Math 273 Lecture 24

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April 25, 2012

In the final lecture of the semester, we will introduce Heegaard Floer homology and talk about its relation to contact geometry. We can not hope to cover the subject with any level of reasonable detail in such a short period, so all proofs will be sketched at best. For a more complete introduction to Heegaard Floer homology, one could read Ozsváth and Szabó's original papers on the subject [7, 6] or their introduction [8] and expository lectures [9].

Definition 1. Let Y be a closed 3-manifold. A *Heegaard splitting* of Y is a decomposition $Y = H_\alpha \cup H_\beta$, where H_α and H_β are handlebodies joined along their common boundary $\Sigma = \partial H_\alpha = \partial H_\beta$. The surface Σ is called a *Heegaard surface*.

Example 2. Let Y be S^3 or any lens space. Then Y has a Heegaard splitting consisting of a pair of solid tori. For example, when $Y = S^3 \subset \mathbb{C}^2$, we can take $H_\alpha = \{(z_1, z_2) \in S^3 \mid |z_1|^2 \leq \frac{1}{2}\}$ and $H_\beta = \{|z_2|^2 \geq \frac{1}{2}\}$, and the corresponding Heegaard surface is the torus $|z_1|^2 + |z_2|^2 = \frac{1}{2}$.

Proposition 3. *Every closed* 3*-manifold admits a Heegaard splitting.*

Proof 1. Take a triangulation of Y, let H_α be a neighborhood of the 1-skeleton, and let H_β be a neighborhood of the 1-skeleton of the dual triangulation. □

Proof 2. Take a self-indexing Morse function $f : Y \to [0, 3]$, so that if $p \in \mathrm{Crit}(f)$ then $f(p) = \mathrm{Ind}_f(p)$, and suppose that there is a single critical point of index 0 and a single one of index 3. Then $H_\alpha = f^{-1}([0, \frac{3}{2}])$ is a genus g handlebody, where g is the number of index 1 critical points, and $H_\beta = f^{-1}([\frac{3}{2}, 1])$. The Heegaard surface is $\Sigma_g = f^{-1}(\frac{3}{2})$. □

Proof 3. Let (B, π) be an open book decomposition of Y, and let $\Sigma_{\frac{1}{2}}, -\Sigma_1 \cong -\Sigma_0$ be two pages whose union is a smooth closed surface Σ. Then $Y \backslash \Sigma$ is a union of the handlebodies $H_\alpha = \pi^{-1}([0, \frac{1}{2}])/\sim$ and $H_\beta = \pi^{-1}([\frac{1}{2}, 1])/\sim$, giving rise to a Heegaard splitting with Heegaard surface Σ. □

Given a Heegaard decomposition $Y = H_\alpha \cup_\Sigma H_\beta$, we can describe the decomposition completely in terms of curves on the surface Σ. We construct H_α by attaching g 1-handles to $\Sigma \times [0, 1]$ along curves $\alpha_1 \times \{0\}, \dots, \alpha_g \times \{0\}$ and filling in the resulting S^2 on the boundary with a ball, and similarly we construct H_β

by attaching g 2-handles to $\Sigma \times [0,1]$ along curves $\beta_1 \times \{1\}, \ldots, \beta_g \times \{1\}$ and filling in the remaining S^2 with a ball. In the Morse theory picture, the α_i are the intersection of the stable manifolds of the index 1 critical points with Σ, and similarly the β_i come from the unstable manifolds of the index 2 points.

Definition 4. A *Heegaard diagram* for Y is a tuple $(\Sigma_g, \{\alpha_1, \ldots, \alpha_g\}, \{\beta_1, \ldots, \beta_g\}, z)$, where the α_i are mutually disjoint simple closed curves in Σ_g; the β_i are as well; these curves describe a Heegaard decomposition of Y as explained above; and z is a point in $\Sigma_g \backslash (\alpha_1 \cup \cdots \cup \beta_g)$.

The Heegaard surface Σ_g is a Riemann surface, and its g-fold symmetric product $\mathrm{Sym}^g(\Sigma_g)$ admits a natural symplectic structure and compatible almost complex structure for which the tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ are totally real, meaning that $T\mathbb{T} \cap J(T\mathbb{T}) = 0$.

