Math 273 Lecture 21

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At the beginning of the course, we used open book decompositions to show that every closed 3-manifold admits a contact structure. There is a strong relationship between these two notions, due to Giroux [1], which we will now investigate; first, we recall some definitions as well as Thurston and Winkelnkemper's proof that contact structures exist.

Definition 1. An *open book decomposition* of a closed 3-manifold Y is a pair $(B, \pi : Y \backslash B \to S^1)$, where $B \subset Y$ is an oriented link called the *binding* and π is a fibration. The fibers $\pi^{-1}(\theta)$ are the interiors of compact surfaces Σ_θ called *pages*, which satisfy $B = \partial\Sigma_\theta$ for all θ.

Theorem 2 ([3]). *An open book decomposition (B, π) gives rise to a contact structure on Y.*

Proof (sketch). Let Σ be the page of the open book, and write $Y \backslash N(B)$ as the mapping torus of $h : \Sigma \to \Sigma$. Take a 1-form λ on Σ such that $d\lambda$ is an area form and $\lambda = (1 + t)d\theta$ on a neighborhood $[-1, 0]_t \times S^1_\theta$ of each component of $\partial\Sigma$; the space of all such λ is convex and nonempty. Then the 1-form

$$\alpha = \phi\lambda + (1 - \phi)h^*\lambda + Kd\phi$$

is a contact form on $\Sigma \times [0, 1]_\phi$ for K large, and it descends to a contact form on the mapping torus $Y \backslash N(B)$. Near $\partial N(B)$ it is equal to $(1 + t)d\theta + Kd\phi$.

Now on each solid torus component $S^1_\theta \times D^2_{(r,\phi)}$ of $N(B)$, we need to identify a contact form α which is equal to $(2 - r)d\theta + Kd\phi$ near $r = 1$; we will take $\alpha = f(r)d\theta + g(r)d\phi$ for some functions f and g which equal $2 - r$ and K respectively near $r = 1$. We pick f and g to satisfy $f(r) = 1$ and $g(r) = r^2$ near $r = 0$ and then choose any extension of (f, g) over the rest of $[0, 1]$ which satisfies $fg' - gf' > 0$; this is equivalent to the contact condition $\alpha \wedge d\alpha > 0$, so we are done. □

Near any component of the binding B we have $\alpha = d\theta + r^2 d\phi$, where B given in coordinates by $r = 0$ and ∂_θ is tangent to B with the same orientation it inherits as $\partial\Sigma_\theta$. Since $\alpha(\partial_\theta) = 1$, B is a positively transverse link. Furthermore, on any page Σ_θ we have $d\alpha|_{\Sigma_\theta} = \phi d\lambda + (1-\phi)h^*d\lambda$, which is an area form because both $d\lambda$ and $h^*d\lambda$ are. We describe these properties as follows:

Definition 3. A contact structure ξ on Y is *supported* by the open book (B, π) if up to isotopy it admits a contact form α such that B is a positively transverse link and $d\alpha$ is a positive area form on each page Σ_θ.

Thus Theorem 2 actually provides a contact structure supported by the given open book.

Example 4. Consider S^3 as the unit sphere in $\mathbb{C}^2$, with the standard contact form $\alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2$. The positive Hopf link $H \subset S^3$ can be expressed as the zero set of the polynomial $f(z_1, z_2) = z_1 z_2$, and then the map

$$\pi(z_1, z_2) = \frac{z_1 z_2}{|z_1 z_2|}$$

defines a fibration $\pi : S^3 \backslash H \to S^1$. Each page $\Sigma_\phi = \pi^{-1}(\phi)$ can be parametrized by a map

$$f(r, \theta) = (\sqrt{t}, \theta, \sqrt{1-t}, \phi - \theta)$$

in polar coordinates, i.e. $z_1 = \sqrt{t}e^{i\theta}$ and $z_2 = \sqrt{1-t}e^{i(\phi-\theta)}$; here $0 < t < 1$ and $\theta \in S^1$, so the pages are annuli. Then

$$f^*\alpha = td\theta + (1-t)(-d\theta) = (2t-1)d\theta$$

and so $f^*d\alpha = 2dt \wedge d\theta$, which is an area form on $[0,1]_t \times S^1_\theta$, so $d\alpha$ is an area form on each page. Furthermore, the components $\{r_1 = 0\}$ and $\{r_2 = 0\}$ of H have oriented tangent vectors ∂_{θ_2} and ∂_{θ_1} respectively, at which points $\alpha(\partial_{\theta_2}) = r_2^2 \iota_{\partial_{\theta_2}} d\theta_2 = 1$ and $\alpha(\partial_{\theta_1}) = r_1^2 \iota_{\partial_{\theta_1}} d\theta_1 = 1$ are positive. Thus (H, π) supports the standard tight contact structure ξ_{st} on S^3.

