Math 273 Lecture 3

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February 1, 2012

Our goal in this lecture is to begin the study of embedded surfaces in contact manifolds, which will allow us to better understand tight contact structures.

Definition 1. Let $\Sigma \subset (M, \xi = \ker(\alpha))$ be a compact embedded surface with area form ω. The *characteristic foliation* of Σ, denoted Σ_ξ, is the singular foliation determined by the vector field X on Σ which satisfies $\iota_X \omega = \alpha|_\Sigma$.

The foliation Σ_ξ only depends on the vector field X up to a nonzero scalar function, so it is independent of both ω and α. Note also that at a point $p \in \Sigma$, we have $\alpha_p(X_p) = \omega_p(X_p, X_p) = 0$, so $X_p \in \xi_p$, and $X_p = 0$ if and only if $\xi_p = T_p\Sigma$; in other words, at each nonsingular point p of X the characteristic foliation spans the line $T_p\Sigma \cap \xi_p$.

Example 2. Let $\alpha = dz + r^2 d\theta$ be a contact form for $(\mathbb{R}^3, \xi_{\text{st}})$ in cylindrical coordinates, and let $\Sigma = S^2$ be the unit sphere. Then ξ_{st} is spanned by the vectors ∂_r and $r^2\partial_z - \partial_\theta$ away from the origin, while $T_p\Sigma$ at a point (r, θ, z) is orthogonal to $r\partial_r + z\partial_z$, so $T_p\Sigma = \xi_{\text{st}}$ iff $r = 0$. Thus the characteristic foliation on Σ has singular points at the poles $(0, 0, \pm 1)$ and is directed at other points (r, θ, z) by the line through $rz\partial_r + \partial_\theta - r^2\partial_z$.

Example 3. The contact structure $\xi = \ker(\cos(r)dz + r\sin(r)d\theta)$ on $\mathbb{R}^3$ has overtwisted disk

$$D = \{(r, \theta, 0) \mid r \leq \pi\}.$$

At each point we have $\xi_p = \text{span}(\partial_r, r\sin(r)\partial_z - \cos(r)\partial_\theta)$, so D_ξ consists of radial line segments with singularities at the origin and along ∂D.

Definition 4. The *divergence* of a vector field X with respect to Σ is the function satisfying

$$\text{div}_\omega(X) \cdot \omega = \mathcal{L}_X\omega = d(\iota_X\omega).$$

The set of $p \in \Sigma$ where $X_p = 0$ and $\text{div}_\omega(X) \neq 0$ only depends on X up to nonzero scalars: if $f \neq 0$ then

$$\text{div}_\omega(fX) \cdot \omega = df \wedge \iota_X\omega + \text{div}_\omega(X) \cdot f\omega = \text{div}_\omega(X) \cdot f\omega,$$

and both ω and $f\omega$ are area forms.

Lemma 5. *A vector field X defines the characteristic foliation Σ_ξ for some contact structure ξ on a neighborhood of Σ if and only if $X_p = 0$ implies $\mathrm{div}_\omega(X) \neq 0$ at p.*

Proof. If $X = \Sigma_\xi$ and $\xi = \ker(\alpha)$, then $X_p = 0$ implies $T_p\Sigma = \xi_p$, so in particular $(\iota_X\omega)_p = \alpha_p$ and $(d\iota_X\omega)_p = d\alpha_p$. This is an area form on $\xi_p = T_p\Sigma$, so $\mathrm{div}_\omega(X) \neq 0$ at p.

Conversely, suppose that $\mathrm{div}_\omega(X) \neq 0$ whenever $X_p = 0$. Let $\beta = \iota_X\omega$ and $f = \mathrm{div}_\omega X$, so that $d\beta = f\omega$; if $\beta_p = 0$ then by assumption $f(p) \neq 0$. Choose a 1-form γ on Σ so that $\beta \wedge \gamma \geq 0$, with $\beta \wedge \gamma > 0$ if $\beta_p \neq 0$, and define

$$\alpha = \beta + t(df - \gamma) + f\,dt$$

on a neighborhood $\Sigma \times \mathbb{R}_t$. Then

$$d\alpha = f\omega - dt \wedge \gamma - t d\gamma$$

and so

$$\begin{aligned}\alpha \wedge d\alpha &= (\beta + t(df - \gamma)) \wedge (-dt \wedge \gamma) + f\,dt \wedge (f\omega - td\gamma) \\ &= ((\beta \wedge \gamma + tdf \wedge \gamma + f(f\omega - td\gamma)) \wedge dt.\end{aligned}$$