Definition 5. The Heegaard Floer hat chain complex is defined as

$$\widehat{CF}(\Sigma_g, \{\alpha_i\}, \{\beta_i\}, z) = \bigoplus_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{F}\mathbf{x}$$

where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. We remark that there are several variations, including CF^+, CF^-, and CF^∞, and that these can be defined with other coefficient rings including $\mathbb{Z}$, but we will not discuss these.

Note that an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be viewed as a tuple of points $(x_1, \ldots, x_g)$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma \in S_g$.

Definition 6. Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A Whitney disk from $\mathbf{x}$ to $\mathbf{y}$ is a holomorphic map

$$u : (D^2, \partial D^2) \to (\mathrm{Sym}^g(\Sigma_g), \mathbb{T}_\alpha \cup \mathbb{T}_\beta)$$

sending i to $\mathbf{x}$, $-i$ to $\mathbf{y}$, and such that for any $x \in \partial D$ we have $u(x) \in \mathbb{T}_\alpha$ if $\mathrm{Re}(x) < 0$ and $u(x) \in \mathbb{T}_\beta$ otherwise. Let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the space of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$. Define $n_z(u)$ to be the number of points of intersection of u with the hypersurface $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$; this only depends on the homotopy class of u.

Definition 7. For any point $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, define the differential $\widehat{\partial}$ by the formula

$$\widehat{\partial}\mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1 \\ n_z(\phi) = 0}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot \mathbf{y}$$

where $\mathcal{M}(\phi)$ is the moduli space of holomorphic representatives of ϕ, and $\mu(\phi)$, the Maslov index of ϕ, is its expected dimension. The $\mathbb{R}$-action comes from viewing D^2 equivalently as an infinite strip in $\mathbb{C}$, say $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and using the translation $t \cdot u(z) = u(z + it)$.

Theorem 8. *The homology group* $\widehat{HF}(Y) = H_*(\widehat{CF}(\Sigma_g, \{\alpha_i\}, \{\beta_i\}, z))$ *is an invariant of* Y *for any Heegaard splitting satisfying an admissibility condition and for a generic perturbation of the almost complex structure.*

The proof is long and difficult, but the main idea is that any two Heegaard splittings can be related by a sequence of moves called isotopies, handleslides, and stabilizations, and in each case one can construct a chain map between the two chain complexes involved which gives an isomorphism on homology. The admissibility condition is needed to ensure that the moduli spaces involved in defining $\widehat{\partial}$ are compact.

Example 9. The 3-sphere has a Heegaard diagram consisting of a solid torus T^2 with α a meridian and β a longitude. The complex $\widehat{CF}(T^2, \alpha, \beta, z)$ has a single generator $\mathbf{x} = \alpha \cap \beta$, so $\widehat{HF}(S^3) = \mathbb{F}$.

Example 10. The product $S^1 \times S^2$ has an admissible Heegaard diagram of the form (T^2, α, β, z), where α and β are parallel curves which have been perturbed to intersect twice. The complex then has a pair of generators $\mathbf{x}$ and $\mathbf{y}$, for which there are two holomorphic disks connecting $\mathbf{x}$ to $\mathbf{y}$, so the differential is zero mod 2 and $\widehat{HF}(S^1 \times S^2) = \mathbb{F}^2$.

Heegaard Floer homology comes with some extra structure. For example, it has a natural decomposition in terms of Spin^c-structures,

$$\widehat{HF}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s}),$$

and given a smooth cobordism W from Y_1 to Y_2 there is an induced map

$$\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$$

which respects this decomposition and is defined by counting holomorphic triangles rather than bigons with given boundary conditions in some appropriate manifold. A knot $K \subset Y$ may be described by a doubly pointed Heegaard diagram $(\Sigma, \{\alpha_i\}, \{\beta_i\}, z, w)$, where the knot is determined by arcs connecting z and w in each handlebody (or equivalently by the union of the Morse flowlines through each basepoint), and the function $n_w(u)$ which counts intersections of a Whitney disk u with the hypersurface $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$ provides a filtration on $\widehat{CF}(Y)$ whose associated graded object is the knot Floer homology $\widehat{HFK}(Y, K)$.

