Math 273 Lectures 15 and 16

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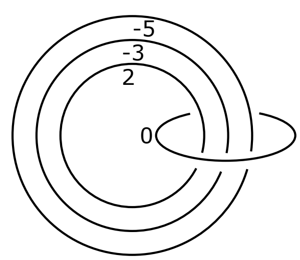
Let $P = -\Sigma(2,3,5)$ be the Poincaré homology sphere with reversed orientation. In this lecture we will see that P does not admit any tight contact structures, following a proof by Etnyre and Honda [1] and its generalization by Lisca and Stipsicz [4, Section 2]. We will use the description of P as a Seifert fibered space over S^2 with three singular fibers and Seifert invariants $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$, but in general we recommend the paper [3] for an introduction to many different constructions of the Poincaré sphere.

Definition 1. A *Seifert fibered space* M over S^2 with invariants $(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$ is a 3-manifold containing solid tori $V_1, \dots, V_n$ whose complement is $M \backslash \cup V_i = S^1 \times \Sigma$, where Σ is a sphere with n punctures. Each V_i is glued to $S^1 \times \Sigma$ by a map $A_i : \partial V_i \to -\partial(M \backslash V_i)$ of the form

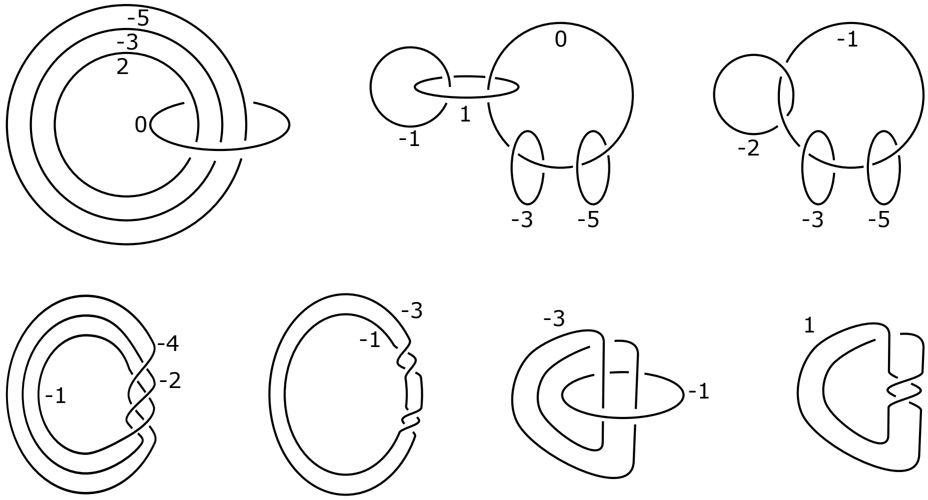
$$A_i = \begin{pmatrix} \alpha_i & \gamma_i \\ -\beta_i & \delta_i \end{pmatrix},$$

where $\gamma_i, \delta_i \in \mathbb{Z}$ and $A_i \in SL_2(\mathbb{Z})$. Here the coordinates on V_i are $(1,0)$ in the meridional direction and $(0,1)$ in the longitudinal direction, and the coordinates on $-\partial(M \backslash V_i)$ are $(0,1)$ in the S^1-direction and $(1,0)$ along the Σ-direction. The cores F_i of each torus V_i are called the *singular fibers* of M.

It is straightforward to check from the definition that P can be described by the following surgery diagram:



In fact, by the series of Kirby moves shown below we see that P can also be described as the result of $+1$-surgery on the right handed trefoil, although we will not need this fact.



For P we can choose the three singular fibers to have gluing maps

$$A_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 5 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let ξ be a tight contact structure on P. If we take disjoint Legendrian representatives F_i of the singular fibers with twisting number $m_i < 0$ and let V_i be a standard tubular neighborhood of F_i, then each ∂V_i is a convex torus with two parallel dividing curves of slope $\frac{1}{m_i}$, and so as seen on $-\partial(P \backslash V_i)$ the dividing slopes $A_i \cdot (m_i, 1)^{\mathsf{T}}$ are $\frac{m_1}{2m_1-1}$, $-\frac{m_2}{3m_2+1}$, and $-\frac{m_3}{5m_3+1}$. Our first goal is to find a standard form for each F_i and V_i.