In particular, α is a contact form on some sufficiently small neighborhood $\Sigma \times (-\epsilon, \epsilon)$ if $\beta \wedge \gamma + f^2\omega > 0$. Both terms are nonnegative, and whenever $\beta \wedge \gamma = 0$ we have $\beta = 0$, hence $f \neq 0$ and thus $f^2\omega > 0$, so the sum is positive as desired. $\square$

Suppose that Σ_0 and Σ_1 are surfaces in contact manifolds (M_i, ξ_i), and that there is a diffeomorphism $\phi : \Sigma_0 \to \Sigma_1$ carrying one characteristic foliation to the other. Giroux showed that one can then extend ϕ to a contactomorphism $\tilde{\phi} : N(\Sigma_0) \to N(\Sigma_1)$: to prove this, one takes contact forms α_i on neighborhoods $N(\Sigma_i) = \Sigma_i \times \mathbb{R}$, extends ϕ to these neighborhoods, rescales the forms so that α_0 and $\phi^*\alpha_1$ agree on $\Sigma_0 \times \{0\}$, and then uses Moser's trick as before to construct an isotopy between the contact structures on a neighborhood $\Sigma_0 \times (-\epsilon, \epsilon)$. Thus the diffeomorphism class of Σ_ξ determines ξ on a neighborhood of Σ.

A generic foliation induced by a vector field X has isolated singular points, so in order to discuss the singularities we can assume that the singular point in question is $p = (0, 0)$ in $\mathbb{R}^2$ with area form $\omega = dx \wedge dy$. If $X = f\partial_x + g\partial_y$, then the foliation is determined by the flow lines of X, i.e. by the equation

$$\dot{\gamma} = X \circ \gamma,$$

with linearization $A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$, and we can compute $\mathrm{div}_\omega X = \mathrm{tr}(A)$.

Definition 6. A singular point is *elliptic* if $\det(A) > 0$ and *hyperbolic* if $\det(A) < 0$.

Definition 7. Suppose the foliation on (Σ, ω) is induced by a contact structure ξ and has a singular point at p. We define the *sign* of p to be the sign of $\mathrm{div}_X \omega$.

Since the contact form restricts to $\iota_X\omega$ on Σ, ξ is oriented by $d\iota_X\omega = (\mathrm{div}_\omega X)\omega$, and so $\mathrm{sign}(p)$ is 1 iff ξ_p and $T_p\Sigma$ have the same orientation. If p is elliptic then it's either a source or a sink depending on the sign of $\mathcal{L}_X\omega = (\mathrm{div}_\omega X)\omega$, hence $\mathrm{sign}(p) = 1$ for a source and -1 for a sink. If p is hyperbolic then the eigenvalues of A are real and have opposite signs, and $\mathrm{div}_\omega X$ is their sum, so $\mathrm{sign}(p)$ is the sign of whichever eigenvalue is larger in magnitude.

Definition 8. Let $\gamma \subset \Sigma$ be a periodic orbit of the flow of the vector field X. On a sufficiently small neighborhood $S^1 \times (-\epsilon, \epsilon)$ of γ, the flow line leaving a point (p, t) will first hit $\{p\} \times (-\epsilon, \epsilon)$ at another point $(p, h(t))$; the function h is called the *Poincaré return map*, and if $h'(0) \neq 1$ then we say that γ is nondegenerate.

Definition 9. A foliation is *Morse-Smale* if the singularities and closed orbits are nondegenerate, the α– and ω–limit set of each flow line (i.e. the set of all limit points $\lim \gamma_{t_i}(x)$ with $t_i \to -\infty$ or $t_i \to +\infty$) is either a singular point or a closed orbit, and there are no flow lines connecting pairs of hyperbolic singularities.

