Math 273 Lectures 13 and 14

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Last time we used Legendrian surgery to construct all $\prod(-r_i - 1)$ tight contact structures on the lens space $L(p,q)$, where $-\frac{p}{q} = [r_0, \dots, r_k]$. Our goal in this lecture is to study Legendrian surgery in more detail. First, we review the definition of contact surgery.

Definition 1. Let $K \subset (Y, \xi)$ be a Legendrian knot with meridian μ and contact framing λ on $\partial N(K)$ where λ lies in a Seifert surface for K. We perform *contact* $\frac{p}{q}$*-surgery* on K by constructing a 3-manifold

$$Y' = (S^1 \times D^2) \cup_f \overline{Y \backslash N(K)},$$

where the gluing map $f : S^1 \times \partial D^2 \to \partial N(K)$ sends $\{*\} \times \partial D^2$ to the curve $a\mu + b\lambda$, and extending the contact structure $\xi|_{\overline{Y \backslash N(K)}}$ to all of Y' by choosing a tight contact structure on $S^1 \times D^2$ which agrees with ξ along the boundary.

Recall that this is not defined if $\frac{p}{q} = 0$, and it is uniquely defined if $\frac{p}{q} = \frac{1}{n}$ for any $n \in \mathbb{Z}$; otherwise we know from the classification of tight contact structures on $S^1 \times D^2$ that we have a choice of contact structure. The case $\frac{p}{q} = -1$ is known as *Legendrian surgery*, and has a special interpretation in terms of symplectic manifolds originally due to Weinstein [5] which we will now examine.

Definition 2. A vector field v on a symplectic manifold (X, ω) is said to be *Liouville* if $\mathcal{L}_v \omega = \omega$. A hypersurface $Y \subset X$ of codimension 1 is *of contact type* if there is a Liouville vector field transverse to Y on a neighborhood of Y.

An arbitrary hypersurface $Y \subset (X, \omega)$ determines a bundle

$$L_Y = TY^{\perp} = \{v \in TX|_Y \mid \omega(v, x) = 0 \text{ for all } x \in TY\}$$

which is a subbundle of TY, and since ω is nondegenerate L_Y is a line bundle. If Y is a level set $\phi^{-1}(c)$, where c is a regular value of $\phi : X \to \mathbb{R}$, and v_ϕ is the vector field satisfying $\iota_{v_\phi} \omega = d\phi$, then

$$\omega(v_\phi, x) = d\phi(x) = 0$$

for all $x \in TY$ and so L_Y is generated by v_ϕ.

Proposition 3. *The submanifold $Y \subset X$ is of contact type if and only if there is a 1-form α on Y such that $d\alpha = \omega|_Y$ and $\alpha|_{L_Y} \neq 0$. Furthermore, if α exists then it is a contact form.*

Proof. Suppose that $Y = \phi^{-1}(c)$ as above. If Y is of contact type with Liouville vector field v, then let $\alpha = \iota_v\omega|_Y$. Then $\omega = \mathcal{L}_v\omega = d(\iota_v\omega)$, and so $\omega|_Y = d\alpha$. Furthermore, let v_ϕ satisfy $\iota_{v_\phi}\omega = d\phi$; then

$$\alpha(v_\phi) = \omega(v, v_\phi) = -\omega(v_\phi, v) = -d\phi(v)$$

which is nonzero since v is transverse to the level set $Y = \phi^{-1}(c)$, hence $\alpha|_{L_Y} \neq 0$.

Conversely, suppose that $\alpha \in \Omega^1(Y)$ satisfies $\omega|_Y = d\alpha$ and $\alpha|_{L_Y} \neq 0$. Extend α to a 1-form α' on a neighborhood $N(Y) \subset X$ satisfying $\omega = d\alpha'$, and let v be a vector field satisfying $\iota_v\omega = \alpha'$. Then

$$\mathcal{L}_v\omega = d\iota_v\omega = d\alpha' = \omega$$

and furthermore

$$d\phi(v) = \omega(v_\phi, v) = -\omega(v, v_\phi) = -\alpha'(v_\phi) = -\alpha(v_\phi)$$

which is nonzero since $\alpha|_{L_Y} \neq 0$, so v is transverse to Y and therefore Y is of contact type.