Lemma 2. *We can find representatives F_i of the singular fibers F_2 and F_3 so that $m_2 = m_3 = -1$.*

Proof. Assume that m_2 and m_3 are both negative. Use Giroux flexibility to give both $\partial(P\backslash V_2)$ and $\partial(P\backslash V_3)$ vertical ruling curves, and let A be a properly embedded annulus in $S^1 \times \Sigma$ whose boundary consists of a single Legendrian ruling curve $S^1 \times \{*\}$ on each of these tori. Then ∂A intersects the dividing sets of $\partial(P\backslash V_2)$ and $\partial(P\backslash V_3)$ in $2 \cdot |3m_2 + 1|$ and $2 \cdot |5m_3 + 1|$ points, respectively, so if these are not equal then by the Imbalance Principle we can find a bypass attached along a vertical ruling curve of one of these tori.

If $m_2 < -1$ and there is a bypass along $\partial(P\backslash V_2)$, then on V_2 its ruling curve is in the homology class $A_2^{-1}(0,1)^{\mathsf{T}} = (-1,3)^{\mathsf{T}}$, so it has slope -3. We now recall how the dividing slope changes on a torus after a bypass attachment: we move counterclockwise along the Farey tessellation from -3 to $\frac{1}{m_2}$, and the new slope is the first point we reach which is connected to $\frac{1}{m_2}$ by an edge, namely $\frac{1}{m_2+1}$. This gives us a standard neighborhood of a curve F_2' with $tw(F_2') = m_2 + 1$. Similarly, if $m_3 < -1$ and the bypass is along $\partial(P\backslash V_3)$ then we can find a curve F_3' isotopic to F_3 with $tw(F_3') = m_3 + 1$.

Now suppose that instead $3m_2 + 1 = 5m_3 + 1$ and there are no bypasses along A, so that its dividing set consists entirely of horizontal curves. If we cut $P\backslash(V_2 \cup V_3)$ open along A and round edges, the result contains a torus (the boundary of $V_2 \cup V_3 \cup A$) whose dividing curves intersect an S^1 longitude in $\#\Gamma_A = |3m_2 + 1| = |5m_2 + 1|$ points and a meridian (i.e. a closed curve in Σ) in $|m_2 + m_3 + 1|$ points. In other words, the boundary slope on this torus is

$$-\frac{m_2 + m_3 + 1}{3m_2 + 1} = -\frac{\frac{8}{5}m_2 + 1}{3m_2 + 1} = -\left(\frac{1}{2} + \frac{m_2 + 5}{30m_2 + 10}\right).$$

Note that $P\backslash(V_2 \cup V_3 \cup A)$ is topologically an $S^1 \times D^2$ which retracts onto V_1, and if $-\frac{p}{q}$ is the boundary slope then the slope as seen from V_1 is determined by $A_1^{-1}(q,p) = (p, 2p-q)$, meaning it is $\frac{2p-q}{p} = 2 + \frac{1}{-p/q}$. If $m_2 = -5$ then the boundary slope is $-\frac{1}{2}$, meaning it is 0 with respect to $S^1 \times D^2$; but then there is an overtwisted disk in $S^1 \times D^2$, which cannot happen. If instead $m_2 < -5$ then this slope (say, r) is strictly less than $-\frac{1}{2}$, which is positive (say, $r' > 0$) with respect to $S^1 \times D^2$. But then $(S^1 \times D^2)\backslash V_1$ is a tight $T^2 \times I$ with boundary slopes $r' > 0$ and $r < 0$, so we can find a convex torus inside it parallel to $S^1 \times \partial D^2$ with boundary slope ∞. This gives rise to a vertical Legendrian curve on that torus which misses the dividing set completely and cobounds an annulus with one of the vertical dividing curves on V_2, and so the Imbalance Principle guarantees a bypass along V_2 and we can proceed as before. □

Proposition 3. *Suppose we have $m_2 = m_3 = -1$ as in the previous lemma. We can also find a Legendrian representative of F_1 with $m_1 = 0$ and a (not necessarily standard) neighborhood $V_i' \supset V_i$ $(1 \leq i \leq 3)$ for which each $\partial(P\backslash V_i)$ has infinite slope.*