Theorem 10. *Given a closed, orientable surface $\Sigma \subset (M, \xi)$, there is a C^∞-small perturbation Σ' of Σ so that Σ'_ξ is Morse-Smale.*

Proof. Write $\xi = \ker(\alpha)$. The condition $\alpha \wedge d\alpha$ is a C^r-open condition for $r \geq 1$, and Morse-Smale vector fields are dense in the C^∞ topology by a theorem of Peixoto, so we can pick an arbitrarily small 1-form β for which $\xi_t = \ker(\alpha + t\beta)$ is a contact structure for $0 \leq t \leq 1$ and Σ_{ξ_1} is Morse-Smale. We use Gray's stability theorem to find a small isotopy of M for which $\psi_t^*\xi_t = \psi_0$, and then take $\Sigma' = \psi_1^{-1}(\Sigma)$. □

We can now use the characteristic foliation to investigate the topology of Σ.

Proposition 11. *Let $\Sigma \subset (M, \xi)$ be closed and orientable, with Morse-Smale characteristic foliation. If $e_\pm$ and $h_\pm$ count the number of elliptic and hyperbolic singularities of each sign, then*

$$\begin{aligned} \chi(\Sigma) &= (e_+ - h_+) + (e_- - h_-) \\ \langle e(\xi), [\Sigma] \rangle &= (e_+ - h_+) - (e_- - h_-). \end{aligned}$$

Proof. The characteristic foliation is determined by a Morse-Smale vector field X on Σ which lies in $\xi_p \cap T_p\Sigma$ at each point p. We compute $\chi(\Sigma) = \langle e(T\Sigma), [\Sigma] \rangle$ and $\langle e(\xi), [\Sigma] \rangle$ by counting the number of zeros of X with appropriate signs. Relative to $T\Sigma$, the index of each singular point p is 1 if p is elliptic and -1 if p is hyperbolic, and relative to ξ we must change the signs of the indices of the negative points, since those are the points where ξ has the opposite orientation. □

In particular we have an equality

$$\chi(\Sigma) + \langle e(\xi), [\Sigma] \rangle = 2(e_+ - h_+).$$

Eventually we will show that if ξ is tight and $\Sigma \neq S^2$ then we can perturb Σ so that $e_+ = 0$, leading to the inequality

$$|\langle e(\xi), [\Sigma]\rangle| \leq -\chi(\Sigma).$$

This will show that there can only be finitely many Euler classes of tight contact structures, even though every even class represents an overtwisted structure.

It turns out that we often don't need all the information in a characteristic foliation to determine the contact structure near Σ. The point of convex surface theory, originally developed by Giroux, is to be able to describe ξ near a surface in terms of very simple information.

Definition 12. A *contact vector field* v is a vector field on M whose flow preserves ξ.

Write $\xi = \ker(\alpha)$. If the flow ϕ_t of v preserves ξ then $\phi_t^*\alpha = f\alpha$ for some nonzero function f; since $\mathcal{L}_v\alpha = \frac{d}{dt}\, \phi_t^*\alpha|_{t=0}$, we see that v is contact if and only if $\mathcal{L}_v\alpha = g\alpha$ for some function g.

Example 13. Let R be the Reeb vector field of α, i.e. the unique vector field satisfying $\alpha(R) = 1$ and $\iota_R d\alpha = 0$. Then

$$\mathcal{L}_R\alpha = d(\iota_R\alpha) + \iota_R d\alpha = d(1) + 0 = 0,$$

so R is a contact vector field.

In fact, let v be a contact vector field which is transverse to ξ. Then $f(p) = \alpha_p(v)$ is nonzero for all p, and one can show that v is the Reeb vector field for the contact form $\frac{1}{f}\alpha$. Thus the set of contact vector fields transverse to ξ is precisely the set of Reeb fields of contact forms for ξ.

Lemma 14. *Any function $H : M \to \mathbb{R}$ determines a unique contact vector field v satisfying*

$$\begin{aligned} \iota_v\alpha &= -H \\ \iota_v d\alpha &= dH - (\iota_{R_\alpha} dH)\alpha \end{aligned}$$

where R_α is the Reeb vector field for the contact form α.

Proof. Write $v = fR_\alpha + w$, where f is a function and $w \in \xi$. Then the first equation gives $f = -H$, and the second gives $\iota_w d\alpha = dH - (\iota_{R_\alpha} dH)\alpha$, which is uniquely solvable for w because $d\alpha$ is nondegenerate on ξ. Thus v is unique, and it is contact since $\mathcal{L}_v\alpha = -(\iota_{R_\alpha} dH)\alpha$. □

Similarly, given a contact vector field v, we can define $H = -\iota_v\alpha$ and H, v will satisfy these equations. Thus if we have a locally defined contact vector field v, the function H will be defined on the same domain and we can use any extension of H to the rest of M to define v globally.

