Math 273 Lecture 8

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Our goal in this lecture is to classify tight contact structures on $S^2 \times [0,1]$, with similar theorems for S^3, $S^1 \times S^2$, B^3, and $\mathbb{R}^3$ following immediately. Given a tight contact structure ξ on $\Sigma \times [0,1]$, we can perturb it to assume that any given level set $\Sigma_t = \Sigma \times \{t\}$ has Morse-Smale characteristic foliation, but this is not true in general for 1-parameter families of surfaces such as $\Sigma \times [0,1]$ itself. Giroux [1] used the term "tomography" to describe the study of families of characteristic foliations.

Definition 1. A foliation $\mathcal{F}$ on a surface Σ satisfies the *Poincaré–Bendixson property* if the α– and ω–limit sets of each orbit are either a singular point, a closed orbit, or a poly-cycle (a union of singular points and flow lines connecting them).

Theorem 2 (Poincaré–Bendixson)**.** *Every foliation with isolated singular points of a sphere or of a planar region has this property.*

Remark 3. A surface $\Sigma \subset (M, \xi)$ whose characteristic foliation has the Poincaré–Bendixson property is convex if and only if all closed orbits of Σ_ξ are nondegenerate and no flow line goes from a negative hyperbolic point to a positive one. This follows from the same construction we used to show that Morse–Smale implies convex.

Proposition 4. *Let ξ be a contact structure on $\Sigma \times [-1,1]$ with $\Sigma \times \{\pm 1\}$ convex. There is an isotopy rel boundary of $\Sigma \times [-1,1]$ so that every non-convex surface Σ_t satisfies the Poincaré-Bendixson property.*

Proof. Let us first suppose that Σ_t is convex for all $\frac{1}{2} \leq |t| \leq 1$, with characteristic foliation $(\Sigma_t)_\xi$ independent of t for $\frac{1}{2} \leq |t| \leq \frac{3}{4}$, and that Σ is divided into regions $\Sigma_+ \cup \Sigma_-$ by a curve Γ (not necessarily a dividing set!) where each component of $\Sigma_\pm$ is planar and $\Sigma_\pm \times \{\frac{1}{2}\}$ is transverse to the characteristic foliation $(\Sigma_{\pm 1/2})_\xi$. Let $G_\pm$ be retractions of $\Sigma_\pm$ whose boundaries are isotopic to $\Gamma_{1/2}$ through curves transverse to $(\Sigma_{1/2})_\xi$.

Let $h : [0,1] \to [\frac{1}{2}, 1]$ be an odd, strictly increasing function with $h(t) = t$ for $t \geq \frac{3}{4}$. We define an isotopy ϕ_s supported on $\Sigma \times [-\frac{3}{4}, \frac{3}{4}]$ which moves points vertically, such that ϕ_1 sends $G_+ \times \{t\}$ to $G_+ \times \{h(t)\}$ for $t \geq 0$ and sends $G_- \times \{t\}$ to $G_- \times \{-h(-t)\}$ for $t \leq 0$. (We can also insist that $\phi_s(G_\pm \times \{t\})$ is always some parallel $G_\pm \times \{t'\}$.) If we let $\xi' = \phi_1^*(\xi_0)$, then every characteristic foliation $(\Sigma_t)_{\xi'}$ for $-\frac{3}{4} \le t \le \frac{3}{4}$ has the Poincaré–Bendixson property.

To see this, take $0 \le t \le \frac{3}{4}$ without loss of generality. Then $\xi'|_{G_+ \times \{t\}} = \xi|_{G_+ \times \{h(t)\}}$, and $\frac{1}{2} \le h(t) \le \frac{3}{4}$, so by assumption $\partial G_+ \times \{t\}$ divides $\Sigma \times \{t\}$ into planar regions along which the characteristic foliations are independent of t. At $t \ge \frac{3}{4}$ the surface Σ_t is convex and fixed by ϕ_s, so the surfaces Σ_t must all have the Poincaré–Bendixson property for $t \ge 0$. We proceed analogously when $t \le 0$.