Exercise 5. Show that the open book decomposition of S^3 whose binding is the unknot $U = \{r_1 = 0\}$ and whose pages are the disks $\{\theta_1 = \text{const.}\}$ also supports ξ_{st}.

Proposition 6. *Let ξ and ξ' be contact structures supported by the same open book (B, π). Then ξ is isotopic to ξ'.*

Proof. We will first construct isotopies of these contact structures so that they have sufficiently nice contact forms. Apply an isotopy so that ξ has a contact form α with the desired properties. Then we let $d\phi$ be an area form on S^1 and pull it back to get a form $\pi^* d\phi$ on $Y \backslash B$. In a neighborhood $N = S^1 \times D^2$ of any component B_i of the binding, we can choose coordinates $(\theta, (r, \phi))$ so that $d\phi$ agrees with the form $\pi^* d\phi$, and since $\alpha(\partial_\theta) > 0$ along B_i we can also insist that $\alpha(\partial_\theta) > 0$ on all of N by possibly shrinking it. Take $\epsilon > 0$ small enough that the solid torus $N' = \{r < \epsilon\}$ lies inside N, and define a smooth nondecreasing function $f : [0, \epsilon] \to [0, 1]$ so that $f(r) = r^2$ near $r = 0$ and $f(r) = 1$ near $r = \epsilon$. Then f defines a function on N' which we can extend to all of Y by letting $f = 1$ outside any of the solid tori N'.

For any $t \geq 0$, we define a new 1-form $\alpha_t = \alpha + tf(r)d\phi$. We can check that

$$\alpha_t \wedge d\alpha_t = \alpha \wedge d\alpha + tf(r) d\phi \wedge d\alpha + t\alpha \wedge df \wedge d\phi.$$

Note that $tf(r)d\phi \wedge d\alpha \geq 0$, because $d\alpha$ is a volume form on each page and $d\phi$ is dual to a vector which is positively transverse to each page. Furthermore, $\alpha \wedge df \wedge d\phi$ is zero outside the neighborhoods N' (where $df = 0$) and equal to $tf'(r) \cdot \alpha \wedge dr \wedge d\phi$ on N'; since $dr \wedge d\phi$ is nonnegative on D^2 and zero on ∂_θ, while $\alpha(\partial_\theta) > 0$, we conclude that $t\alpha \wedge df \wedge d\phi \geq 0$ as well. Therefore α_t is actually a contact form for any $t > 0$, and since $\alpha = \alpha_0$ these all define contact structures isotopic to ξ. We use the same construction, with the same neighborhoods N' of B, to find contact forms α'_t for ξ' up to isotopy.

Finally, we take t to be very large and define an isotopy from ξ to ξ' by the family of contact forms $\alpha_s = (1-s)\alpha_t + s\alpha'_t$. Then we compute

$$\alpha_s \wedge d\alpha_s = \left[(1-s)^2\alpha_t \wedge d\alpha_t + s^2\alpha'_t \wedge d\alpha'_t\right] + s(1-s)\left[\alpha_t \wedge d\alpha'_t + \alpha'_t \wedge d\alpha_t\right]$$

and the first term in brackets is clearly positive, so we must show that the second term in brackets is nonnegative. Outside the neighborhoods N', where $f = 1$ and $df = 0$, we have

$$\alpha_t \wedge d\alpha'_t + \alpha'_t \wedge d\alpha_t = (\alpha + td\phi) \wedge d\alpha' + (\alpha' + td\phi) \wedge d\alpha = td\phi \wedge (d\alpha + d\alpha') + O(1)$$

which is positive for t large enough. Inside the neighborhoods N' , we have

$$\begin{aligned} \alpha_t \wedge d\alpha'_t + \alpha'_t \wedge d\alpha_t &= (\alpha + tfd\phi) \wedge (d\alpha' + tf'dr \wedge d\phi) + (\alpha' + tfd\phi) \wedge (d\alpha + tf'dr \wedge d\phi) \\ &= t(f' \cdot (\alpha + \alpha') \wedge dr \wedge d\phi + fd\phi \wedge (d\alpha + d\alpha')) + O(1). \end{aligned}$$

