Math 273 Lecture 6

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Giroux's criterion can be used to study Legendrian knots in tight contact structures; we will use this to develop our first complete classification of a contact geometric object, namely Legendrian representatives of the unknot in a tight contact structure.

Definition 1. Let $K \subset (M, \xi)$ be an oriented, nullhomologous Legendrian knot in a contact manifold, and let Σ be a Seifert surface for K. The *Thurston-Bennequin invariant* of K, denoted $tb(K)$, is equal to $tw(K, \Sigma)$, the twisting of $\xi|_K$ with respect to $T\Sigma|_K$.

Since Σ is a surface with boundary, we can choose a trivialization $\xi|_\Sigma \cong \Sigma \times \mathbb{R}^2$ and let $\pi : \xi|_\Sigma \to \mathbb{R}^2$ be the projection onto the $\mathbb{R}^2$ factor. Parametrize K by a map $\gamma : S^1 \to M$. The *rotation number* of K with respect to Σ, denoted $r(K, \Sigma)$, is the degree of the map $\pi \circ \gamma' : S^1 \to \mathbb{R}^2 \backslash \{0\}$, i.e. the winding number of $\pi(\frac{d\gamma}{dt})$ around the origin.

We remark that $\gamma'(t)$ is a nonzero section of $\xi|_K$, and one can show that $r(K, \Sigma)$ is the obstruction to extending it to a nonzero section of ξ along all of Σ. If Σ_ξ has nondegenerate critical points then we can write

$$r(K, \Sigma) = r_+ - r_-$$

where $r_+ = (e_+^{\text{int}} - h_+^{\text{int}}) + \frac{1}{2}(e_+^\partial - h_+^\partial)$, with each term counting the number of positive elliptic or hyperbolic points on the interior or boundary of Σ, and likewise for r_-; the proof is similar to the one where we showed that $\langle e(\xi), [\Sigma] \rangle = (e_+ - h_+) - (e_- - h_-)$ for closed Σ.

Example 2. Let $K \subset (\mathbb{R}^3, \xi_{\text{st}})$ be an oriented Legendrian knot, and consider the front projection (i.e. the xz-projection) of K. Then ∂_y is a section of ξ, and we can define a nonzero section v of $\xi|_\Sigma$ away from the cusps of K by letting $v = \gamma'(t)$ along $\gamma(t) \in K$, letting $v = \partial_y$ outside a neighborhood of K, and interpolating between the two inside that neighborhood. Extending v to a neighborhood of each cusp, we can force v to have a zero precisely at the cusp, and a model computation shows that the sign of this zero is $+1$ for cusps which are oriented downward and -1 for upward cusps. We conclude that

$$r(K, \Sigma) = \frac{1}{2} \left(\#(\downarrow \text{cusps}) - \#(\uparrow \text{cusps}) \right).$$

It is similarly easy to compute $tb(K)$. Let K' be a pushoff of K along the Reeb vector field ∂_z: in the front projection, this is just a slight vertical translation of K. Then $tb(K) = lk(K, K')$. In the Lagrangian (xy-) projection, we used this to argue that $lk(K, K')$ was the writhe of that projection. A nearly identical argument applies here, except that now each cusp contributes a $-\frac{1}{2}$-twist, and so

$$tb(K) = \text{writhe}(K) - \frac{1}{2}\#\text{cusps}.$$

Definition 3. Let K be a oriented, nullhomologous Legendrian knot. The *positive and negative stabilizations* of K, denoted K^+ and K^-, are operations in a Darboux ball around a point of K which replace a smooth arc of K in the front projection with a zig-zag oriented either up or down. The orientations are chosen so that the stabilizations satisfy

$$\begin{aligned} tb(K^{\pm}) &= tb(K) - 1 \\ r(K^{\pm}) &= r(K) \pm 1 \end{aligned}$$

with respect to some Seifert surface.

