Math 273 Lecture 4

Steven Sivek

February 3, 2012

Last time we showed that an embedded surface $\Sigma \subset (M, \xi)$ is convex if and only if it has dividing curves Γ, and any two sets of dividing curves are isotopic through curves transverse to Σ_ξ. In order to replace characteristic foliations with convex surfaces for many applications, we will need two more results: first, that any closed surface is C^∞-close to a convex one, and second, that if Σ is convex then we can realize any foliation divided by Γ_Σ by perturbing Σ.

We begin by recalling the definition of a Morse-Smale foliation.

Definition 1. A foliation $\mathcal{F}$ on Σ is Morse-Smale if its singularities and closed orbits are nondegenerate, no two hyperbolic singularities are connected by a flow line, and the α– and ω–limit set of each flow line is either a singular point or a closed orbit.

Theorem 2. *If a surface $\Sigma \subset (M, \xi)$ has Morse-Smale characteristic foliation, then Σ is convex.*

Proof. We will construct dividing curves Γ for Σ. Let Σ_+ consist of a small disk around each positive elliptic point, a band along the stable trajectory of each positive hyperbolic point, and an annulus around each repelling closed orbit of Σ_ξ (i.e. orbits for which the Poincaré return map h satisfies $h'(0) > 1$). Similarly, let Σ_- consist of a small disk around each negative elliptic point, a band along the unstable trajectory of each negative hyperbolic point, and an annulus around each attracting closed orbit ($h'(0) < 1$). We make each disk, band, or annulus small enough that Σ_+ and Σ_- are disjoint and Σ_ξ is transverse to the boundary of each.

Let ω be a volume form on Σ, and let v satisfy $\mathcal{L}_v \omega = \alpha|_\Sigma$ so that v directs Σ_ξ. If $\Sigma_\pm$ are determined by a dividing set Γ, then ω must satisfy $\pm \mathrm{div}_\omega v > 0$ on $\Sigma_\pm$. One can check by computing in local coordinates that if $\omega' = e^f \omega$ then $\mathrm{div}_{\omega'} v = \mathrm{div}_\omega v + df(v)$, so we can rescale ω by some e^f to achieve this condition. Namely, on Σ_+ we pick f to be zero near each elliptic singularity and closed orbit and then increasing along flow lines so that $df(v)$ is sufficiently large, and similarly for Σ_- (with f decreasing along flow lines instead).

Let $A = \Sigma \backslash (\Sigma_+ \cup \Sigma_-)$. Then the characteristic foliation on A is nonsingular and transverse to the boundary and it does not contain any closed orbits, so A must be a union of annuli and bands foliated by arcs of A_ξ which are oriented from Σ_+ to Σ_-. We let Γ be the cores of these annuli and bands, and pick

$\omega' = e^f\omega$ so that $\operatorname{div}_{\omega'} v = 0$ precisely along Γ. It follows that Γ divides Σ_ξ, as desired. $\square$

Finally, we show that the precise characteristic foliation doesn't matter: once we know the dividing curves Γ on Σ, we can achieve any foliation divided by Γ by an isotopy of Σ.

Theorem 3 (Giroux flexibility)**.** *Let $i : \Sigma \to M$ be an embedding of Σ into (M, ξ) with convex image, and let $\mathcal{F}$ be a foliation of Σ divided by $i^{-1}(\Gamma_\Sigma)$. Given any neighborhood U of $i(\Sigma)$ in M, there is an isotopy $\phi_s : \Sigma \to M$ supported in U so that $\phi_0 = i$, each $\phi_s(\Sigma)$ is convex with dividing set Γ_Σ, ϕ_s fixes $i^{-1}(\Gamma_\Sigma)$ for all s, and $\phi_1(\mathcal{F})$ is the characteristic foliation of $\phi_1(\Sigma)$.*

Proof. Since $i(\Sigma)$ is convex we may pass to a vertically invariant neighborhood $\Sigma \times \mathbb{R}$, with $\Sigma = \Sigma \times \{0\}$, and contact form

$$\alpha_0 = \eta_0 + f_0 dt$$

for which $\Gamma_\Sigma = f^{-1}(0)$. Furthermore, $\mathcal{F}$ defines a vertically invariant contact structure ξ' on $\Sigma \times \mathbb{R}$ for which Σ is convex, so let $\alpha_1 = \eta_1 + f_1 dt$. Since Γ_Σ divides both Σ_ξ and $\mathcal{F}$, we can take $\Gamma_\Sigma = f^{-1}(0) = (f')^{-1}(0)$, and then by a small isotopy we can arrange for ξ and ξ' to agree on a neighborhood of Γ_Σ. Away from this neighborhood we rescale the contact forms so that $f_0 = f_1 = \pm 1$ on $\Sigma_\pm$.

