Math 273 Lecture 20

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In this lecture we continue the classification of overtwisted contact structures on a closed 3-manifold Y. Let ξ and ξ' be overtwisted contact structures which are homotopic as plane fields. Our strategy will be to find a nice triangulation T of Y, in which the 1-skeleton $T^{(1)}$ is Legendrian, the 2-skeleton $T^{(2)}$ is convex, and each 3-ball is overtwisted in both ξ and ξ'. We call such T an *overtwisted* triangulation *of* Y. *If* we can get ξ and ξ' to agree on the 2-skeleton, then we can reduce the problem to studying overtwisted contact structures on B^3.

Proposition 1. *Let Y have overtwisted contact structures ξ and ξ' and a fixed triangulation T. If ξ and ξ' have the same Euler class, then we can find an isotopy ϕ_t of Y such that $(\phi_1)_*\xi = \xi'$ in a neighborhood of $T^{(2)}$ and T is overtwisted with respect to both $(\phi_1)_*\xi$ and ξ'.*

Proof. We will perform the isotopy by moving T around with respect to one contact structure or the other and then applying Gray stability to realize this as an isotopy of Y. We apply a C^0-small perturbation of T to make $T^{(1)}$ Legendrian with respect to either contact structure, stabilizing edges as needed so that $\xi = \xi'$ near $T^{(1)}$ and each face D of $T^{(2)}$ has $tb(\partial D) < 0$. Then a C^∞-small perturbation lets us make each face convex with dividing set Γ_D having endpoints on the interior of an edge. It will suffice to show that for every D, Γ_D is the same up to isotopy for both contact structures, and that Γ_D contains a simple closed curve; then by Giroux's criterion either 3-ball adjacent to D will be overtwisted.

We can assume by an isotopy of either contact structure that there is an overtwisted disk Δ in some fixed simplex σ of the triangulation. Let σ' be another simplex which shares a face $D \subset T^{(2)}$ with σ. Take a parallel copy Δ' of Δ in an I-invariant neighborhood, and push D by an isotopy fixing ∂D so that it contains Δ' and is still convex. Now D splits $\sigma \cup \sigma'$ into two new simplices $\sigma_1 \cup \sigma_1'$, both of which are overtwisted because they contain I-invariant neighborhoods of the overtwisted Δ', and we achieved this by an isotopy which fixed all of $T^{(2)}$ except the interior of D. Repeating this finitely many times since Y is compact, we can now assume that every simplex σ is overtwisted.

The last step is to ensure that the dividing sets on $T^{(2)}$ with respect to ξ and ξ' are identical. Note that if we stabilize some arc of $T^{(1)}$, this adds a boundary-parallel arc to either the positive region or the negative region of each adjacent face D depending on the sign of the stabilization; thus we can change $\chi(R_+(\Gamma_D)) - \chi(R_-(\Gamma_D))$ by ± 1 for each face D containing the stabilized arc.

We define a 2-cocycle δ on the faces of T by the formula

$$\delta(D) = [\chi(R_+(\Gamma_D^\xi)) - \chi(R_-(\Gamma_D^\xi))] - [\chi(R_+(\Gamma_D^{\xi'})) - \chi(R_-(\Gamma_D^{\xi'}))].$$

Then $[\delta] = e(\xi) - e(\xi') = 0$, and δ is even since an Euler class of a contact structure is even, so $\delta = 2d\theta$ for some 1-cocycle $\theta \in \mathrm{Hom}(C_1(M), \mathbb{Z})$. For a fixed 1-simplex $\gamma \subset T^{(1)}$, we let $n = \theta(\gamma)$. If $n > 0$, we change the triangulation with respect to ξ by stabilizing γ positively n times and with respect to ξ' by stabilizing n times; if $n < 0$ then we stabilize each in the opposite direction. Repeating this for each arc of $T^{(1)}$ preserves the property that $\xi = \xi'$ in a neighborhood of $T^{(1)}$, and also ensures that

$$\chi(R_+(\Gamma_D^\xi)) - \chi(R_-(\Gamma_D^\xi)) = \chi(R_+(\Gamma_D^{\xi'})) - \chi(R_-(\Gamma_D^{\xi'}))$$

for each face D of $T^{(2)}$.

At this point we use the generalized Right-to-Life Principle to get the dividing sets on each face to agree. Fix a face D along the boundary of the overtwisted simplex σ. We can find a series of bypass moves which convert Γ_D^ξ into $\Gamma_D^{\xi'}$, and since each of these corresponds to a real bypass, we can achieve these moves by an isotopy of D which fixes its boundary. Repeating this procedure along each face of $T^{(2)}$ completes the proof. □

Now suppose we have homotopic overtwisted contact structures ξ and ξ' and an overtwisted triangulation T of Y such that ξ and ξ' agree on a neighborhood N of $T^{(2)}$. Then $Y\backslash N$ is a union of overtwisted balls, and if we connect pairs of them by standard neighborhoods of Legendrian arcs until they are all connected then we get an overtwisted 3-ball B with respect to either structure so that $\xi = \xi'$ on its complement.