Definition 15. A surface $\Sigma \subset (M, \xi)$ is *convex* if there is a contact vector field which is transverse to Σ.

Proposition 16. *A surface $\Sigma \subset (M, \xi)$ is convex iff there is an embedding $\phi : \Sigma \times \mathbb{R} \hookrightarrow M$, with $\Sigma = \phi(\Sigma \times \{0\})$, such that $\phi^*(\xi)$ is invariant in the $\mathbb{R}$ direction.*

Proof. If ϕ exists and t is the $\mathbb{R}$ coordinate then ∂_t is a contact vector field on $\Sigma \times \mathbb{R}$, so $\phi_*(\partial_t)$ is the desired field and it is transverse to Σ. Conversely, suppose Σ is transverse to the contact vector field v. Take the corresponding function H and cut it off smoothly outside a small neighborhood of Σ to get a new contact vector field v' which is still transverse to Σ; if we make the neighborhood small enough then the flow of v' will be well-defined and not have any closed orbits. Then we define $\phi(p, t)$ to be the time t flow of the point $p \in \Sigma$ along v'. □

Given the $\Sigma \times \mathbb{R}$ coordinates on a neighborhood of a convex surface Σ, then, we can write the contact structure locally as

$$\alpha = \eta + f dt,$$

where η and f are a 1-form and function on Σ and the condition $\alpha \wedge d\alpha > 0$ is equivalent to $f d\eta - df \wedge \eta > 0$ on Σ. If Σ has area form ω then the characteristic foliation is determined by $\iota_X \omega = \eta$, so Σ_ξ is singular exactly where $\eta = 0$.

Definition 17. Let Σ be a surface with singular foliation $\mathcal{F}$. A multi-curve $\Gamma \subset \Sigma$ transverse to $\mathcal{F}$ *divides* $\mathcal{F}$ if Σ admits a volume form ω and a vector field v directing $\mathcal{F}$ such that $\Sigma \backslash \Gamma = \Sigma_+ \sqcup \Sigma_-$ with $\pm \mathcal{L}_v \omega > 0$ on $\Sigma_\pm$ and v points out of Σ_+ along Γ.

Example 18. If Σ is convex and has contact form $\alpha = \eta + f dt$ on a neighborhood $\Sigma \times \mathbb{R}$, then the curve $\Gamma_\Sigma = f^{-1}(0)$ divides Σ. Indeed, we take $\omega_0 = f d\eta - df \wedge \eta$ and v_0 to be the vector field X with $\iota_X \omega_0 = \eta$, which directs Σ_ξ by definition, and $\Sigma_\pm = \{p \in \Sigma \mid \pm f(p) > 0\}$. If Γ_Σ is not transverse to Σ_ξ then there is a point $p \in f^{-1}(0)$ where $df_p(X)$=0, and then $X \in \ker(f d\eta - df \wedge \eta)$ which contradicts the fact that α is contact.

Let A be a product neighborhood $\Gamma_\Sigma \times [-1, 1]$ of Γ_Σ on which the characteristic foliation consists of arcs $\{*\} \times [-1, 1]$. On $\Sigma \backslash A$ we take the area form $\omega = \frac{1}{|f|} \omega_0$ and $v = v_0$. Then one can compute

$$\pm \mathcal{L}_v \omega = \frac{1}{f^2} \omega > 0$$

on $\Sigma_\pm$, and we leave it as an exercise to extend ω, v smoothly across A.

Remark 19. Using the $\Sigma \times \mathbb{R}$ model neighborhood with contact vector field $v = \partial_t$, we have $v_p \in \xi_p$ if and only if $f = 0$. In other words, we can identify $\Gamma_\Sigma = \{p \in \Sigma \mid v_p \in \xi_p\}$. We also observe that Γ_Σ is nonempty: if $f \neq 0$ then we can rescale and take $\alpha' = \eta' + dt$ with $d\eta' > 0$, and then $\int_{\partial \Sigma} \alpha' = \int_\Sigma d\alpha' = \int_\Sigma d\eta' > 0$, which is a contradiction whenever $\partial \Sigma$ is nonempty or Legendrian.