For the rescaled contact forms $\eta_0 + dt$ and $\eta_1 + dt$, we observe that $d\eta_0$ and $d\eta_1$ are area forms on Σ, so if $\eta_s = (1-s)\eta_0 + s\eta_1$ then any linear combination

$$\alpha_s = \eta_s + dt$$

is a contact form as well. We now apply Moser's trick one more time to realize this by an isotopy ϕ_s: given the equation $\dot{\alpha}_s = \mathcal{L}_v \alpha_s$, by setting $v = g\partial_t + w$ with $w \in T\Sigma$, we get

$$\eta_1 - \eta_0 = d(g + \iota_w \eta_s) + \iota_w d\eta_s$$

which we can solve by finding w such that $\iota_w d\eta_s = \eta_1 - \eta_0$ and then setting $g = -\iota_w \eta_s$. Since v does not depend on the t coordinate at all, the image of Σ under this isotopy remains transverse to ∂_t, and it is fixed near Γ_Σ. $\square$

Corollary 4. *Let $\Sigma \subset (M, \xi)$ be a convex surface. Then*

$$\langle e(\xi), [\Sigma] \rangle = \chi(\Sigma_+) - \chi(\Sigma_-),$$

from which it follows that

$$\chi(\Sigma) + |\langle e(\xi), [\Sigma] \rangle| = 2 \max(\chi(\Sigma_+), \chi(\Sigma_-)).$$

Proof. Perturb Σ so that Σ_ξ is Morse-Smale and directed by a vector field v, and then construct the dividing set Γ as above. In particular, Σ_+ contains exactly the positive singularities of v, so $\chi(\Sigma_+) = e_+ - h_+$, and similarly $\chi(\Sigma_-) = e_- - h_-$. The corollary follows once we recall that

$$\langle e(\xi), [\Sigma] \rangle = (e_+ - h_+) - (e_- - h_-). \quad \square$$

At this point we should remark that we will also wish to consider convex surfaces with Legendrian boundary. Suppose K is a component of $\partial\Sigma$; then by the contact neighborhood theorem, K has a neighborhood N contactomorphic to $S^1 \times \mathbb{R}^2$ with contact form

$$\alpha = \cos(\theta)dx - \sin(\theta)dy$$

and $K = S^1 \times \{(0,0)\}$. We can perform a C^0-small isotopy of $\Sigma \cap N$ fixing K so that Σ wraps uniformly around K in a smaller neighborhood N' of K. Then since

$$\xi = \mathrm{span}(\partial_\theta, \sin(\theta)\partial_x + \cos(\theta)\partial_y)$$

the foliation Σ_ξ has singularities along K whenever $\sin(\theta)\partial_x + \cos(\theta)\partial_y$ is tangent to Σ. One can compute that if Σ twists in the left handed direction around K, then a singular point p of Σ_ξ on K is positive iff it is a source of the flow along K, whereas the opposite is true if Σ twists in the right handed direction. In particular, if $tw(K,\Sigma) > 0$ then we cannot make Σ convex, whereas if $tw(K,\Sigma) \leq 0$ we can by putting Σ in a standard form near K via a C^0-small isotopy. This standard form is given by

$$\alpha = \cos(n\theta)dx - \sin(n\theta)dy$$

with Σ given locally by $x \geq 0$, $y = 0$ and $n = -tw(K,\Sigma)$.

The rest of the theorems we have proved about convex surfaces and characteristic foliations still hold for surfaces with Legendrian boundary: we need only allow a C^0-small isotopy near the boundary and a C^∞-small isotopy everywhere else.