Proof. Since $\partial(P\backslash V_2)$ and $\partial(P\backslash V_3)$ now have slopes $-\frac{1}{2}$ and $-\frac{1}{4}$, we can again find vertical Legendrian ruling curves and an annulus A whose boundary consists of one ruling curve from each torus. Then ∂V_2 intersects the dividing set Γ_A in four points, so there are at most two bypasses in A along ∂V_2. Cutting $P\backslash(V_1 \cup V_2)$ open along A as before, we compute the boundary slope on $V_2 \cup V_3 \cup A$: if the number of bypasses is zero then it is $-\frac{2}{4} = -\frac{1}{2}$, and if the number is one then it is $-\frac{2}{2} = -1$. Just as before, these cases lead to an overtwisted disk, which is impossible, and a convex torus $S^1 \times D^2$ with boundary slope ∞. Similarly, if the number of bypasses is two then $\partial(V_2 \cup V_3 \cup A)$ has boundary slope ∞. Thus in either of the allowable cases we find a torus parallel to ∂V_1 with dividing curves $A_1^{-1} \cdot (0,1)^\intercal = (1,2)^\intercal$ with respect to V_1, i.e. with slope 2. This torus bounds $V_1' \supset V_1$ such that $\partial(P\backslash V_1')$ has slope ∞, and the same argument as before (sometimes known as the "Twist Number Lemma") now lets us find a curve isotopic to F_1 with strictly larger twisting number as long as $m_1 \leq -1$. Thus we can find a representative of F_1 with $m_1 = 0$.

Take a vertical ruling curve on $\partial(P\backslash V_1')$ and find an annulus A whose boundary consists of this curve and a vertical Legendrian curve on $\partial(P\backslash V_2)$. Since $\Gamma_A \cap \partial V_1'$ is empty, A contains two bypasses along ∂V_2; if we enlarge V_2 to V_2'

by pushing its boundary across these bypasses, then $\partial V_2'$ has vertical dividing curves. We repeat the same procedure with V_3, pushing it across four such bypasses to get the desired $\partial V_3'$. □

We remark that $V_1' \backslash V_1$ is minimally twisting since $\xi|_{V_1'}$ is tight, and its boundary slopes as seen from $P \backslash V_1$ are 0 and ∞, so it is a basic slice. Similarly $V_2' \backslash V_2$ is a union of 2 basic slices which commute with each other (i.e. the order in which they are attached does not matter), and $V_3' \backslash V_3$ is a union of 4 such basic slices. Each contact structure $\xi|_{V_i' \backslash V_i}$ is thus uniquely determined by the number q_i of positive basic slices, and these satisfy

$$0 \leq q_1 \leq 1, \quad 0 \leq q_2 \leq 2, \quad 0 \leq q_3 \leq 4.$$

Let $\Sigma' \times S^1 = P \backslash \cup V_i'$, where Σ' is a sphere with three punctures (or equivalently a pair of pants). Then $\partial(\Sigma' \times S^1)$ consists of three tori which each have two vertical dividing curves.

Proposition 4. *Fix a convex surface $\Sigma' = \Sigma' \times \{*\}$ with Legendrian boundary. The dividing set $\Gamma_{\Sigma'}$ consists of three arcs, each of which connects a different pair of components of $\partial \Sigma'$, and furthermore the contact structure $\xi|_{\Sigma' \times S^1}$ is uniquely determined by the dividing set $\Gamma_{\Sigma'}$ up to isotopy rel boundary.*

Proof. Given a convex $\Sigma' = \Sigma' \times \{*\}$, its dividing set must intersect $\partial \Sigma'$ in six points, i.e. $\Gamma_{\Sigma'}$ contains three arcs. If one such arc is boundary-parallel, then there is a bypass along V_i' which we may use to thicken it to V_i'' with slope 0 as viewed from $\partial(P \backslash V_i'')$. But then we can take vertical Legendrian curves on $\partial(P \backslash V_i'')$ and $\partial(P \backslash V_j')$ for some $j \neq i$, and by the Imbalance Principle we find another bypass along $\partial V_i''$, so we attach it to get an even bigger V_i''' with slope ∞. In particular, we can now find a convex torus parallel to $\partial V_i'$ in $V_i''' \backslash V_i'$ whose slope, viewed with respect to V_i', is zero, and this gives rise to an overtwisted disk. We conclude that no dividing arc on Σ' can be boundary-parallel, and therefore (using Giroux's criterion to eliminate the possibility of closed dividing curves) $\Gamma_{\Sigma'}$ consists of three arcs which each connect a different pair of components of $\partial \Sigma'$.