We now claim that such α is a contact form on Y if it exists. Indeed, we can compute

$$\alpha \wedge d\alpha = (\iota_v\omega) \wedge \omega|_Y = \frac{1}{2}\iota_v(\omega \wedge \omega)$$

and since $\omega \wedge \omega$ is a volume form on X and Y is transverse to v, this must be nonzero on Y.

Finally, in general we cannot assume that $Y = \phi^{-1}(c)$ because Y may be nonseparating. However, we can always assume that there is a function $\phi : X \to \mathbb{R}$ for which Y is a connected component of the inverse image of a regular value, and then the above proof still works as expected. □

It turns out that the symplectic structure is uniquely determined near the boundary by the Liouville vector field v and the contact structure $\xi = \ker(\alpha)$ on Y. Indeed, there is a small collar neighborhood of Y along which the flow Φ_t of v determines a diffeomorphism $\varphi : Y \times (-\epsilon, \epsilon) \to N(Y)$ by the formula $\varphi(x, t) = \Phi_t(x)$. But the symplectic structure $\omega' = d(e^t\alpha)$ on $Y \times \mathbb{R}$, also known as the *symplectization* of (Y, ξ), has Liouville vector field ∂_t and $Y = Y \times \{0\}$ is of contact type with $\omega'|_Y = d\alpha$, so it is easy to check that $\omega' = \varphi^*\omega$. In other words, a contact type hypersurface $(Y, \xi) \subset (X, \omega)$ always has a neighborhood symplectomorphic to the symplectization of (Y, ξ).

We will be concerned with symplectic 4-manifolds whose boundaries are of contact type; we say that the induced contact structure is *ω-convex* if the Liouville vector field points out of $Y = \partial X$ along Y. In general we can cut and paste along such manifolds as follows.

Proposition 4. *Let $U_i \subset (X_i, \omega_i)$ be codimension-0, ω_i-convex submanifolds with contact type boundaries $(Y_i, \xi_i) = \partial U_i$. If there is a contactomorphism $f : (Y_1, \xi_1) \to (Y_2, \xi_2)$, then the manifold*

$$(X_1 \backslash U_1) \cup_f U_2$$

admits a symplectic structure.

Proof. We need to patch together the symplectic forms ω_i along neighborhoods of Y_i. Let $\alpha_i = \iota_{v_i}\omega_i$ be the contact forms on Y_i, and suppose that $f^*\alpha_2 = g\alpha_1$ where $g : Y_1 \to \mathbb{R}$ is nonzero; by rescaling ω_2 we can ensure that $0 < g < 1$. In the symplectization of (Y_1, ξ_1), we have $Y_1 \cong Y_1 \times \{1\}$ and $Y_2 \cong \mathrm{graph}(\ln(g))$. Each of these is of contact type and has a neighborhood N_i symplectomorphic to a neighborhood of ∂U_i. If these neighborhoods cobound a region $V \subset Y_1 \times \mathbb{R}$ which is diffeomorphic to $Y_1 \times \mathbb{R}$, then using these symplectomorphisms gives us a symplectic manifold $(X_1 \backslash U_1) \cup V \cup U_2$, which is diffeomorphic to $(X_1 \backslash U_1) \cup U_2$, as desired. □

One of the most interesting special cases of this procedure is gluing a Weinstein 2-handle to a symplectic manifold with contact type boundary. Given the standard symplectic structure $\omega = \sum dx_i \wedge dy_i$ on $\mathbb{R}^4$, consider the region H defined by the inequalities

$$\begin{aligned} f = (x_1^2 + x_2^2) - \frac{1}{2}\left(y_1^2 + y_2^2\right) &\geq -1 \\ g = (x_1^2 + x_2^2) - \frac{\epsilon}{6}\left(y_1^2 + y_2^2\right) &\leq \frac{\epsilon}{2}. \end{aligned}$$

The gradient of f is $v = 2x_1\partial_{x_1} - y_1\partial_{y_1} + 2x_2\partial_{x_2} - y_2\partial_{y_2}$, and

$$\iota_v\omega = 2x_1 dy_1 + y_1 dx_1 + 2x_2 dy_2 + y_2 dx_2$$

satisfies $\mathcal{L}_v\omega = d\iota_v\omega = \omega$, so $v = \nabla f$ is a Liouville vector field. It is transverse to the hypersurface $f^{-1}(-1)$ since $df(v) = \langle \nabla f, v \rangle = |v|^2 > 0$ away from the origin. Furthermore, v is transverse to $g^{-1}(\frac{\epsilon}{2})$ since we can compute