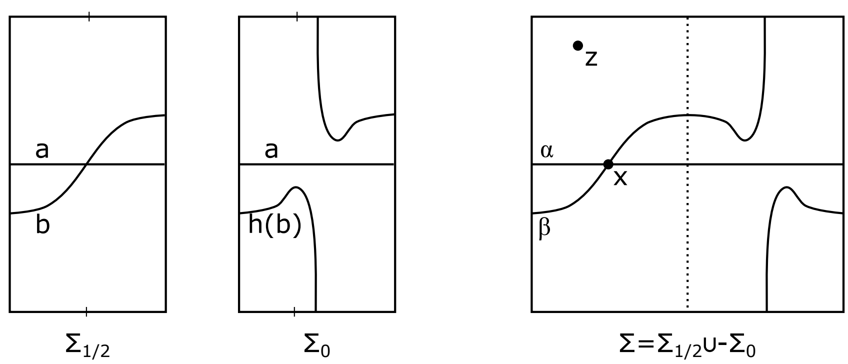
Using the Giroux correspondence, Ozsváth and Szabó [4] associated an invariant $c(\xi) \in \widehat{HF}(-Y)$ to any contact structure ξ on Y. Honda, Kazez, and Matić [3] gave an alternate construction of $c(\xi)$, which we will discuss here.

Let (B, π) be an open book decomposition supporting (Y, ξ) and take the Heegaard surface $\Sigma = \Sigma_{\frac{1}{2}} \cup -\Sigma_0$ as above, giving a Heegaard splitting $H_\alpha = \pi^{-1}([0, \frac{1}{2}])/\sim$ and $H_\beta = \pi^{-1}([\frac{1}{2}, 1])/\sim$. Let $a_1, \ldots, a_r$ be a set of arcs which cut Σ_0 into a disk. Define a set of arcs $b_1, \ldots, b_r$ such that each b_i is isotopic to a_i by a small isotopy; this isotopy pushes ∂a_i in the direction determined by the orientation of $\partial \Sigma_0$ to get ∂b_i; and $a_i \cap b_i$ consists of a single point x_i where the arcs intersect transversely with positive sign. Then the curves $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ in Σ specify a handle decomposition of H_α, and the curves $\beta_i = \partial(b_i \times [\frac{1}{2}, 1])$ likewise give a handle decomposition of H_β.

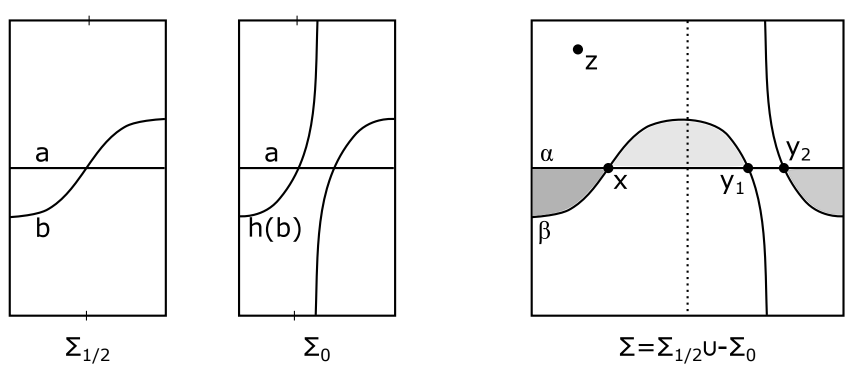
In terms of a Heegaard diagram, if we view Σ as the boundary of $H_\alpha = (\Sigma_0 \times [0, \frac{1}{2}])/\sim$ then each curve α_i is the union of $a_i \subset \Sigma_0$ and $a_i \subset \Sigma_{\frac{1}{2}}$, and each curve β_i is the union of $b_i \subset \Sigma_{\frac{1}{2}}$ and $h(b_i) \subset \Sigma_0$. (Recall that we glue the mapping torus of h together by identifying $b_i \subset \Sigma_1$ with $h(b_i) \subset \Sigma_0$.) We can place a basepoint z in $\Sigma_{\frac{1}{2}}$ outside all of the thin strips cobounded by α_i and β_i, and identify a distinguished generator $\mathbf{x} \in \widehat{CF}(\Sigma, \{\beta_i\}, \{\alpha_i\}, z)$ as the set $\{x_1, \ldots, x_r\}$ where we view each $x_i = a_i \cap b_i$ inside $\Sigma_{\frac{1}{2}}$.