Now in the general case we perform an isotopy of ξ in order to construct the curve Γ. Since $\Sigma_{\pm 1}$ are convex, they have dividing curves $\Gamma_{\pm 1}$. We can take a multi-curve $K \subset \Sigma$ whose complement is a union of planar regions, and by an isotopy we can also insist that every component of K intersects and is transverse to both Γ_1 and Γ_{-1}. Now we can apply the Legendrian realization principle to $K_\pm = K \times \{\pm\frac{1}{2}\} \subset \Sigma \times \{\frac{1}{2}\}$ and then take Σ_+ to be a tubular neighborhood of K and Σ_- its complement. □

In particular, if $(\Sigma \times I, \xi)$ has convex boundary then the non-convex surfaces $\Sigma_t = \Sigma \times \{t\}$ have either degenerate closed orbits or "retrograde connections" from negative to positive hyperbolic points. Generically we can assume that these retrograde connections happen at finitely many times $t_1, \dots, t_n$, and that at those times, all critical points are nondegenerate and the retrograde connection is the only orbit connecting two hyperbolic points.

Proposition 5. *The set of times t where Σ_t has a retrograde orbit has no accumulation points.*

Proof. Suppose there was an accumulation point at $t = 0$ and let $t_i \to 0$ be a times whose retrograde orbits limit to the one at time $t = 0$. Choose coordinates locally so that we are working on $\mathbb{R}^2 \times [-1, 1]$ with area form $\omega = dx \wedge dy$ at each slice and so that the positive and negative hyperbolic points at each time including $t = 0$ are at $(\pm 1, 0, t_i) \in \mathbb{R}^2 \times [-1, 1]$, with the retrograde orbit equal to the line segment of the x-axis between them at height t_i. Then the characteristic foliation along each orbit is directed by a positive multiple v of ∂_x, so if the contact form is

$$\alpha = \eta_t + f_t dt$$

then $\eta_{t_i} = \iota_v \omega = g_{t_i}(x) dy$ for some positive function g_{t_i} on each orbit.

Along the limiting separatrix γ at $t = 0$ it follows that $\frac{d\eta_t}{dt}|_{t=0}$ is a multiple of dy, hence $(\eta_t \wedge \frac{d\eta_t}{dt})|_{t=0} = 0$ along γ. On the other hand, f_t has sign ± 1 at the endpoints $(\pm 1, 0, 0)$, so there must be a point p on the interior of γ where $f_0 = 0$ and f_0 is increasing along γ, hence $df_0(\partial_x) > 0$ at p. But $\alpha \wedge d\alpha > 0$ is equivalent to

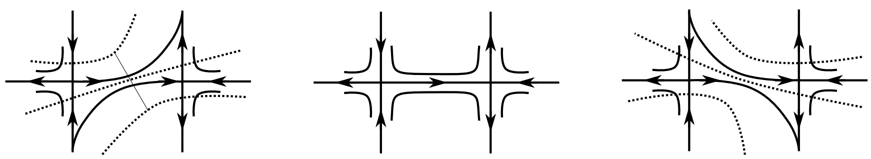
$$f_t d\eta_t + \eta_t \wedge (df_t - \dot{\eta}_t) > 0$$

and at p we have $f_0 = \eta_0 \wedge \dot{\eta}_0 = 0$, so it follows that $g_0(p) df_0(\partial x) \cdot dy \wedge dx = (\eta_0 \wedge df_0)_p > 0$, which is a contradiction. □

In fact, we can learn more from this argument. Giroux used a careful analysis to show the following:

Lemma 6 (Crossing lemma). *Suppose that $(\Sigma_{t_0})_\xi$ contains a retrograde saddle-saddle connection, i.e. a flow line from a negative hyperbolic point p_- to a positive hyperbolic point p_+. There is a neighborhood $(t_0 - \epsilon, t_0 + \epsilon)$ on which this retrograde orbit corresponds to a pair of flow lines which cross at $t = t_0$, with the stable separatrix c_+ of p_+ passing above the negative separatrix c_- of p_- for $t > t_0$ and vice versa for $t < t_0$.*

Here is a picture of the characteristic foliation on Σ_t near a retrograde orbit, at times $t_0 - \epsilon$, t_0, and $t_0 + \epsilon$.

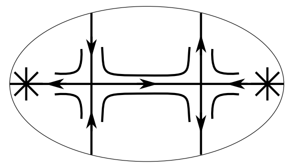


The dotted lines in the "before" and "after" pictures are a local picture of a dividing set at each time. The dividing sets change when crossing $t = t_0$ by the same picture as a bypass attachment along the thin diagonal arc in the "before" picture; if we perturb it slightly to make that arc Legendrian, then it is not hard to see that this neighborhood of the retrograde orbit really does contain a bypass. In particular, retrograde saddle–saddle connections are equivalent to bypass attachments.