$$dg(v) = \langle \nabla g, v \rangle = 4(x_1^2 + x_2^2) - \frac{\epsilon}{3}(y_1^2 + y_2^2) = 4g + \frac{\epsilon}{3}(y_1^2 + y_2^2)$$

and so along $g^{-1}(\frac{\epsilon}{2})$ we have $dg(v) = 2\epsilon + \frac{\epsilon}{3}(y_1^2 + y_2^2) > 0$. Therefore $Y = \partial H$ is of contact type, with contact form $\alpha = \iota_v\omega|_Y$, and the Liouville vector field v points out of H along $g^{-1}(\frac{\epsilon}{2})$ and into H along $f^{-1}(-1)$.

We now claim that the attaching circle

$$K = \{x_1 = x_2 = 0, y_1^2 + y_2^2 = 2\} \subset f^{-1}(-1) \cap \partial H$$

is Legendrian. Indeed, at a point $(0, 0, y_1, y_2)$ it has tangent vector $w = -y_2\partial_{y_1} + y_1\partial y_2$, and we evaluate

$$\alpha(w) = \iota_w(2x_1 dy_1 + y_1 dx_1 + 2x_2 dy_2 + y_2 dx_2) = 0.$$

Similarly, the circle

$$K' = \{x_1^2 + x_2^2 = \frac{\epsilon}{2}, y_1 = y_2 = 0\} \subset g^{-1}(\frac{\epsilon}{2}) \cap \partial H$$

is Legendrian since $\alpha(w') = 0$ where w' is the tangent vector $-x_2\partial_{x_1} + x_1\partial_{x_2}$ at $(x_1, x_2, 0, 0)$.

Theorem 5. *Let (Y, ξ) be the ω-convex boundary of (X, ω), and let $L \subset Y$ be a Legendrian knot. Then we can attach a 2-handle H to X along L so that the resulting $X' = X \cup H$ is symplectic with ω'-convex boundary Y'.*

Proof. Give H the model symplectic structure discussed above. Since $L \subset Y$ and $K \subset \partial H$ are both Legendrian, they have contactomorphic neighborhoods $N(L)$ and $N(K)$, and by choosing ϵ sufficiently small we can make $f^{-1}(1) \cap \partial H$ lie inside $N(K)$. The above gluing proposition now says that we can form a symplectic manifold $(X \cup H, \omega')$ by using the contactomorphism $N(L) \xrightarrow{\sim} N(K)$, and since the Liouville vector field on H points out along $g^{-1}(\frac{\epsilon}{2})$ the new boundary Y' will be ω'-convex. □

This gluing operation replaces a neighborhood $N(L) \subset Y$ with the neighborhood $g^{-1}(\frac{\epsilon}{2})$ of $K' \subset \partial H$; this is a solid torus, so evidently Y' is the result of surgery along L. Furthermore, the contactomorphism $N(L) \xrightarrow{\sim} N(K)$ is determined up to isotopy rel boundary by the contact framings of L and K, so the surgery coefficient is uniquely determined as well.

The contact framing along L can be thought of as a nonzero section of $\xi|_L$ transverse to TL. In the attaching circle $K \subset \partial H$, which is defined by $x_1 = x_2 = 0$ and $y_1^2 + y_2^2 = 2$, the tangent bundle TL is spanned by $-y_2\partial_{y_1} + y_1\partial_{y_2}$. Since $\alpha = y_1 dx_1 + y_2 dx_2$ along K, the framing is specified by the section

$$-y_2\partial_{x_1} + y_1\partial_{x_2}.$$

On the other hand, the framing we use to perform surgery is specified by a vector field along K which lies in $T(\partial H)$; for this we can use the constant vector field ∂_{x_1}. As we travel along K, the contact framing $-y_2\partial_{x_1} + y_1\partial_{x_2}$ makes a full positive twist with respect to ∂_{x_1}, and so the framing we get by pushing K off of itself inside H is one less than the contact framing. But this is the framing which we use to fill $\partial N(L)$, so we conclude:

Theorem 6. *Attaching a Weinstein 2-handle to (X, ω) along a Legendrian knot L in its ω-convex boundary Y gives a symplectic manifold (X', ω') whose ω'-convex boundary Y' is the result of a Legendrian surgery on $L \subset Y$.*