Theorem 4 (Thurston-Bennequin inequality)**.** *Let K be a Legendrian knot with Seifert surface Σ. Then $r(K, \Sigma) = \langle e(\xi), [\Sigma]\rangle$ if Σ is convex, and if ξ is tight then*

$$tb(K) + |r(K, \Sigma)| \leq -\chi(\Sigma).$$

Proof. We will prove the inequality for $tb(K) + r(K, \Sigma)$ first. Note that $tb + r$ is preserved under positive stabilization, so we will apply enough of these stabilizations to assume that $tb(K) \leq 0$. This allows us to make Σ convex.

Since $tb(K) = -\frac{1}{2}|\Gamma_\Sigma \cap K|$, there are $-tb(K)$ arcs of Γ_Σ with both endpoints on K. We can compute $\chi(\Sigma)$ from $\chi(\Sigma_+)$ and $\chi(\Sigma_-)$ by observing that we glue $\Sigma_\pm$ together along a set of arcs and closed circles; the circles have Euler characteristic 0, but we must subtract $1 = \chi(\delta)$ from $\chi(\Sigma_+) + \chi(\Sigma_-)$ for each arc δ involved in the gluing, and so

$$\chi(\Sigma) = \chi(\Sigma_+) + \chi(\Sigma_-) + tb(K).$$

Next, we observe that $r(K, \Sigma) = \chi(\Sigma_+) - \chi(\Sigma_-)$. Indeed, if we glue two copies of Σ_+ along the $-tb(K)$ arcs of $\partial\Sigma \cap \Sigma_+$ then the resulting surface Σ' satisfies $\chi(\Sigma') = 2\chi(\Sigma_+) - tb(K)$. Now $\partial\Sigma'$ is transverse to the characteristic foliation of Σ', and in particular Σ' has two singularities for each interior singular point of Σ_+ and one for each boundary singular point, so $\chi(\Sigma') = 2r_+$. Similarly we have $2\chi(\Sigma_-) - tb(K) = 2r_-$, and combining these we get

$$r(K, \Sigma) = r_+ - r_- = \chi(\Sigma_+) - \chi(\Sigma_-) = \langle e(\xi), [\Sigma]\rangle.$$

Finally, we combine these equations to get

$$tb(K) + r(K, \Sigma) = \chi(\Sigma) - 2\chi(\Sigma_-).$$

Now by Giroux's criterion, any disk components of $\Sigma\backslash\Gamma_\Sigma$ must intersect $\partial\Sigma$, so Σ_+ has at most $-tb(K)$ disks; since these are the only components with positive Euler characteristic, it follows that $\chi(\Sigma_+) + tb(K) \leq 0$. In particular, we have $\chi(\Sigma) - \chi(\Sigma_-) \leq 0$, hence

$$tb(K) + r(K, \Sigma) = -\chi(\Sigma) + 2(\chi(\Sigma) - \chi(\Sigma_-)) \leq -\chi(\Sigma).$$

To finish the proof, we note that reversing the orientation of K changes the sign of $r(K, \Sigma)$ but preserves $tb(K)$ and $-\chi(\Sigma)$, hence $tb(K) - r(K, \Sigma) \leq -\chi(\Sigma)$ as well. □

For Legendrian knots in $(\mathbb{R}^3, \xi_{\text{st}})$, the Thurston-Bennequin inequality implies that $tb(K) \leq 2g(K) - 1$, and so we can speak of the *maximal Thurston-Bennequin number* $\overline{tb}(K)$ for a topological knot K. This is an interesting invariant in its own right, and there are a wide variety of techniques to compute it, including refinements of this inequality using the smooth 4-ball genus or the s and t invariants in Khovanov and knot Floer homology, and bounds from knot polynomials including the Kauffman and HOMFLY polynomials. For more on many of these bounds, see [1].

Remark 5. We showed in the course of the proof that $r(K, \Sigma) = \langle e(\xi), [\Sigma] \rangle$ for convex Σ, which we can always achieve by stabilizing K. In particular, $r(K, \Sigma)$ does not depend on Σ if either $e(\xi) = 0$ or $H_2(M, K) \cong \mathbb{Z}$ (in which case all Seifert surfaces are homologous). This is true for the standard tight contact structure on $\mathbb{R}^3$ or S^3, or indeed on any homology sphere, so in this case we will unambiguously write $r(K)$.