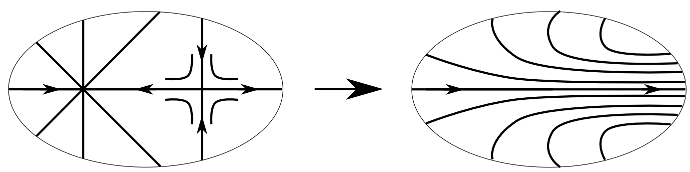
Suppose that $\Sigma = S^2$, and that $(\Sigma \times I, \xi)$ is tight with convex boundary. Then we can perturb ξ rel boundary so that there is a finite set of times t_i when Σ has a retrograde saddle–saddle connection, and since Γ_{Σ_t} must be connected there cannot be any degenerate closed orbits: Giroux showed that these cause the death or birth of a pair of nondegenerate closed orbits, which would change $\#\Gamma_{\Sigma_t}$. Every other surface $\Sigma_t = \Sigma \times \{t\}$ must be convex, since $(\Sigma_t)_\xi$ satisfies the Poincaré–Bendixson property. This means that we can construct ξ by taking a contact structure in which every Σ_t is convex and attaching a series of bypasses; last time we observed that these must all be trivial bypasses. Thus it remains to be seen that if we take a convex S^2 and attach a bypass B, then a neighborhood of $S^2 \cup B$ is diffeomorphic to $S^2 \times I$ where each $S^2 \times \{t\}$ is convex. This justifies the claim that trivial bypasses really are trivial, at least on S^2.

Proposition 7. *Let D be a neighborhood of a retrograde orbit γ in an S^2 in a tight contact structure ξ for which all singular points are isolated and no other arcs of $(S^2)_\xi$ connect pairs of hyperbolic points. Then $D \times I$ can be isotoped rel boundary so that each $D \times \{t\}$ is convex.*

Proof. Let $p_\pm$ be the hyperbolic points of each sign at either end of γ. If γ' is the other half of the stable separatrix of p_+, then by the Poincaré–Bendixson property, γ' limits to either a singular point, a closed orbit, or a poly-cycle. The closed orbit cannot exist because ξ is tight, and no vertex of a limit poly-cycle can be elliptic, but a poly-cycle in S^2 cannot have two connected hyperbolic vertices by assumption so γ' must in fact limit to a singular point, and this point is then positive elliptic; call it e_+. Similarly, the other half of the unstable separatrix of p_- limits to a negative elliptic point e_-. We can include these points in the disk D and arrange D so that D_ξ has no other singularities:



We will focus on a neighborhood D' of the flow line connecting e_- to h_-. Since there are only finitely many unstable separatrices of negative hyperbolic points, we can make D' small enough so that it misses all of them except the ones emanating from h_-, and the one which leaves D' never returns. If this neighborhood were embedded in the negative region of a convex surface $\Sigma \subset \Sigma \times \mathbb{R}$, one could find a perturbation of D' rel boundary so that the characteristic foliation changes inside D' as follows:



But on the other hand we don't actually need Σ to be convex, because the characteristic foliation determines the contact structure in a neighborhood of ξ, so we can always change the foliation in this way. This is called the Elimination Lemma, because it allows us to eliminate pairs of elliptic and hyperbolic singularities of the same sign from a characteristic foliation, and we have already implicitly proved it by Giroux flexibility: in a neighborhood $D' \times I$ where our disk is at $D' \times \{0\}$, we can find some isotopy ϕ_s sending $(x, 0)$ to $(x, f(x))$ where graph(f) has the desired foliation, and we can fix $f = 0$ outside D'.

For our situation we need a stronger version of the Elimination Lemma, however, since we want to isotope $D' \times I$ rel boundary so that no surface $\phi_1(D' \times \{t\})$ has a retrograde orbit. In order to do so, let $\psi : (-\epsilon, \epsilon) \to [0, 1]$ be an even, compactly supported bump function with $\psi(0) = 1$, and define the isotopy

$$\phi_s(x, t) = (x, t + \psi(t)f(x)).$$

(We need to check that this is well-defined, but $\frac{d}{dt}(t+\psi(t)f(x)) = 1+\psi'(t)f(x) > 0$ as long as we rescale f to make it sufficiently small.) We will also arrange f and $(S^2)_\xi$ so that the horizontal line through e_- and p_- is a union of flow lines of $\phi_s(D \times \{0\})$ at all times, and that the orbit leaving D' along e_- does not connect to another hyperbolic point.