If ∂B has more than one dividing curve (note that $\Gamma_{\partial B}$ is the same for both ξ and ξ'), then we claim it has at least three. Indeed, if $\#\Gamma_{\partial B} = 2$ then ∂B has positive and negative regions an annulus and a pair of disks, in some order, so that $\langle e(\xi), \partial B\rangle = \chi((\partial B)_+) - \chi((\partial B)_-) = \pm 2$, but $\langle e(\xi), \partial B\rangle = 0$ because ∂B is null-homologous. Therefore we can find a bypass arc α which intersects three different dividing curves (if not, say if $(\partial B)_+$ is a pair of pants, then a similar Euler characteristic argument applies). Since we made the faces of $T^{(2)}$ have closed dividing curves, $Y\backslash B$ is overtwisted, so there exists a bypass in $Y\backslash B$ along α, and we can add this bypass to B to decrease $\#\Gamma_{\partial B}$ by two. Again, this works simultaneously in both contact structures because they coincide on $Y\backslash B$. By repeating this process until a single dividing curve remains, we have shown:

Lemma 2. *If ξ and ξ' are overtwisted and have the same Euler class then there is an embedded ball $B \subset M$ with convex boundary such that $\xi|_{Y\backslash B} = \xi'|_{Y\backslash B}$, there is a single dividing curve on ∂B with respect to both contact structures, and each contact structure is overtwisted on both B and $Y\backslash B$.*

Take Darboux balls inside B with respect to either contact structure; these are isotopic rel boundary, so if we remove them as well as $Y\backslash B$ we are left with homotopic overtwisted contact structures ξ and ξ' on $S^2 \times I$ which agree on a neighborhood of $S^2 \times \partial I$ and have one dividing curve on each boundary component. Now we have shown that ξ is stably isotopic to some multiple bypass triple attachment Δ^{n_0}, i.e. $\xi \circ \Delta^a = \Delta^{a+n_0}$ for some a, and similarly $\xi \circ \Delta^b = \Delta^{b+m_0}$ for some b and m_0. Since $Y\backslash B$ is overtwisted, we may take any admissible arc on $\partial B = S^2 \times \{1\}$ and use the Right-to-Life Principle to find $\max(a, b)$ bypass triples along that arc inside $Y\backslash B$; attaching them all to B, we now have $\xi|_{S^2\times I} = \Delta^n$ and $\xi'|_{S^2\times I} = \Delta^m$ up to actual isotopy.

We now recall the Pontryagin-Thom construction, generalized to manifolds M with boundary. Given a trivialization of the tangent bundle TM, a co-oriented plane field ξ on M determines a smooth map $f : M \to S^2$ by sending each point p to the oriented unit normal to ξ_p. We pick a regular value $c \in S^2$ and a basis $\mathfrak{b}$ of T_cS^2, and we associate to f the framed link $L_f = f^{-1}(c)$ with basis $\mathfrak{b}_f = f^*\mathfrak{b}$ for its normal bundle. Note that $\xi \cong f^*(TS^2)$, so $e(\xi) = f^*(e(TS^2))$; in other words, $f^* : \mathbb{Z} \to H^2(M;\mathbb{Z})$ sends 2 to the Euler class of M. Since $1 \in H^2(S^2, \mathbb{Z})$ is Poincaré dual to a regular value c of f, it follows that $f^*(1)$ is dual to the preimage $f^{-1}(c) = L_f$ and so $e(\xi) = 2 \cdot PD(L_f)$.

Theorem 3 ([3])**.** *Two plane fields ξ, ξ' on M are homotopic rel boundary if and only if for any common regular value c of the corresponding maps $f, f' : M \to S^2$, the framed links $(L_f, \mathfrak{b}_f)$ and $(L_{f'}, \mathfrak{b}_{f'})$ are relatively framed cobordant, meaning that they are related by a framed cobordism $(\Sigma, \mathfrak{b}) \subset M \times I$ which is constant along $\partial M \times I$. Furthermore, it suffices to check this for a single common regular value.*

If two plane fields ξ, ξ' agree on the complement of a ball B^3, we may consider their relative Pontryagin submanifolds $(L, \mathfrak{b}), (L', \mathfrak{b}') \subset B^3$. We have a relative cobordism Σ from L to L', since B^3 is contractible, but we may not be able to extend the framings across Σ. Thus we remove a small disk with unknotted boundary U from $\mathrm{int}(\Sigma)$, and we can give U a framing δ for which $(L, \mathfrak{b})$ is relatively framed cobordant to $(L', \mathfrak{b}') \cup (U, \delta)$.