Remark 6. The Thurston-Bennequin inequality (proved in full generality by Eliashberg) gives a knot-theoretic characterization of tightness: if ξ is tight then $tb(K) + |r(K, \Sigma)| \leq -\chi(\Sigma)$, but if ξ is overtwisted then the boundary of an overtwisted disk violates this inequality. In fact, Bennequin first proved that the standard contact structure on $\mathbb{R}^3$ is tight by proving this inequality, or more accurately its analogue $sl(K) \leq -\chi(\Sigma)$ for transverse knots, for ξ_{st}.

From the equation $tb(K)+r(K, \Sigma) = \chi(\Sigma)-2\chi(\Sigma_-)$ and $\chi(\Sigma) = 1-2g(\Sigma)$ we see that $tb \pm r$ is always odd. For a Legendrian representative of an unknot in the tight $\mathbb{R}^3$, this and the Thurston-Bennequin inequality are the only restrictions on tb and r: any values for which $tb + |r| \leq -1$ and $tb + r$ is odd can be achieved by an appropriate stabilization of the unknot with $(tb, r) = (-1, 0)$. It turns out that for unknots the values of tb and r uniquely determine the Legendrian knot type:

Theorem 7 (Eliashberg–Fraser)**.** *If two Legendrian representatives of the unknot in $(\mathbb{R}^3, \xi_{\text{st}})$ have the same values of tb and r, then they are Legendrian isotopic.*

Remark 8. This is definitely not true for other knot types: the first example is 5_2, for which Chekanov identified two distinct Legendrian representatives with $tb = 1$ and $r = 0$.

Proof. By the Thurston-Bennequin inequality, we know that if K is a Legendrian representative of the unknot then $tb(K) \leq -1$, so we can find a convex disk D bounded by K. We will study this disk to see that first of all, if $tb(K) < -1$ then K can be destabilized, and second, that if $tb(K) = -1$ then K is unique. We observe that Γ_D cannot contain any closed curves, or else we could use the Legendrian Realization Principle to find a parallel curve with $tb = 0$, which is impossible. Therefore Γ_D consists of $-tb(K)$ properly embedded arcs.

Suppose $tb(K) < -1$. Then some dividing curve $a \subset \Gamma_D$ is boundary-parallel; let Δ be the disk it bounds. By Giroux flexibility, we can pick the characteristic foliation on Δ so that it has an elliptic point p on $\partial\Delta\backslash a$ whose leaves all exit Δ transverse to a, and the adjacent elliptic points on K are connected by a union of flow lines parallel to a. We let Δ' be the disk bounded by this union of flow lines and containing Δ. Then $D\backslash\Delta'$ has Legendrian boundary with two corners and one less dividing curve, so if we smooth this boundary at the corners then we get an unknot K' with $tb(K') = tb(K)+1$. It is straightforward to check that K is a stabilization of K', i.e. that K is Legendrian isotopic to one of $(K')^{\pm}$ depending on the sign of p.

Now suppose that $tb(K) = -1$. Then Γ_Δ is a single arc, and we can arrange the characteristic foliation on Δ to have two elliptic singularities of opposite signs on $\partial\Delta$, with Δ_ξ a family of arcs connecting them. These arcs are all Legendrian, so if we fix one such arc a then by an isotopy we can make Δ_ξ an arbitrarily small neighborhood of a. But any two Legendrian arcs are isotopic, so any $tb = -1$ unknot can be isotoped into a small neighborhood of, say, the arc $a_0 = [0,1]_x \times \{(0,0)\}$ in $\mathbb{R}^3$. Given any disks in this neighborhood with the same characteristic foliation Δ_ξ, which contain a_0 as a leaf, we can construct a Legendrian isotopy between their boundaries, so K is unique. $\square$

References

[1] Lenhard Ng, *A skein approach to Bennequin-type inequalities*, Int. Math. Res. Not. (2008), Art. ID rnn116, 18 pp.