In particular, consider the case of an overtwisted disk $D \subset (M,\xi)$. The boundary of D is Legendrian by definition, and $tw(\partial D, \Sigma) = 0$, so we can make D convex. By taking the model disk of radius π in the xy-plane in the standard overtwisted structure

$$\xi_{ot} = \ker(\cos(r)dz + r\sin(r)d\theta)$$

on $\mathbb{R}^3$ and pushing the interior up slightly, we see that D can be made convex with characteristic foliation having a unique singular point at the origin; this singularity is positive and elliptic, and the leaves of D_ξ spiral outward toward their limit cycle ∂D. We have now shown by Giroux flexibility that every overtwisted contact structure contains an overtwisted disk of this form.

With this description of an overtwisted disk in hand, we are finally ready to sketch a proof that tight contact structures exist. This technique of "filling by holomorphic disks" was used successfully by both Eliashberg and Gromov.

Theorem 5. *Let (X,J) be a Stein domain with plurisubharmonic exhaustion function $\phi : X \to \mathbb{R}$ having regular value c, and let $M = \phi^{-1}(c)$ have contact structure $\xi = TM \cap J(TM)$. Then ξ is tight.*

Proof. Suppose ξ is overtwisted, and let Σ be an overtwisted disk with the characteristic foliation described above and singular point p. Bishop's theorem says that since p is elliptic and $T_p\Sigma = \xi_p$ is a complex line in T_pX, there is a family of holomorphic disks

$$\psi_s : (D^2, \partial D^2) \to (X, \Sigma)$$

for $0 < s < \delta$ such that the images $D_s = \psi_s(D^2)$ are all disjoint, and there is a neighborhood $N \subset \Sigma$ of p such that the boundaries $\partial D_s = \psi_s(\partial D)$ foliate $N - p$. Some hard analysis shows that the moduli space $\mathcal{M}$ of such holomorphic disks is 1-dimensional, and so given any disk ψ_{s_0} we can extend it to a family ψ_s for $s \in (s_0 - \epsilon, s_0 + \epsilon)$: i.e. $\mathcal{M}$ is an open interval. We will assume for now that these holomorphic disks do not fill all of the interior of Σ, so $\{\psi(\partial D^2) \mid \psi \in \mathcal{M}\}$ avoids a neighborhood of $\partial\Sigma$. Let $\omega = -d(d\phi \circ J)$ be the symplectic form induced by the complex structure.

First, we wish to investigate the behavior of each holomorphic disk in $\mathcal{M}$. We claim that the function $f_s = \phi \circ \psi_s$ is subharmonic, meaning $\Delta f_s \geq 0$ and Δf_s does not vanish everywhere. Indeed, if x and y are the coordinates on D^2, then using the formula

$$2\Delta f \cdot dx \wedge dy = d(f_x dy - f_y dx) = -d(df \circ i)$$

with $f = \phi \circ \psi_s$ and noting that ψ_s is holomorphic gives

$$2\Delta f \cdot dx \wedge dy = -d(d\phi \circ d\psi_s \circ i) = -d(d\phi \circ J \circ d\psi_s) = \psi_s^*(-d(d\phi \circ J)) = \psi_s^*\omega.$$

This last quantity is equal to

$$\omega((D\psi_s \cdot \partial_x, D\psi_s \cdot \partial_y)dx \wedge dy = \omega(D\psi_s \cdot \partial_x, J(D\psi_s \cdot \partial_x))dx \wedge dy$$

and since $\omega(v, Jv) > 0$ whenever $v \neq 0$, we have $\Delta f \geq 0$ with equality if and only if $D\psi_s = 0$. Thus f_s satisfies a strong maximum principle: it is maximized only along ∂D^2, and it is increasing in the outward normal direction along ∂D^2.

From the strong maximum principle we conclude that each D_s is transverse to M along its boundary. Furthermore, we claim that ∂D_s is transverse to the characteristic foliation Σ_ξ, which along ∂D_s is the real line $\xi \cap T\Sigma$. If this were not the case at some $x \in \partial D_s$, then $T_x D_s$ would contain the line $\xi_x \cap T_x\Sigma$, and since $T_x D_s$ is a complex line containing some $v \in \xi$ it also contains $Jv \in \xi$, hence $T_x D_s = \xi_x \subset T_x M$ which contradicts the transversality.