Remark 11. Note that we have switched the order of $\{\alpha_i\}$ and $\{\beta_i\}$ in the chain complex. If $(\Sigma, \{\alpha_i\}, \{\beta_i\})$ is a Heegaard diagram for Y, then $(\Sigma, \{\beta_i\}, \{\alpha_i\})$ is a Heegaard diagram for $-Y$: if the former comes from a self-indexing Morse function $f : Y \to [0, 3]$, then the latter comes from $3 - f : -Y \to [0, 3]$.

As an example, we consider two open books for S^3. The first is an annulus with monodromy a right-handed Dehn twist, which supports the standard tight contact structure ξ_{std}, and the Heegaard surface is a torus.



Here the element $\mathbf{x}$ is the unique generator of $\widehat{CF}(T^2, \{\beta\}, \{\alpha\}, z)$, so $\widehat{\partial}\mathbf{x} = 0$ and $[\mathbf{x}]$ generates $\widehat{HF}(-S^3) = \mathbb{F}$. On the other hand, if we take the monodromy to be a left-handed Dehn twist, this open book still corresponds to S^3 and supports some contact structure ξ_{ot}, but this time we get a larger chain complex:



Now $\widehat{CF}(T^2, \{\beta\}, \{\alpha\}, z)$ has three generators $\mathbf{x}, \mathbf{y_1}, \mathbf{y_2}$, and we have highlighted Whitney disks which show that $\widehat{\partial}\mathbf{y_1} = \widehat{\partial}\mathbf{y_2} = \mathbf{x}$. In this chain complex, the class $[\mathbf{y_1} + \mathbf{y_2}]$ generates $\widehat{HF}(-S^3)$ and $[\mathbf{x}] = 0$.

Proposition 12. *The element* $\mathbf{x}$ *is a cycle in* $\widehat{CF}(\Sigma, \{\beta_i\}, \{\alpha_i\}, z)$*, i.e.* $\widehat{\partial}\mathbf{x} = 0$.

Proof. Consider a holomorphic disk $u : D^2 \to \mathrm{Sym}^g(\Sigma)$ from $\mathbf{x}$ to another generator $\mathbf{y}$ which contributes to $\widehat{\partial}\mathbf{x}$. By identifying $u(p) \in \mathrm{Sym}^g(\Sigma)$ with a set of g points of Σ, some of which may coincide, we lift u to a map $\hat{u} : \hat{D} \to \Sigma$ where $\hat{D}$ is a branched cover of D^2. Since $u(\partial D^2) \subset \mathbb{T}_\alpha \cup \mathbb{T}_\beta$ we know that $\hat{u}(\partial\hat{D}) \subset (\bigcup \alpha_i) \cup (\bigcup \beta_i)$, and since $n_z(u) = 0$ we also see that z is not in the image of $\hat{u}$. In particular, if the image of $\hat{u}$ intersects in a region of $\Sigma \backslash (\alpha_1 \cup \cdots \cup \beta_g)$ then it must contain that whole region, since holomorphic maps are open, and since it avoids the region of $\Sigma_{\frac{1}{2}}$ containing z we conclude that $\mathrm{Image}(\hat{u}) \cap \Sigma_{\frac{1}{2}}$ is a union of the thin strips between each pair a_i and b_i.

We now claim that the image of $\hat{u}$ has the "wrong" orientation to be counted as a holomorphic disk. Let δ be an oriented arc of $\partial\hat{D}$ which includes in its interior a corner p of $\partial\hat{D}$ such that $\widehat{u}(p) = x_i \in \mathbf{x}$ for some i. Since u is a disk from $\mathbf{x}$ to $\mathbf{y}$, as we travel along δ the image of $\hat{u}$ must pass from α_i to β_i. However, the thin strips are oriented in the opposite way: if u is orientation-preserving, then along any region of the image of $\widehat{u}$ inside a thin strip, the boundary orientation forces us to travel from β_i to α_i near x_i. This means that there are no disks counted in the definition of $\widehat{\partial}\mathbf{x}$ after all. □

Theorem 13. *The homology class* $c(\xi) = [\mathbf{x}] \in \widehat{HF}(-Y)$ *is an invariant of the contact structure* ξ*; that is, it does not depend on the open book decomposition or the choice of arcs* a_i.

In particular, for the contact structures ξ_{std} and ξ_{ot} on S^3 supported by the above open books we have $c(\xi_{\mathrm{std}}) = 1 \in \mathbb{F}$ and $c(\xi_{\mathrm{ot}}) = 0$.