Example 20. Consider $(\mathbb{R}^3, \xi_{\text{st}})$ with the contact form

$$\alpha = dz + x dy - y dx = dz + r^2 d\theta$$

discussed earlier. This has a contact vector field $v = x\partial_x + y\partial_y + 2z\partial_z$ (in fact, $\mathcal{L}_v\alpha = 2\alpha$), and since $\xi = \text{span}(x\partial_z - \partial_y, y\partial_z + \partial_x)$, it follows that $v \in \xi$ iff $z = 0$. Thus the unit sphere Σ is convex and divided by the equator $\Gamma_\Sigma = \{z = 0\}$.

Theorem 21. *Let $\Sigma \subset (M, \xi)$ be orientable with Legendrian boundary. Then Σ is convex if and only if the characteristic foliation Σ_ξ admits dividing curves.*

Proof. We have shown one direction: if Σ is convex, then $\Gamma_\Sigma = \{p \in \Sigma \mid v_p \in \xi_p\}$ divides Σ_ξ. Now suppose instead that Σ_ξ is divided by Γ, with area form ω and vector field v. Given a function f on Σ, the 1-form

$$\alpha = \iota_v\omega + f dt$$

defines an $\mathbb{R}$-invariant plane field on a neighborhood $\Sigma \times \mathbb{R}$ which is contact iff

$$f\mathcal{L}_v\omega - df \wedge \iota_v\omega > 0$$

on Σ, or equivalently iff $f \cdot \text{div}_\omega v - df(v) > 0$. Remove a tubular neighborhood A of Γ, and let $f = \pm 1$ on $\Sigma_\pm \backslash A$; then $df(v) = 0$ there and $f \cdot \text{div}_\omega v = \pm\text{div}_\omega v > 0$, so α is contact away from A.

Parametrize a slightly enlarged $A = \Gamma \times I$ which is still foliated by arcs $\{p\} \times [-\epsilon, \epsilon]$ so that each (p, s) is the time-s flow under $-v$ of $(p, 0)$; in other words, $v = -\partial_s$. Let

$$h(p, s) = \exp\left(-\int_0^s \text{div}_\omega v(p, t) dt\right)$$

and set $f = g \cdot h$ for some function $g(p, s)$; then

$$f\text{div}_\omega v - df(v) = gh\text{div}_\omega v + \frac{\partial(gh)}{\partial s} = \frac{\partial g}{\partial s} \cdot h$$

and so we want $\frac{\partial g}{\partial s} \cdot h > 0$, with $gh = \pm 1$ near $s = \pm\epsilon$. This can be accomplished by setting $g = \pm\frac{1}{h}$ near $s = \pm\epsilon$ and insisting that g be a smooth, increasing function in between. If we further require g to satisfy $g(p, 0) = 0$, then we will also have $\Gamma = f^{-1}(0)$. □

Theorem 22. *Let Σ be convex with two sets of dividing curves Γ_0 and Γ_1. Then Γ_0 and Γ_1 are isotopic through curves transverse to Σ_ξ.*

Proof. In the proof of the previous theorem we constructed contact forms $\alpha_i = \iota_{\partial_t}\omega + f_i dt$ on $\Sigma \times \mathbb{R}$ with $\Gamma_i = f_i^{-1}(0)$. Let $\alpha_t = \iota_{\partial_t}\omega + ((1-t)f_0 + tf_1)dt$ and check that each α_t is a contact form. Then by the Moser's trick argument mentioned above there is an isotopy $\psi_t \times \text{id}_\mathbb{R}$ of $\Sigma \times \mathbb{R}$ carrying α_0 to α_1, and each $f_t^{-1}(0)$ remains transverse to Σ_ξ, so ψ_t is the desired isotopy. □

Next time we will show that convex surfaces are everywhere, in the sense that any surface with Morse-Smale characteristic foliation has dividing curves and thus is convex. Furthermore, if Σ is convex with dividing set Γ and $\mathcal{F}$ is any foliation divided by Γ, then Σ may be perturbed to have characteristic foliation $\mathcal{F}$. Thus the dividing curves are in some sense all we need to understand the contact structure near Σ.