We now claim that $\xi|_{\Sigma' \times S^1}$ is unique up to an isotopy rel boundary. To see this, cut open along Σ' and then take a pair of convex disks $\gamma \times I \subset \Sigma' \times I$ such that after edge rounding, each $\partial(\gamma \times I)$ intersects the dividing set of $\partial(\Sigma' \times I)$ twice. The result is a tight 3-ball, which is unique up to isotopy rel boundary, and since each $\gamma \times I$ had a single dividing arc, there is a unique way to glue the ball back together along these disks up to isotopy. In particular, the contact structure on $\Sigma' \times S^1$ depends only on $\Gamma_{\Sigma'}$ as desired: for example, we can apply an isotopy away from Σ' so that it is invariant in the S^1-direction. □

At this point we understand ξ completely once we know the integers q_i and the dividing set $\Gamma_{\Sigma'}$. Our goal now is to show that any choice of q_1, q_2, q_3 must actually result in an overtwisted contact structure. The key observation is that if a neighborhood of F_i has a torus boundary with slope 0 as seen from V_i, then this neighborhood has an overtwisted disk. As seen from $P\backslash\cup V_i'$, these "critical slopes" are

$$c_1 = \tfrac{1}{2}, \quad c_2 = -\tfrac{1}{3}, \quad c_3 = -\tfrac{1}{5}$$

and so any time we can find a convex torus parallel to $\partial(P\backslash V_i')$ with slope c_i we know that ξ must be overtwisted. We will find these tori by repeating a strategy we have already used several times: we look for vertical annuli connecting pairs of neighborhoods V_i'' and V_j'' and cut P along these annuli to get boundary tori which achieve the critical slopes.

Proposition 5. *Let Σ be a pair of pants and $(\Sigma\times S^1, \xi)$ a tight contact manifold with three boundary tori T_0, T_1, T_2 each having two dividing curves of slopes $\frac{p_0}{q}$, ∞, and $\frac{p_2}{q}$. Let $(T_0\times I)\cup(T_2\times I)$ be minimally twisting collar neighborhoods of $T_0\cup T_2$ with complement $(\Sigma'\times S^1, \xi|_{\Sigma'\times S^1})$ as above. If either $p_0 = p_2 = -1$ and $\xi|_{T_0\times I}$ is isotopic to $\xi|_{T_2\times I}$, or $\frac{p_2}{q} < 0$ and both $\xi|_{T_0\times I}$ and $\xi|_{T_2\times I}$ decompose into basic slices of the same sign, then there is a convex annulus $A\subset\Sigma\times S^1$ whose boundary consists of vertical ruling curves of T_0 and T_2, and which does not have any boundary-parallel dividing curves.*

Proof. Take a tight contact structure on $T^2\times I$ isotopic to $\xi|_{T_0\times I}$, and along a torus $T^2\times\{\epsilon\}$ inside an invariant neighborhood of $T^2\times\{0\}$ we can remove a standard neighborhood U' of a vertical Legendrian ruling. We know that this ruling intersects $\Gamma_{T^2\times\{\epsilon\}}$ in $2q$ points, so it has twisting number $-q$ and thus $\partial U'$ has boundary slope $-\frac{1}{q}$. Using an annulus whose boundary consists of vertical ruling curves on $\partial U''$ and $T^2\times\{1\}$, the Imbalance Principle provides a series of bypasses along $\partial U'$ and we can enlarge U' until the resulting U'' has boundary slope ∞. But then $\partial U''$ and $\partial U'$ cobound a tight $T^2\times I$ with boundary slopes $-\frac{1}{q}$ and ∞, and since $\frac{p_2}{q}$ lies in between them we can find a convex torus in between them with slope $\frac{p_2}{q}$. Let U be the solid torus we have found with boundary slope $\frac{p_2}{q}$, and let ξ' be the contact structure on $(T^2\times I)\backslash U$.

