$ with the following nonzero differentials [**11**]:

$$\begin{aligned}
\partial x_2 = \partial x_3 &= x_1,\\
\partial x_7 &= x_4 + x_5(1 + (x_2 + x_3)x_4),\\
\partial x_8 &= x_6,\\
\partial x_9 &= x_6(1 + (x_2 + x_3)x_4),\\
\partial x_{10} &= x_9 + x_8(1 + (x_2 + x_3)x_4),\\
\partial x_{13} &= x_6(x_2 + x_3) + x_{11}(1 + x_5(x_2 + x_3)),\\
\partial x_{14} &= (1 + (x_2 + x_3)x_4)x_{12},\\
\partial x_{15} &= x_{12}x_{11},\\
\partial x_{16} &= x_{14}x_{11} + (1 + (x_2 + x_3)x_4)x_{15},\\
\partial x_{17} &= x_{12}(x_{13} + x_8(x_2 + x_3)) + x_{15}(1 + x_5(x_2 + x_3)),\\
\partial x_{19} &= (1 + (x_2 + x_3)x_4) + cx_{18},\\
\partial x_{20} &= x_{18}x_{12},\\
\partial x_{21} &= x_{14} + x_{19}x_{12} + cx_{20},
\end{aligned}$$

$$\begin{aligned}
\partial x_{22} &= (1 + x_5(x_2 + x_3))x_{18}, \\
\partial x_{23} &= 1 + x_{11}x_{22} + (x_{13} + x_8(x_2 + x_3))x_{18}, \\
\partial x_{24} &= 1 + x_{22}x_{12} + (1 + x_5(x_2 + x_3))x_{20}, \\
\partial x_{25} &= 1 + c,
\end{aligned}$$

where

$$c = x_2+x_3+(1+(x_2+x_3)x_4)x_{17}+x_{14}(x_{13}+x_8(x_2+x_3))+x_{16}(1+x_5(x_2+x_3)).$$

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