The independence of the choice of arcs a_i follows from relating the invariant for one choice of arcs to the invariant for another by a series of handleslides, which give isomorphisms of $\widehat{HF}(-Y)$. At this point we can use the Giroux correspondence: it is enough to show invariance under positive stabilization of the open book. This is not the strategy used in [3], but Honda, Kazez, and Matić prove in [2, Section 3.2] in slightly more generality (for partial open books and sutured Floer homology) that a positive stabilization of an open book corresponds to a series of isotopies, handleslides, and stabilizations of the associated Heegaard diagram, and that the corresponding isomorphisms of $\widehat{HF}(-Y)$ carry the contact element for the original open book to the contact element for the stabilized one.

Proposition 14. *Let* (Y', ξ') *be obtained from* (Y, ξ) *by Legendrian surgery on a knot* $K \subset Y$*. Then the map*

$$\widehat{HF}(-Y') \to \widehat{HF}(-Y)$$

corresponding to the associated 2*-handle cobordism sends* $c(\xi')$ *to* $c(\xi)$.

Proof (sketch). We can construct an abstract open book decomposition (S, h) for (Y, ξ) for which K is a nonseparating curve in a page, by using a contact cell decomposition with K in the 1-skeleton. Then one can show that (Y', ξ') is supported by the open book $(S, h \circ \tau_K)$, where τ_K is a right-handed Dehn twist along $K \subset S$. (We leave it as an exercise to prove this, at least topologically; note that since the Reeb vector field is transverse to the page containing K, both the page framing and the contact framing of K agree.)

Take a basis of arcs $a_1, \dots, a_r$ for S with push-offs $b_1, \dots, b_r$ so that K intersects b_1 exactly once, is parallel to b_2, and misses $b_3, \dots, b_r$. Then if (S, h) gives us a Heegaard diagram $(\Sigma, \{\beta_i\}, \{\alpha_i\}, z)$ for $-Y$, the Heegaard diagram $(S, h \circ \tau_K)$ associated to $-Y'$ has the form $(\Sigma, \{\beta_1', \beta_2, \dots, \beta_r\}, \{\alpha_i\}, z)$ where β_1 and β_1' differ only inside $-S_0 \subset \Sigma$. The surgery cobordism map then has a very simple description, and one can verify that it takes $c(\xi')$ to $c(\xi)$. □

Corollary 15. *If (Y, ξ) is overtwisted then $c(\xi) = 0$.*

Proof. Since ξ is overtwisted, there is a Legendrian link $L \subset (Y, \xi)$ on which Legendrian surgery gives us the contact manifold (S^3, ξ_{ot}), and so repeated application of the above proposition gives a map $\widehat{HF}(-S^3) \to \widehat{HF}(-Y)$ carrying $c(\xi_{\mathrm{ot}}) = 0$ to $c(\xi)$. □

Proposition 16. *If (Y, ξ) is Stein fillable then $c(\xi) \neq 0$.*

Proof. Stein fillable contact structures are the result of Legendrian surgery on a Legendrian link $L \subset (\#^k(S^1 \times S^2), \xi_{\mathrm{std}})$ for some $k \geq 0$, so it suffices to check that

$$c(\xi_{\mathrm{std}}) \in \widehat{HF}(-\#^k(S^1 \times S^2))$$

is nonzero. When $k = 0$ this is the fact that $(S^3, \xi_{\mathrm{std}})$ has nonzero invariant, which we have already seen. For $k = 1$ we can use the fact that the open book with annular pages and trivial monodromy supports a contact structure on $S^1 \times S^2$ with nonzero invariant, so it must be ξ_{std} because all others are overtwisted. Similarly, for $k > 1$ the desired contact structure comes from an open book with trivial monodromy and one can check in this case that $c(\xi_{\mathrm{std}}) \neq 0$. □

Thus the Heegaard Floer contact invariant detects tight contact structures: if $c(\xi) \neq 0$ then ξ must be tight, and for example this is always the case when ξ is Stein fillable. This nonvanishing result has been strengthened to strongly symplectically fillable contact structures [1], and to weakly fillable structures if one uses an appropriate system of twisted coefficients [5], but there are also many nonfillable tight contact structures for which $c(\xi) \neq 0$. This has played a very prominent role in contact geometry in recent years, both in helping to classify tight contact structures and in topological applications such as a proof of the fact that Heegaard Floer homology detects the Thurston norm [5].

References

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