Definition 4. Given plane fields ξ, ξ' as above, let d be the divisibility of $e = e(\xi) = e(\xi')$. We define the obstruction class

$$d^3(\xi, \xi') \in \mathbb{Z}/d\mathbb{Z}$$

to be the linking number $l(U, U')$, where U' is a push-off of U determined by the framing δ.

We need to check that this is really well-defined modulo d, which is an easy extension of the case $\partial M = \emptyset$ proved in [1] as follows. Suppose that $(L, \mathfrak{b})$ is framed cobordant to $(L, \mathfrak{b} + n)$ for some n. The framed cobordism $\Sigma \subset M \times I$ can be glued together to give a closed surface $\tilde{\Sigma} \subset M \times S^1$ of self-intersection n. Now the homology class $\alpha = [\tilde{\Sigma}] - [L \times S^1] \in H_2(M \times S^1; \mathbb{Z})$ has trivial intersection with $Y \times \{0\}$, so it is the image of some class $\tilde{a}$ in $H_2(M)$. This means that $\alpha^2 = 0$ and $[L \times S^1]^2 = 0$, so

$$n = (\alpha + [L \times S^1])^2 = 2\alpha \cdot [L \times S^1].$$

On the other hand $2\alpha \cdot [L \times S^1]$ in $M \times S^1$ is the same as $\tilde{\alpha} \cdot 2[L] = \langle 2 \cdot PD([L]), \tilde{\alpha}\rangle$ in M, so $n = \langle e(\xi), \tilde{\alpha}\rangle$. It follows that n is a multiple of d. Similarly, given a class $\tilde{\alpha} \in H_2(M)$ with $\langle e(\xi), \tilde{\alpha}\rangle = d$, one can use $\tilde{\alpha}$ to construct such a framed cobordism by hand. Therefore $d^3(\xi, \xi')$ is well-defined modulo d, and two plane fields with the same Euler class e agree if and only if $d^3(\xi, \xi') = 0$ in $\mathbb{Z}/d\mathbb{Z}$.

We now reduced the theorem to the following statement: if the contact structures are isotopic to Δ^m and Δ^n on $S^2 \times I$ as plane fields, then they are isotopic on all of Y. Huang [3] has shown that if η' is the result of attaching a bypass triple to η, then $d^3(\eta, \eta') = -1$. Letting d denote the divisibility of the Euler class $e = e(\xi) = e(\xi')$, and recalling that $d^3(\xi, \xi') = 0$ because ξ and ξ' are homotopic, it follows that d divides $m - n$. Since we can move some bypass triples from $S^2 \times I$ to $Y \backslash B$ so that the contact structures on $S^2 \times I$ are now both isotopic to $\Delta^{\min(m,n)}$ but differ on $Y \backslash B$ by these bypass triples, it will now suffice to prove the following.

Proposition 5. *Let Σ be a closed surface and η an I-invariant contact structure on $\Sigma \times I$. Then $\eta \circ \Delta^l$ is stably isotopic rel boundary to η, where $l = \langle e(\eta), \Sigma\rangle$.*

Proof. We can change the dividing set on Σ by any number of bypass moves: given an arc, we simply attach the associated bypass triangle to Σ and push the last two bypasses in the triangle away from Σ, and this preserves the stable isotopy type of η. Thus we take $g(\Sigma)+1$ oriented nonseparating dividing curves $\gamma_1, \dots, \gamma_{g+1}$, which collectively split Σ into two connected genus-0 components Σ_0 and Σ_1 which are exchanged by an involution of Σ; and then we add another p dividing curves inside Σ_1 and q inside Σ_0, each of which bounds a small disk with no other dividing curves inside. It follows that

$$\langle e(\eta), \Sigma\rangle = 2(p - q)$$

so we will take p and q positive such that $2(p-q) = l$, and the resulting surface is a convex representative of Σ. Its positive region Σ_+ is equal to Σ_0 minus the q disks inside it, plus the p disks that were placed inside Σ_1, and similarly for the negative region Σ_-.

Fix one of the p dividing curves, and let $\alpha_1, \dots, \alpha_{p-1}$ be disjoint bypass arcs with one endpoint on the fixed curve γ_+ and the other points of intersection with Γ_Σ both on the same contractible dividing curve so that each one of these p dividing curves intersects some α_i. Similarly, fix a curve γ_- among the other set of q dividing curves and use it to identify disjoint bypass arcs $\beta_1, \dots, \beta_{q-1}$. It is easy to check that a bypass triple along any α_i or β_j consists of three consecutive trivial bypasses, hence Δ_{α_i} and Δ_{β_j} are isotopic to contact structures induced by isotopies.

