Math 273 Lecture 11

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Last time we claimed that the models of a basic slice, namely the submanifolds $T_1 = T^2 \times [0, \frac{1}{8}]$ and $T_2 = T^2 \times [\frac{1}{2}, \frac{5}{8}]$ of

$$(T^2 \times \mathbb{R}, \xi = \ker(\sin(2\pi z)dx + \cos(2\pi z)dy))$$

with boundary tori perturbed in each case to be convex with boundary slopes 0 and -1, really are basic slices. We still need one fact to complete this claim:

Proposition 1. *T_1 and T_2 are minimally twisting.*

Proof. Suppose that T_1 contains a convex torus T of slope $s \notin [-1, 0]$ (the proof will be the same for T_2). Observe that each torus $T^2 \times \{z_0\} \subset T_1$ has characteristic foliation directed by

$$\cos(2\pi z)\partial_x - \sin(2\pi z)\partial_y$$

up to sign, so that $T^2 \times \{z_0\}$ is foliated by lines of slope $-\tan(2\pi z)$, which decreases from 0 at $z_0 = 0$ to $-\infty$ at $z_0 = \frac{1}{4}$. Suppose that there is a convex torus $T \subset M \cong T_1$ whose dividing curves have slope $s \notin (-1, 0)$, and let s' be a slope satisfying $s < s' < -1 < 0$ on the boundary of the Farey tessellation such that s and s' are connected by a geodesic. Pick an element of $SL_2(\mathbb{Z})$ sending s to $\frac{0}{1}$ and s' to $\frac{1}{0}$, so that the boundary slopes -1 and 0 of T_1 both become negative. The corresponding diffeomorphism of T_1 sends it to some $T^2 \times [a, b]$ with $[a, b] \subset (0, \frac{1}{4})$.

Now consider the standard tight contact structure $(\mathbb{R}^3, \xi_{\text{st}})$ with contact form

$$\alpha = dz + r^2 d\theta,$$

and pass to the quotient under $z \mapsto z + 1$. The complement of the z-axis is foliated by tori $\Sigma_{r_0} = \{r = r_0\}$, each of which is convex because it is transverse to the contact vector field ∂_r. On each torus Σ_{r_0}, the contact planes are spanned by ∂_r and $-r_0^2\partial_z + \partial_\theta$, so Σ_{r_0} has a characteristic foliation consisting of lines of slope $-r_0^2$. In particular, there is a contact embedding

$$\phi : T^2 \times (0, \frac{1}{4}) \hookrightarrow M = \bigcup_{0<r<\infty} \Sigma_r$$

which sends each $T^2 \times \{z\}$ with slope $-\tan(2\pi z)$ to $\Sigma_{\sqrt{\tan(2\pi z)}}$, preserving the directed characteristic foliation of each such torus, and thus ϕ is a contactomorphism by a standard argument involving Moser's trick. (We will need to rotate T^2 by π for one of T_1 or T_2 to fix the direction of the foliation, but otherwise the argument is the same in either case.)

The image $\phi(T)$ has dividing curves of slope 0, and since $\phi(T)$ is parallel to $\phi(T^2 \times \{a\})$ it bounds a solid torus for which the lines of slope 0 are meridians, so we can find a Legendrian curve γ in $\phi(T)$ parallel to the dividing curves which bounds a disk in that solid torus. In particular, γ is an unknot with $tb(\gamma) = 0$, and this violates the Thurston-Bennequin inequality since ξ_{st} is tight, so it cannot exist. □

Corollary 2. *There are exactly two basic slices with $s_0 = 0$ and $s_1 = -1$.*

Corollary 3. *Given any basic slice $(T^2 \times [0,1], \xi)$ with boundary slopes s_0 and s_1, and a rational number s between s_1 and s_0, we can find a convex torus parallel to $T^2 \times \{0\}$ with slope s.*

Proof. Reduce to the case $(s_0, s_1) = (0, -1)$ and find the torus in either of the two model contact structures by perturbing an appropriate $T^2 \times \{z\}$. □

Finally, we claim that these basic slices correspond to bypass attachments.

Proposition 4. *Let T be a convex torus with two dividing curves of slope 0, and let D be a bypass attached to T along a curve of slope $-\frac{p}{q}$ in some contact manifold, with $p > q > 0$. Then some neighborhood $(T^2 \times [0,1], \xi_D)$ of $T \cup D$ is a basic slice.*

Proof. We already showed that in such a neighborhood $T^2 \times \{1\}$ has two dividing curves of slope -1, so we only need to see that ξ_D is minimally twisting, which we will do by embedding it inside a minimally twisting contact structure. Take the contact structure

$$(T^2 \times \mathbb{R}, \xi = \ker(\sin(2\pi z)dx + \cos(2\pi z)dy))$$

and perturb $T_0 = T^2 \times \{0\}$ and $T_{1/8} = T^2 \times \{\frac{1}{8}\}$ to be convex with dividing curves of slope 0 and -1 and characteristic foliations consisting of ruling curves of slope $-\frac{p}{q}$. Let A be an annulus with one boundary component a ruling curve of T_0 and one a ruling curve of $T_{1/8}$. Then A intersects Γ_{T_0} in $2p$ points and $\Gamma_{T_{1/8}}$ in $2(p-q)$ points, and $q > 0$, so by the Imbalance Principle A contains a bypass D_0 along T_0. Now by Giroux flexibility we can arrange the characteristic foliation on $T_0 \cup D_0$ to match the one on $T \cup D$, so they have a contactomorphic neighborhood with contact structure ξ_D. Then ξ_D embeds in $T^2 \times [-\epsilon, \frac{1}{8} - \epsilon]$ for an arbitrarily small $\epsilon > 0$, and this is minimally twisting by the same proof as when $\epsilon = 0$. □