Similarly, we can take a vertical annulus between vertical ruling curves on $T^2\times\{0\}$ and $\partial U''$ to find bypasses along $T^2\times\{0\}$, and if we push $T^2\times\{0\}$ across them we will get a parallel torus T with boundary slope ∞. Let C be the collar neighborhood of $T^2\times\{0\}$ with boundary T, and note that C is disjoint from U''. Now $(T^2\times I)\backslash U$ is diffeomorphic to $\Sigma\times S^1$, and we can take this diffeomorphism to send $T^2\times\{0\}$ to T_0, $T^2\times\{1\}$ to $-T_1$, and ∂U to T_2. We can also arrange for it to send the collar C to $T_0\times I$ and the neighborhood $U''\backslash U$ to $T_2\times I$; then $(T^2\times I)\backslash(U''\cup C)$ is identified with $\Sigma'\times S^1$, both of which have dividing curves of infinite slope on each boundary torus.

Now we can find an annulus A of the desired form inside $((T^2\times I)\backslash U, \xi')$, where it should connect vertical ruling curves of $T^2\times\{0\}$ and ∂U: since U is a neighborhood of a ruling curve $\gamma\times\{\epsilon\}\subset T^2\times\{\epsilon\}$, where $T^2\times\{\epsilon\}$ lies in an invariant neighborhood of $T^2\times\{0\}$, we can take A to be intersection of the invariant annulus $(\gamma\times[0,\epsilon])$ with $(T^2\times I)\backslash U$. Thus it only remains to be shown that ξ is isotopic to ξ'.

Consider a convex surface Σ' inside $Y = (T^2\times I)\backslash(U''\cup C)\cong\Sigma'\times S^1$. The dividing set $\Gamma_{\Sigma'}$ intersects each component of $\partial\Sigma'$ in two points, so it contains three arcs. If one of these arcs is boundary-parallel, it would create a bypass along a component of ∂Y, so if we push the boundary along that bypass we get a torus of slope 0. If this torus is parallel to $\partial U''$, then since $(Y, \xi'|_Y)$ embeds into $T_0 \times I$ and the image of $\Sigma' \cap \partial U''$ bounds a disk there, this torus would give rise to an overtwisted disk in $T_0 \times I \subset \Sigma \times S^1$. Otherwise this torus is parallel to a component of $T^2 \times \partial I$. In that case we can repeat the construction of C to find a torus parallel to and arbitrarily close to T_0 with slope ∞, and the slope 0 torus we have found lies in between that one and $T_0 \times \{1\}$, which also has slope ∞. But this contradicts the assumption that $\xi|_{T_0 \times I}$ is minimally twisting, so $\Gamma_{\Sigma'}$ has no boundary-parallel arcs. In particular Proposition 4 tells us that (Y, ξ') is isotopic to $(\Sigma' \times S^1, \xi)$.

Pushing ξ' forward to $\Sigma \times I$ for ease of notation and letting A now denote the image of $A \subset (T^2 \times I) \backslash U$ inside $\Sigma \times I$, we now need to see that $\xi'|_{T_0 \times I}$ and $\xi'|_{T_2 \times I}$ are isotopic rel boundary to $\xi|_{T_0 \times I}$ and $\xi|_{T_2 \times I}$, respectively, and we will use their relative Euler classes to see this. Let A_0 and A_2 be vertical annuli with one boundary component on T_0 or T_2 and the other on T_1; we have $\langle e(\xi'), A_0 \rangle = \langle e(\xi'), A_2 \rangle + \langle e(\xi'), A \rangle = \langle e(\xi'), A_2 \rangle$ because $\langle e(\xi'), A \rangle = \chi(A_+) - \chi(A_-) = 0$. If $B_i = A_i \cap (T_i \times I)$ for $i = 0, 2$, then since $\langle e(\xi'), A_i \backslash B_i \rangle = 0$ we have

$$\langle e(\xi'), B_0 \rangle = \langle e(\xi'), B_2 \rangle$$

where in either case we have restricted ξ' to $T_i \times I$. On the other hand, we know that

$$\langle e(\xi), B_2 \rangle = \langle e(\xi), B_0 \rangle = \langle e(\xi'), B_0 \rangle$$

because $\xi|_{T_0 \times I}$ is isotopic to $\xi|_{T_2 \times I}$ and ξ and ξ' agree on B_0, so in particular ξ and ξ' have the same Euler classes on $T_0 \times I$ and likewise on $T_2 \times I$. Since they are minimally twisting contact structures on $T^2 \times I$, we know that they must therefore be isotopic and we are done. □