Next, we consider the topology of the moduli space $\mathcal{M}$. Given a disk $D_s = \psi_s(D^2)$ with boundary in Σ, the boundary bounds another disk $D'_s \subset \Sigma$ such that $D_s \cup D'_s$ is a sphere bounding a ball B_s. We have

$$\int_{D_s \cup D'_s} \omega = \int_{B_s} d\omega = 0$$

and so $\int_{D_s} \omega$ is bounded above by $\int_\Sigma |\omega|$. Thus we can apply the Gromov compactness theorem and consider the limit curve D at the end ψ_{s_0} of the compactification $\overline{\mathcal{M}}$. There cannot be any bubbling on the interior of D: if there is a sphere bubble S, then

$$\int_S \omega = -\int_{\partial S} d\phi \circ J = 0$$

contradicting the fact that holomorphic curves have positive symplectic area. If there is bubbling on the boundary then ∂D_s is transverse to Σ_ξ, so outside a fixed neighborhood of $\partial\Sigma$ the angle between ∂D_s and Σ_ξ is bounded away from 0. Thus the limit $\psi_{s_0} : (D^2, \partial D^2) \to (X, \Sigma)$ is also an embedded holomorphic disk. One can show that it is smooth, so it really is an element of $\mathcal{M}$ and then we can extend $\mathcal{M}$ to the open interval $(s_0 - \epsilon, s_0 + \epsilon)$. This contradicts the claim that $\psi_{s_0} \in \partial\overline{\mathcal{M}}$, unless ψ_{s_0} touches $\partial\Sigma$ after all.

We conclude that ψ_{s_0} has a point where $\psi_{s_0}(\partial D^2)$ is tangent to $\partial\Sigma$, and hence to the characteristic foliation along $\partial\Sigma$. But we just showed that $\psi_{s_0}(\partial D^2)$ is transverse to Σ_ξ, so we have a contradiction and Σ cannot exist. □

Corollary 6. *The standard contact structure* $(\mathbb{R}^3, \xi_{\mathrm{st}} = \ker(dz - ydx))$ *is tight.*

Proof. Since the Stein fillable structure (S^3, ξ_{st}) we have seen on $S^3 = \partial B^4$ (with $\phi(z_1, z_2) = |z_1|^2 + |z_2|^2$) is tight, any Darboux ball around a point in (S^3, ξ_{st}) is tight, and so some neighborhood N of the origin in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ is tight as well. If $(\mathbb{R}^3, \xi_{\mathrm{st}})$ contains an overtwisted disk D, then we observe that for r large enough the image of N under the contactomorphism

$$\phi(x, y, z) = (rx, ry, r^2 z)$$

of $(\mathbb{R}^3, \xi_{\mathrm{st}})$ will contain D, contradicting the tightness of $\xi_{\mathrm{st}}|_N$. □

Remark 7. One can also prove this by showing that $(\mathbb{R}^3, \xi_{\mathrm{st}})$ is contactomorphic to the complement of any point of (S^3, ξ_{st}) by using stereographic projection.

Corollary 8. $S^2 \times S^1$ *has a tight contact structure.*

Proof. Last time we found a convex S^2 in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ with a single dividing curve. This S^2 has an $\mathbb{R}$-invariant neighborhood $(S^2 \times \mathbb{R}, \xi)$ which must be tight since it embeds in ξ_{st}, so the translation $t \mapsto t + 1$ induces a contact structure ξ_0 on the quotient $S^2 \times S^1$. If ξ_0 had an overtwisted disk then it would lift to the cover $(S^2 \times \mathbb{R}, \xi)$, which contradicts the fact the ξ is tight. □

Gromov and Eliashberg actually showed something stronger than this theorem: let (M, ξ) be a contact manifold in the boundary of a symplectic manifold (X, ω) such that $\omega|_\xi > 0$. Then the holomorphic filling argument can be generalized to show that ξ is tight. (Alternative proofs using Seiberg–Witten theory and Heegaard Floer homology were given by Kronheimer–Mrowka and Ozsváth–Szabó, respectively.) Such contact structures are called *weakly symplectically semi-fillable*, and there is a hierarchy of different types of fillable contact structures between these and the Stein fillable ones.