If (Y, ξ) is the ω-convex boundary of (X, ω), then we know that Y is strongly symplectically fillable (which was defined using the criterion $d\alpha = \omega|_Y$) and hence tight. This verifies that we are actually performing Legendrian surgery, rather than filling in $S^1 \times D^2$ with an overtwisted contact structure. It also shows that Legendrian surgery preserves symplectic fillability. In fact, Eliashberg [2] showed that the same is true for Stein fillability:

Theorem 7. *If (X, ω) is a Stein domain with boundary Y, then the plurisubharmonic exhaustion function $\phi : X \to \mathbb{R}$ can be extended across the handle H so that $Y' = \partial(X \cup H)$ is the level set of a regular value. Therefore Legendrian surgery preserves Stein fillability.*

In fact, Etnyre and Honda [3] showed that Legendrian surgery preserves *weak symplectic fillability*, which is defined as being the boundary of a symplectic manifold (X, ω) for which $\omega|_\xi > 0$ and which also implies tightness. The proof is similar to the proof that it preserves strong fillability: we note that we only need the condition $\omega|_Y = d\alpha$ on a small standard neighborhood of L, and so the claim follows from:

Lemma 8. *Let (X, ω) be a symplectic manifold with boundary Y, and let ξ be a contact structure on Y such that $\omega|_\xi > 0$. Let $L \subset Y$ be a Legendrian knot. Then ξ admits an arbitrarily small perturbation in a neighborhood N of L so that $\xi = \ker(\iota_v \omega|_Y)$ for some Liouville vector field v defined in X near N.*

Proof. Let (X', ω') be a strong filling of some other (Y', ξ') and let $L' \subset Y'$ be Legendrian with standard neighborhood N'. We can then find a diffeomorphism $f : N \to N'$ which carries L to L' and satisfies $f^*(\xi'|_{L'}) = \xi|_L$ and $f^*(\omega'|_{N'}) = \omega|_N$. Then f can be extended to a symplectomorphism on neighborhoods U and U' of $N \subset X$ and $N' \subset X'$. Now ξ can be perturbed close to L so that $f^*(\xi')$ and ξ actually agree on a small neighborhood of L, and if the perturbation is small enough then the result is still contact isotopic to ξ by Gray stability. But then if ω' has Liouville vector field v', we see that $v = f^*(v')$ is a Liouville vector field in that small neighborhood of L on which $\xi = f^*(\xi')$ is the kernel of $\iota_v \omega = \iota_{f^*(v')} f^*(\omega')$, as desired. □

In conclusion, if (Y, ξ) is Stein fillable, strongly fillable, or weakly fillable, then so is the result of any Legendrian surgery along Y. There are tight contact structures which are not weakly fillable [3], however, and so one can ask whether Legendrian surgery preserves tightness. The answer turns out to be no for 3-manifolds with boundary [4], but for closed 3-manifolds this is still an open question.

We will now make a few remarks about these different types of filling. Although the classes of weakly and strongly symplectically fillable contact structures are different in general, Eliashberg [1] proved that this is not always the case:

Theorem 9. *Let (Y, ξ) be a weakly fillable contact structure on a rational homology sphere. Then ξ is strongly fillable.*

Proof. Let (X, ω) be a weak filling of (Y, ξ), so that $\partial X = Y$ and $\omega|_\xi > 0$, and let $\xi = \ker(\alpha)$. Since both ω and $d\alpha$ are symplectic forms on ξ, we have $\omega|_\xi = f d\alpha|_\xi$ for some positive function $f : Y \to \mathbb{R}$, and then $d(f\alpha)|_\xi = f d\alpha|_\xi + df \wedge \alpha|_\xi = f d\alpha|_\xi$ implies $\omega|_\xi = d(f\alpha)|_\xi$. Replace α with $f\alpha$ so that $\omega|_\xi = d\alpha|_\xi$; then in general we have $\omega|_Y = d\alpha + \alpha \wedge \beta$ for some $\beta \in \Omega^1(Y)$.