Lemma 6. *If $q_2 \leq q_3 \leq q_2 + 2$, then ξ is overtwisted.*

Proof. Let $V_2'' \subset V_2'$ and $V_3'' \subset V_3'$ be neighborhoods of F_2 and F_3 with boundary slopes $-\frac{1}{2}$ as viewed from $P \backslash V_i$. Now $q_3 \geq q_2$ and $4 - q_3 \geq 2 - q_2$ by assumption, so V_3 has at least as many positive basic slices as V_2 and likewise for negative slices. In particular we can assume that $\xi|_{V_2' \backslash V_2''}$ is isotopic to $\xi|_{V_3' \backslash V_3''}$ by shuffling the basic slices of each $V_i' \backslash V_i$ to make sure that the $V_i' \backslash V_i''$ have the same number of basic slices of each sign.

We now have a convex annulus A connecting two ruling curves of $\partial V_2''$ and $\partial V_3''$, and A has no boundary-parallel dividing curves, so if we cut along A then $\partial(V_2'' \cup V_3'' \cup N(A))$ has two dividing curves of slope $-\frac{1}{2}$ in $P \backslash (V_2'' \cup V_3'' \cup N(A))$. After we reverse the orientation of this torus so that it is parallel to $\partial(P \backslash V_1')$, it has the critical slope $c_1 = \frac{1}{2}$ and so ξ is overtwisted. □

In particular, it follows immediately that $q_3 \neq 2$.

Lemma 7. *If $q_1 = 0$ and $q_3 \leq 1$, or $q_1 = 1$ and $q_3 \geq 3$, then ξ is overtwisted.*

Proof. Suppose $q_1 = 0$ and $q_3 \leq 1$. Take a negative stabilization of F_1 with standard neighborhood $V_1'' \subset V_1$; this has slope -1 in V_1, so $\partial(P \backslash V_1'')$ has slope $\frac{1}{3}$. Similarly, F_3 has a neighborhood $V_3'' \subset V_3$ for which $\partial(P \backslash V_3'')$ has slope $-\frac{1}{3}$, and since $q_3 \leq 1$ we can shuffle the basic slices of $V_3 \backslash V_3''$ so that $\xi|_{V_1' \backslash V_1''}$ and $\xi|_{V_3' \backslash V_3''}$ consist entirely of negative basic slices.

We now get a convex vertical annulus A connecting ruling curves of $\partial V_1''$ and $\partial V_3''$ such that Γ_A has no boundary-parallel components. We cut along A and round corners, and $\partial(V_1'' \cup V_3'' \cup N(A))$ has two dividing curves of slope $\frac{1}{3}$ so it yields a torus parallel to $\partial(M \backslash V_2')$ with critical slope $c_2 = -\frac{1}{3}$.

The case $q_1 = 1, q_3 \geq 3$ is identical, with all signs of basic slices reversed. $\square$

Now if $q_1 = 0$ then the last two lemmas show that $q_3 \geq 3$; either $q_3 = 3$ and $q_2 = 0$, or $q_3 = 4$ and $q_2 \in \{0, 1\}$. Similarly, if $q_1 = 1$ then $q_3 \leq 1$ and we must have $q_3 < q_2$.

Lemma 8. *If* (q_1, q_2) *is either* $(0, 0)$ *or* $(1, 2)$*, then* ξ *is overtwisted.*

Proof. Let $V_1'' \subset V_1$ and $V_2'' \subset V_2$ be standard neighborhoods of F_1 and F_2 after stabilizing them two times and once, respectively. By assumption all basic slices in $V_1' \backslash V_1$ and $V_2' \backslash V_2$ have the same sign, so we choose the signs of the stabilizations to agree with this sign. The boundary slope of each V_i'' in V_i' is $-\frac{1}{2}$, so $\partial(P \backslash V_1)$ has slope $\frac{2}{5}$ and $\partial(P \backslash V_2)$ has slope $-\frac{2}{5}$. As in the previous lemma, we can find a vertical annulus A connecting these for which $\partial(V_1'' \cup V_2'' \cup N(A))$ has slope $\frac{1}{5}$, which gives a torus parallel to $\partial(P \backslash V_3')$ with slope $c_3 = -\frac{1}{5}$. $\square$

The only remaining possibilities are $(q_1, q_2, q_3) = (0, 1, 4)$ and $(q_1, q_2, q_3) = (1, 1, 0)$.