Note that f' and f are both nonnegative. Near $r = 0$ we have $f'dr \wedge d\phi = 2rdr \wedge d\theta$, which is a positive area form on D^2 and so the first term in parentheses is positive; then away from $r = 0$ we have $f > 0$ and so the second term is positive. Thus for t large enough this form is positive on all of N', and so α_s is a contact form for $0 \leq s \leq 1$. We conclude that it gives the desired isotopy between ξ and ξ'. □

We have now shown that every open book decomposition of Y supports a unique contact structure up to isotopy. The Giroux correspondence provides a converse: every contact structure is supported by an open book, and furthermore if two open books support the same contact structure then they are related by a series of moves called positive stabilizations. Before proving this, we will consider some equivalent ways to describe the contact structure supported by an open book.

Proposition 7. *The contact structure (Y, ξ) is supported by the open book (B, π) if and only if up to isotopy, ξ admits a contact form whose Reeb vector field R is positively tangent to B and positively transverse to the pages of π.*

Proof. Suppose we have a contact form α with such a Reeb vector field R, and recall that $\alpha(R) = 1$ and $\iota_R d\alpha = 0$ by definition. Then $\alpha(R) > 0$ along B, so B is a positively transverse link. At any point p on a page Σ_θ, we have $\iota_R(\alpha \wedge d\alpha) = (\iota_R\alpha)d\alpha - \alpha \wedge \iota_R d\alpha = d\alpha$, and so given a positive basis (v_1, v_2) for $T_p\Sigma_\theta$ we have

$$d\alpha(v_1, v_2) = (\alpha \wedge d\alpha)(R, v_1, v_2) > 0$$

so $d\alpha$ is an area form on Σ_θ. Therefore (B,π) supports $\xi = \ker(\alpha)$.

Conversely, suppose that (Y,ξ) is supported by (B,π), so that there is a contact form α for which B is positively transverse and $d\alpha$ is positive on each page. Let R be the Reeb vector field of α. Then at any point on a page, the fact that $\iota_R(\alpha\wedge d\alpha) = d\alpha$ is an area form on the page implies that R must be positively transverse to that page. Furthermore, on a neighborhood $S^1_\theta \times D^2$ of a component of B, we can write $R = a\partial_\theta + b\partial_x + c\partial_y$ where (x,y) are rectangular coordinates on D^2. If at $(x,y)=(0,0)$ we have $(b,c)\neq(0,0)$, then either b or c is nonzero on an entire neighborhood of the point $(\theta,0,0)$. The oriented normal vectors to the pages Σ_0, $\Sigma_{\pi/2}$, Σ_π, and $\Sigma_{3\pi/2}$ at points a distance ϵ away from B are given by ∂_y, $-\partial_x$, $-\partial_y$, and ∂_x respectively, and so these pages cannot be positively transverse to R in the respective cases $c<0$, $b>0$, $c>0$, and $b<0$. We conclude that in fact $(b,c)=(0,0)$ along B, so the Reeb vector has the form $R = a\partial_\theta$, and then $a\cdot\alpha(\partial_\theta) = \alpha(R) = 1$ and $\alpha(\partial_\theta) > 0$ together imply that $a>0$, so R is positively tangent to B. □

Definition 8. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with $f(0,0)=0$ and an isolated singularity at the origin. Then for $\epsilon > 0$ small, the sphere S^3 of radius ϵ centered at the origin admits a *Milnor open book* (B,π), where $B = f^{-1}(0)\cap S^3$ and $\pi : S^3\backslash B \to S^1$ is the *Milnor fibration*

$$\pi(z_1,z_2) = \frac{f(z_1,z_2)}{|f(z_1,z_2)|}.$$

Proposition 9. *A Milnor open book on S^3 supports the tight contact structure.*

Proof. We will give a proof in the case where f is homogeneous. Recall that the standard contact structure ξ_{st} is given by $TS^3 \cap i(TS^3)$ corresponding to the strictly plurisubharmonic exhaustion function $\phi(z_1,z_2) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right)$ on B^4.