This isotopy certainly fixes the boundary of $D' \times I$ and eliminates e_- and h_- in $D' \times \{0\}$, so that $\phi_1(D \times \{0\})$ no longer has a retrograde orbit. We may have introduced new retrograde orbits in other disks $\phi_1(D \times \{t\})$ by accident, however, so we need to check that this does not happen. But at all times around $t = 0$ where $\phi_1(D' \times \{t\})$ has no more negative hyperbolic points, this cannot happen because only the horizontal flow line can connect to a hyperbolic point, namely h_+, and as we follow it left out of D' it not limit to another hyperbolic point. Otherwise, when $\phi_1(D' \times \{t\})$ still has a hyperbolic point coming from h_-, we know that the unstable separatrix of h_- will connect to an elliptic point because it was already redirected away from h_+ in the first place.

Since our isotopy ϕ_1 eliminates the retrograde orbit from $D \times I$, we conclude that the image of every surface $S^2 \times \{t\}$ will be convex, as desired. □

Theorem 8. *Any two tight contact structures ξ, ξ' on $S^2 \times I$ with the same characteristic foliation on $S^2 \times \partial I$ are isotopic rel boundary.*

Proof. We have shown that each contact structure can be isotoped rel boundary so that every surface $S^2 \times \{t\}$ is convex in both ξ and ξ'. Now change ξ' by a continuous isotopy fixing $S^2 \times \partial I$ so that each dividing curve on $S^2 \times \{t\}$ is brought to the dividing curve of ξ on $S^2 \times \{t\}$. Since the two contact structures are convex at every level and divided by the same family of curves, they are isotopic rel boundary. □

Theorem 9. *Up to isotopy, there is a unique tight contact structure on B^3 with convex boundary and a given characteristic foliation on ∂B^3. There is a unique tight contact structure up to isotopy on each of S^3, $\mathbb{R}^3$, and $S^1 \times S^2$.*

Proof. We know tight contact structures exist on each of these. Given two tight contact structures ξ, ξ' on B^3 with the same characteristic foliation on the boundary, we can find contactomorphic Darboux balls $B, B' \subset B^3$ and remove them one at a time. The remaining contact manifolds are $\xi|_{S^2 \times I}$ and $\xi'|_{S^2 \times I}$, and both are tight with the same characteristic foliation on $S^2 \times \partial I$, so they are isotopic and this extends over B and B'.

Suppose there are two tight contact structures ξ, ξ' on S^3. Remove contactomorphic Darboux balls from each; the complements are tight contact balls, hence they are isotopic as well.

For ξ, ξ' on $\mathbb{R}^3$, given $n > 0$ we can let Σ and Σ' be perturbations of the sphere of radius n which are convex for ξ and ξ' respectively, and by Giroux flexibility we can arrange for them to have the same characteristic foliations. We identify invariant neighborhoods of Σ and Σ' by a contact isotopy for all $n > 0$, and similarly for Darboux balls centered at the origin, and then the regions between the neighborhoods of Σ at radii n and $n + 1$ (and likewise for Σ') are tight $S^2 \times I$ with convex boundaries, hence they are all isotopic as well.

Finally, for ξ, ξ' on $S^1 \times S^2$ we take convex perturbations of a sphere $\Sigma = \{*\} \times S^2$ with the same characteristic foliation. The complement of a standard neighborhood of Σ is a tight $S^2 \times I$ with a fixed boundary, hence is unique up to isotopy. □

Finally, we note as a corollary that trivial bypasses are trivial when attached to *any* convex surface, not just S^2.

Proposition 10. *Let Σ be a convex surface and B a trivial bypass attached along some arc $\alpha \subset \Sigma$. Then a neighborhood of $\Sigma \cup B$ is contactomorphic rel boundary to $(\Sigma \times I, \xi)$ in which every $\Sigma \times \{t\}$ is convex.*

Proof. Cut out a small disk D around α which contains the disk cobounded by α and one of the dividing curves; we can realize ∂D as a Legendrian curve so that D is convex, and by Giroux's criterion it has a tight neighborhood. The Right-to-Life Principle says that we can find a trivial bypass along α in a vertically invariant neighborhood of D, which is also tight, so a neighborhood of the union of D and the trivial bypass is a topological $D \times I$ with a tight contact structure. But the tight contact structure on $D \times I \cong B^3$ is unique (given the characteristic foliation along its boundary), so we can isotope it fixing $\partial D \times I$ so that each $D \times \{t\}$ becomes convex. This isotopy extends trivially to all of $\Sigma \times I$, and it follows that each $\Sigma \times \{t\}$ becomes convex as well. □

References

[1] Emmanuel Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. 141 (2000), no. 3, 615–689.

[2] Yang Huang, A proof of the classification theorem of overtwisted contact structures via convex surface theory, arXiv:1102.5398.