Lemma 9. *If* $(q_1, q_2, q_3) = (0, 1, 4)$ *or* $(q_1, q_2, q_3) = (1, 1, 0)$ *then* ξ *is overtwisted.*

Proof. Let $V_2'' \subset V_2'$ and $V_3'' \subset V_3'$ be neighborhoods containing V_2 and V_3 which both have boundary slope -1. Then $V_2' \backslash V_2''$ and $V_3' \backslash V_3''$ are basic slices, and since we can shuffle the basic slices in $V_2' \backslash V_2$ (which has one basic slice of each sign) and $V_3' \backslash V_3$ we can pick the slice $V_2' \backslash V_2''$ to match that of $V_3' \backslash V_3''$. Then $\xi|_{V_2' \backslash V_2''}$ is isotopic to $\xi|_{V_3' \backslash V_3''}$, so we can find a vertical convex annulus A connecting them with no boundary-parallel dividing curves, and as usual we cut along A to get a convex torus of slope -1. This gives a convex torus parallel to $\partial(M \backslash V_1')$ with boundary slope 1.

Let $V_1'' \supset V_1'$ be the solid torus bounded by the convex torus of slope 1 in $\partial(P \backslash V_1')$. We can compute that $\partial V_1''$ has slope 1 when viewed from V_1' as well, so V_1'' is a standard neighborhood of a Legendrian knot with twisting number 1; in particular we can destabilize F_1 so that $m_1 = 1$. Having done so, we can stabilize F_1 again with whichever sign we want to find a basic slice inside V_1'' of that sign. We pick the sign to disagree with the sign of $V_1' \backslash V_1$, and then shuffle the basic slices inside V_1' so that the new $V_1' \backslash V_1$ has a different sign from the original $V_1' \backslash V_1$ but $\partial(P \backslash V_1')$ still has slope ∞.

We have shown that there is an isotopy sending the contact structure for the given (q_1, q_2, q_3) to the contact structure corresponding to $(1 - q_1, q_2', q_3')$ for some possibly different q_2' and q_3'. This isotopy fixed V_2'' and V_3'' but not necessarily V_2' and V_3', so we have to construct new neighborhoods $V_2' \supset V_2''$ and $V_3' \supset V_3''$ with infinite slope. If $q_1 = 0$ then q_3' is still at least 3, since the new V_3' still has the three positive basic slices of $V_3'' \backslash V_3$, so the contact structure has the form $(1, q_2', q_3')$ with $q_3' \geq 3$ and we already know that this is overtwisted. Similarly, if $q_1 = 1$ then $q_3' \leq 1$, so we have a contact structure of the form $(0, q_2', q_3')$ with $q_3' \leq 1$ and this is overtwisted as well. □

Since every possible choice of the q_i led to an overtwisted contact structure, we conclude that there are no tight contact structures on P.

Remark 10. Lisca and Stipsicz [4] used this argument to show that for any $n \geq 1$, the Seifert fibered space over S^3 with invariants $(-\frac{1}{2}, \frac{n}{2n+1}, \frac{1}{2n+3})$ (equivalently, the manifold obtained by $(2n-1)$-surgery on the $(2, 2n+1)$ torus knot) does not have a tight contact structure; the manifold P corresponds to $n = 1$.

References

[1] John B. Etnyre and Ko Honda, *On the nonexistence of tight contact structures*, Ann. of Math. (2) 153 (2001), no. 3, 749–766.

[2] Paolo Ghiggini and Stephan Schönenberger, *On the classification of tight contact structures*, Topology and geometry of manifolds (Athens, GA, 2001), 121–151, Proc. Sympos. Pure Math., 71, Amer. Math. Soc., Providence, RI, 2003. arXiv:math/0201099.

[3] R. C. Kirby and M. G. Scharlemann, *Eight faces of the Poincaré homology 3-sphere*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), pp. 113–146, Academic Press, New York-London, 1979.

[4] Paolo Lisca and András I. Stipsicz, *Ozsváth-Szabó invariants and tight contact three-manifolds. II,* J. Differential Geom. 75 (2007), no. 1, 109–141.