We wish to show that the Reeb vector field R is positively transverse to the pages of π, so first we must determine R; we will compute R for the boundary of a general Stein domain and then specialize to the case $S^3 = \partial B^4$. Given an spsh function $\phi : X \to \mathbb{R}$ and complex structure J, the induced contact form on the level set ∂X is $\alpha = -d\phi \circ J$. Write the Reeb vector field in the form $R = Jv$ for some vector v; then we have an induced metric $\langle x,y\rangle = d\alpha(x,Jy)$ on X, so $(\iota_R d\alpha)(x) = d\alpha(Jv,x) = -\langle x,v\rangle$. Since $\iota_R d\alpha = 0$ on TS^3, the vector v must be orthogonal to all of $T(\partial X)$, hence it is a multiple of $\nabla\phi$. Furthermore, R satisfies $\alpha(R) = -d\phi(JR) = 1$, so $\langle \nabla\phi, -J^2 v\rangle = -1$ or simply $\langle\nabla\phi, v\rangle = 1$. Therefore $R = J\frac{\nabla\phi}{|\nabla\phi|^2}$.

Next, we observe that

$$\nabla\phi = \sum_{i=1}^{2}(x_i\partial_{x_i} + y_i\partial_{y_i}),$$

so since f is homogeneous it follows that $df(\nabla\phi) = (\deg f)f$. In particular, along the level set $B = f^{-1}(0)$ we have $df(R) = df(\frac{i}{|\nabla\phi|^2}\nabla\phi) = 0$ because

$df(\nabla\phi) = (\deg f)\cdot 0 = 0$, and so R is tangent to B. Next, by writing f in polar coordinates and taking logarithmic derivatives, we see that $d\pi = \mathrm{Im}(f^{-1}df)$. We want to show that R is transverse to the pages of π, so it will suffice to show that $d\pi(R) > 0$, or equivalently that $d\pi(i\nabla\phi) > 0$ since the two differ by a factor of $|\nabla\phi|^2$. Now we can compute

$$d\pi(i\nabla\phi) = \mathrm{Im}(f^{-1}df(i\nabla\phi)) = \mathrm{Re}(f^{-1}df(\nabla\phi)) = \mathrm{Re}(\deg f) = \deg f > 0$$

as desired. Therefore (B,π) supports ξ_{st}.

Now suppose f is not homogeneous. We replace $\phi(z_1,z_2) = |z_1|^2 + |z_2|^2$ with the function

$$\psi(z_1,z_2) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2 + \frac{1}{c}|f(z_1,z_2)|^2\right)$$

and let S^3 be the level set $\psi^{-1}(\frac{r^2}{2})$, where c and r are some positive constants. The Reeb vector field on S^3 is now $R = i\frac{\nabla\psi}{|\nabla\psi|^2}$, where

$$\nabla\psi = \nabla\phi + \frac{1}{c}f\nabla f,$$

and since $d\pi = \mathrm{Im}(f^{-1}df)$ as before we have

$$d\pi(R) = \frac{1}{|\nabla\psi|^2}\mathrm{Re}\left(f^{-1}df(\nabla\phi + \frac{1}{c}f\nabla f)\right) = \frac{1}{|\nabla\psi|^2}\mathrm{Re}\left(\frac{df(\nabla\phi)}{f} + \frac{df(\nabla f)}{c}\right).$$

Since $df(\nabla f) = |\nabla f|^2$ and $\frac{1}{c} = \frac{r^2-|z|^2}{|f|^2}$ along $S^3 = \psi^{-1}(\frac{r^2}{2})$, we conclude that up to a factor of $|\nabla\psi|^2$, $d\pi(R)$ is the real part of

$$\frac{df(\nabla\phi)}{f} + (r^2 - |z|^2)\left|\frac{\nabla f}{f}\right|^2.$$

We leave it as an exercise to show that this quantity can also be made positive on $S^3\backslash f^{-1}(0)$ by taking r sufficiently small and c sufficiently large compared to r. □

References

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[3] W. P. Thurston and H. E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. 52 (1975), 345–347.