To make these isotopies precise, fix small disks $D_\pm$ in Σ such that $D_\pm \cap \Gamma_\Sigma = \gamma_\pm$. If we fix an oriented curve $c \subset \Sigma$ which intersects one of $D_\pm$ transversely in an arc, and an annular neighborhood $A \supset c$ containing $D_\pm$ for which $A \backslash D_\pm$ does not intersect the dividing set, then there is an isotopy ϕ_t supported in A which transports $D_\pm$ once around c and satisfies $\phi_0 = \text{id}$ and $\phi_1(D_\pm) = D_\pm$. We let $\Phi(\gamma_\pm, D_\pm, c)$ denote the corresponding map $\Sigma \times I \to \Sigma \times I$, and $\xi_{\Phi(\gamma_\pm, D_\pm, c)}$ the induced contact structure. According to [2, Lemma 6.12], if c travels once clockwise around a single dividing curve c', then $\xi_{\Phi(\gamma_\pm, D_\pm, c)}$ is isotopic rel boundary to Δ^2_α where α is a trivial bypass arc which intersects c once and c' twice; the proof uses a local computation which says that both operations lower the d^3 invariant by 2 on a ball supporting them.

Now if α_i intersects γ_+ and some other dividing curve c_i, we let c_i^- be a curve in Σ_- which is parallel to c_i and then Δ_{α_i} is isotopic to $\xi_{\Phi(\gamma_+, D_+, c_i^-)}$. In particular we have an isotopy

$$\Delta^2_{\alpha_1} \circ \ldots \circ \Delta^2_{\alpha_{p-1}} = \xi_{\Phi(\gamma_+, D_+, c_1^-)} \circ \ldots \circ \xi_{\Phi(\gamma_+, D_+, c_{p-1}^-)} = \xi_{\Phi(\gamma_+, D_+, c^-)}$$

where $c^- \subset \Sigma_-$ is a curve homologous to $\gamma_1 \cup \ldots \cup \gamma_{g+1}$, and so

$$\Delta^2_{\alpha_1} \circ \ldots \circ \Delta^2_{\alpha_{p-1}} = \xi_{\Phi(\gamma_+, D_+, \gamma_1^-)} \circ \ldots \circ \xi_{\Phi(\gamma_+, D_+, \gamma_{g+1}^-)}$$

for parallel copies γ_i^- of γ_i in Σ_-. Similarly, if we take parallel copies $\gamma_i^+ \subset \Sigma_+$ then we have a stable isotopy

$$\Delta^{-2}_{\beta_1} \circ \ldots \circ \Delta^{-2}_{\beta_{q-1}} \sim \xi_{\Phi(\gamma_-, D_-, \gamma_1^+)} \circ \ldots \circ \xi_{\Phi(\gamma_-, D_-, \gamma_{g+1}^+)}.$$

In particular, the composition $\Delta^l = \Delta^2_{\alpha_1} \circ \ldots \circ \Delta^2_{\alpha_{p-1}} \circ \Delta^{-2}_{\beta_1} \circ \ldots \circ \Delta^{-2}_{\beta_{q-1}}$ is stably isotopic to the composition

$$\xi_{\Phi(\gamma_+, D_+, \gamma_i^-)} \circ \xi_{\Phi(\gamma_-, D_-, \gamma_i^+)}$$

over $i = 1, \ldots, g+1$; we are allowed to commute structures induced by isotopies $\Phi(\gamma_+, D_+, \gamma_i^-)$ and $\Phi(\gamma_-, D_- \gamma_j^+)$ because the isotopies are supported on disjoint annuli.

It remains to be shown that $\xi_{\Phi(\gamma_+, D_+, \gamma_i^-)} \circ \xi_{\Phi(\gamma_-, D_-, \gamma_i^+)}$ is stably isotopic to an I-invariant contact structure. Fix an annular neighborhood of γ_i containing both D_+ and D_-, and let δ be a bypass arc which intersects Γ_+ once and Γ_- twice. Then if Φ denotes the composition of the isotopies, we have

$$\xi_\Phi \circ \Delta_\delta = \sigma_{\Phi^{-1}(\delta)} \circ \xi_\Phi \circ \sigma_{\delta'} \circ \sigma_{\delta''}$$

with $\Phi^{-1}(\delta)$ isotopic to δ. But in this latter sequence ξ_Φ is isotopic to an I-invariant contact structure, so $\xi_\Phi \circ \Delta_\delta = \sigma_{\Phi^{-1}(\delta)} \circ \sigma_{\delta'} \circ \sigma_{\delta''} = \Delta_{\Phi^{-1}(\delta)}$ and we are done. □

References

[1] Robert E. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. (2) 148 (1998), no. 2, 619–693.

[2] Yang Huang, *A proof of the classification theorem of overtwisted contact structures via convex surface theory*, arXiv:1102.5398.

[3] Yang Huang, *Bypass attachments and homotopy classes of 2-plane fields in contact topology*, arXiv:1105.2348.