Fix a constant $C > 0$ and a function $h : Y \times [0,1] \to \mathbb{R}$ such that $h(x,0) = 0$, $h(x,t) = Ct$ for $1-\epsilon \leq t \leq 1$, and $\frac{\partial h}{\partial t} > 0$, and consider the closed 2-form $\Omega = \omega + d(h\alpha)$ on $Y \times [0,1]$. We compute

$$\Omega = (1+h)d\alpha + \alpha \wedge \beta + d_Y h \wedge \alpha + \frac{\partial h}{\partial t} dt \wedge \alpha$$

and so

$$\Omega \wedge \Omega = 2(1+h)\frac{\partial h}{\partial t} dt \wedge \alpha \wedge d\alpha > 0.$$

Thus Ω is symplectic, $\Omega|_{Y\times\{0\}} = \omega$, and $\Omega|_{Y\times[1-\epsilon,1]} = \omega + Cd(t\alpha)$. In particular, by gluing $(Y \times [0,1], \Omega)$ to (X,ω) along $Y \times \{0\}$, we can replace the symplectic form ω in a neighborhood of Y with $\omega + Cd(t\alpha)$, where α is a contact form such that $\omega|_\xi = d\alpha|_\xi$.

Now if Y is a rational homology sphere then any closed 2-form on it is exact, so in a neighborhood $Y \times [1-\epsilon,1]$ of $Y = \partial X$ we may assume that $\omega = d\lambda$. Let $\varphi : [1-\epsilon,1] \to [0,1]$ be a cutoff function which is 1 near $t = 1-\epsilon$ and 0 near $t = 1$, and consider the symplectic form

$$\omega' = d(\varphi\lambda) + Cd(t\alpha)$$

on $Y \times [1-\epsilon,1]$. Certainly ω' is closed, $\omega' = \omega$ near $Y \times \{1-\epsilon\}$, and $\omega' = Cd(t\alpha)$ near $Y \times \{1\}$, so if ω' is symplectic then we may replace $\omega|_{Y\times[1-\epsilon,1]}$ with ω' to get a strong filling of (Y,ξ). But ω' is symplectic in a fixed neighborhood of $Y \times \{1-\epsilon\}$ since it equals ω there, and in general we compute

$$\omega' \wedge \omega' = C^2 \cdot 2t dt \wedge \alpha \wedge d\alpha + O(C)$$

so that ω' can be made symplectic outside that neighborhood as well simply by taking C large enough. $\square$

For the standard tight (S^3, ξ_{st}), we can actually write down a complete list of fillings. In the Stein case, it is a theorem of Gompf that every Stein domain can be built out of 4-dimensional 0, 1, and 2-handles; turning this upside down, we can build a Stein filling (X,J) of ξ_{st} by attaching 2, 3, and 4-handles to a piece $S^3 \times [0,1]$ of the symplectization of ξ_{st} along the $S^3 \times \{0\}$ side. This means that the map $\pi_1(S^3) \to \pi_1(X)$ is surjective, i.e. that X is simply connected.

Now take a Darboux ball B around a point $x \in \mathbb{CP}^2$, where B has contact type boundary (S^3, ξ_{st}). We can glue $\mathbb{CP}^2 \backslash B$ to X along their contactomorphic S^3 boundaries to get a new symplectic manifold $Z = X \cup_{S^3} (\mathbb{CP}^2 \backslash B)$. Now if we construct another closed 4-manifold $\tilde{X}$ by filling in $\partial X = S^3$ with a ball, then we have $Z = \tilde{X} \# \mathbb{CP}^2$. It is known via Seiberg-Witten theory that a connected sum of two manifolds M_1 and M_2 cannot have a symplectic structure unless $b_2^+(M_i) = 0$ for some i, and since $b_2^+(\mathbb{CP}^2) = 1$ we conclude that $b_2^+(X) = b_2^+(\tilde{X}) = 0$. Donaldson's theorem tells us that X is homeomorphic to a blow-up of B^4 at some finite number of points, but since X is Stein this number must be zero. In particular, $b_2^-(Z) = 0$ and so Z is homeomorphic to $\mathbb{CP}^2$. A theorem of McDuff tells us that Z must be the standard $\mathbb{CP}^2$ with its unique symplectic structure, and X is the complement of a neighborhood of a symplectic $\mathbb{CP}^1$; since such $\mathbb{CP}^1$ are all isotopic to each other, X is uniquely determined. In particular, we have shown the following theorem of Eliashberg:

Theorem 10. *The standard 4-ball $B^4 \subset (\mathbb{C}^2, i)$ is the unique Stein filling of (S^3, ξ_{st}) up to diffeomorphism.*

The same is almost true for strong symplectic fillings X of (B^4, ξ_{st}): Eliashberg showed that they are all diffeomorphic to $B^4 \#^n \overline{\mathbb{CP}^2}$ for some $n \geq 0$.

References

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