We claim that ξ_{st} is the only tight contact structure on S^3 up to isotopy. This will require significant effort to prove, involving an analysis of tight contact structures on $S^2 \times I$, and we will not prove it in this lecture. However, we can draw an interesting topological consequence from this and the method of filling by holomorphic disks; this proof is due to Eliashberg.

Theorem 9 (Cerf's theorem)**.** *Every orientation-preserving diffeomorphism of $S^3 = \partial B^4$ extends to a diffeomorphism of B^4.*

Proof. Let $\phi : S^3 \to S^3$ be a diffeomorphism, and change ϕ by an isotopy so that ϕ is the identity on a neighborhood of the north and south poles. Then $\phi^*\xi_{\mathrm{st}}$ is a tight contact structure, so by our claim of uniqueness, $\phi^*\xi_{\mathrm{st}}$ must be isotopic to ξ_{st}; in other words, modifying ϕ by another smooth isotopy which is fixed near the poles, we can assume that $\phi^*\xi_{\mathrm{st}} = \xi_{\mathrm{st}}$.

Given coordinates (x_1, y_1, x_2, y_2) on $S^3 \subset \mathbb{C}^2$, we can foliate the complement of the poles by the spheres

$$\Sigma_t = \{y_2 = t\}$$

for $-1 < t < 1$. Let $\Sigma_t' = \phi(\Sigma_t)$; we observe that $\Sigma_t' = \Sigma_t$ for t near ± 1, and that ϕ sends the characteristic foliation of Σ_t to that of Σ_t'. We can check that the characteristic foliation on Σ_t has two elliptic singularities $p_t^\pm = (0, 0, \pm\sqrt{1-t^2}, t)$ of sign ± 1, and since the union of all $p_t^\pm$ is an unknot U we can insist that ϕ fixes a ball containing U. Each twice-punctured sphere $\Sigma_t \backslash p_t^\pm$ has a foliation $\mathcal{F}$ by the boundaries of holomorphic disks

$$D_{s,t} = \{(x_1, y_1, s, t)\}.$$

Using Bishop's theorem, we can also find families of holomorphic disks $D_{s,t}'$ around the elliptic points $\phi(p_t^\pm) \in \Sigma_t'$, and the boundaries of these disks give a foliation $\mathcal{F}'$ of $\Sigma_t' \backslash \phi(p_t^\pm)$ while (as we showed in the proof that fillable implies tight) being transverse to the characteristic foliation of Σ_t'.

In particular, we have smooth foliations $\phi(\mathcal{F})$ and $\mathcal{F}'$ of $\Sigma_t' \backslash \phi(p_t^\pm)$ which are transverse to the characteristic foliation $(\Sigma_t')_{\xi_{\mathrm{st}}}$, and so by an isotopy along the leaves of $(\Sigma_t')_{\xi_{\mathrm{st}}}$ we see that ϕ is isotopic to a diffeomorphism of S^3 fixing a neighborhood of U for which $\phi(\mathcal{F}) = \mathcal{F}'$ for each t. There is now a compact disk C of pairs (s, t) for which $\phi(D_{s,t})$ is not necessarily $D_{s,t}'$.

We have reduced the theorem to the following problem: given $\phi(\partial D_{s,t}) = \partial D_{s,t}'$ for all $s, t \in C$, such that ϕ extends to a diffeomorphism $\phi(D_{s,t}) = D_{s,t}'$ for $s, t \in \partial C$, does ϕ extend over all disks $D_{s,t}$? The restriction map $\mathrm{Diff}(D^2) \to \mathrm{Diff}(S^1)$ is a Serre fibration with contractible fiber, so we are done. □

References

[1] Yakov Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, 2 (Durham, 1989), 45–67, London Math. Soc. Lecture Note Ser., 151, Cambridge Univ. Press, Cambridge, 1990.

[2] Hansjörg Geiges and Kai Zehmisch, *Eliashberg's proof of Cerf's theorem*, J. Topol. Anal. 2 (2010), no. 4, 543–579.

[3] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds,* Invent. Math. 82 (1985), 307–347.