Let $p > q > 1$, and let $\text{Tight}(S^1 \times D^2, -\frac{p}{q})$ be the set of tight contact structures on $S^1 \times D^2$ with convex boundary having dividing set Γ, where Γ is a pair of curves of slope $-\frac{p}{q}$. (This means that each component of Γ is in the homology class $-q[\partial D^2]+p[S^1] \in H_1(S^1 \times \partial D^2)$.) Similarly, let $\mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1)$ be the set of minimally twisting tight contact structures with a pair of dividing curves of slope -1 on $T^2 \times \{0\}$ and $-\frac{p}{q}$ on $T^2 \times \{1\}$. We wish to describe the latter set by breaking its members into basic slices, so first we need to see how the boundary slope $-\frac{p}{q}$ changes upon removing a basic slice.

For any rational $-\frac{p}{q} < -1$, consider the continued fraction expansion

$$-\frac{p}{q} = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \frac{1}{r_k}}}}$$

with all $r_i \leq -2$; we will abbreviate this as $-\frac{p}{q} = [r_0, \dots, r_k]$. Let $-\frac{p'}{q'}$ be the fraction obtained by taking $\frac{p'}{q'}$ to be the first point connected to $\frac{p}{q}$ when traveling counterclockwise from $\frac{0}{1}$. Since $\frac{p}{q}$ and $\frac{p'}{q'}$ are connected, the vectors (p,q) and (p',q') are an integral basis of $\mathbb{Z}^2$, and since $\frac{p'}{q'} < \frac{p}{q}$ we conclude that $pq' - qp' = 1$. The three properties

$$pq' - qp' = 1, p' < p, q' \leq q$$

uniquely characterize p' and q' in terms of p and q.

Now let $-\frac{a}{b} = [r_0, \dots, r_{k-1}, r_k + 1]$; if $r_k = -2$ then this is equivalent to $[r_0, \dots, r_{k-1} + 1]$. We claim that $a = p'$ and $b = q'$.

Lemma 5. *Suppose that $-\frac{p}{q}$ and $-\frac{p'}{q'}$, both less than or equal to -1, satisfy $pq' - qp' = 1$, $0 < p' < p$, and $0 < q' < q$. Then for any integer $r < \frac{1}{-p/q}$, so do the rational numbers $-\frac{a}{b} = r - \frac{1}{-p/q}$ and $-\frac{a'}{b'} = r - \frac{1}{-p'/q'}$.*

Proof. We have $-\frac{a}{b} = \frac{rp+q}{p}$ and $-\frac{a'}{b'} = \frac{rp'+q'}{p'}$, so

$$ab' - ba' = -(rp+q)p' + p(rp'+q') = pq' - qp' = 1.$$

Furthermore, $b' < b$ is equivalent to $p' < p$, which is true by assumption, and $a' < a$ is equivalent to $-rp' - q' < -rp - q$, or $-r(p-p') > q - q'$. But then $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix}$ are an integral basis of $\mathbb{Z}^2$, hence $\begin{pmatrix} p-p' \\ q-q' \end{pmatrix}$ and either $\begin{pmatrix} p \\ q \end{pmatrix}$ or $\begin{pmatrix} p' \\ q' \end{pmatrix}$ are as well, so the points $\frac{p}{q}$, $\frac{p'}{q'}$, and $\frac{p-p'}{q-q'}$ form a triangle in the Farey tessellation. This means that $\frac{p}{q} = \frac{p'+(p-p')}{q'+(q-q')}$ lies in between the other two points, hence $\frac{p}{q} < \frac{p-p'}{q-q'}$ and in particular $-r\left(\frac{p-p'}{q-q'}\right) > \frac{1}{p/q}\left(\frac{p}{q}\right) \geq 1$. We conclude that $-r(p-p') > q - q'$ as desired. $\square$

Now suppose $r_k \leq -2$. If $-\frac{p}{q} = r_k = [r_k]$ and $-\frac{p'}{q'} = -\frac{r_k+1}{1} = [r_k+1]$, so that $p = -r_k$, $p' = -r_k - 1$, and $q = q' = 1$, then we have $pq' - qp' = 1$,

$0 < p' < p$, and $0 < q' \leq q$. By repeated use of the lemma, it follows that if

$$\begin{aligned} -\frac{p}{q} &= [r_0, \dots, r_{k-1}, r_k] \\ -\frac{p'}{q'} &= [r_0, \dots, r_{k-1}, r_k + 1] \end{aligned}$$

then $pq' - qp' = 1$, $0 < p', p$, and $0 < q' \leq q$.

Proposition 6. *Let $\xi \in \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1)$ with $p > q > 0$, and suppose that $-\frac{p}{q}$ has continued fraction $[r_0, \dots, r_k]$. Let $-\frac{p'}{q'} = [r_0, \dots, r_{k-1}, r_k + 1]$. Then ξ may be factored into a union $(T^2 \times [0, \frac{1}{2}], \xi') \cup (T^2 \times [\frac{1}{2}, 1], \xi'')$, where ξ'' is a basic slice and*

$$\xi' \in \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p'}{q'}, -1).$$

Proof. Fix the characteristic foliation of $T^2 \times \partial I$ to be ruled by Legendrian curves of slope 0; then as before we can take a convex annulus with one boundary component on each $T^2 \times \{i\}$, $i = 0, 1$, and find a bypass along $T^2 \times \{1\}$ on that annulus by the Imbalance Principle. Some neighborhood of $T^2 \times I$ and that bypass is a basic slice, which then has boundary slopes $-\frac{p}{q}$ and $-\frac{a}{b}$ for some a, b.

To compute $-\frac{a}{b}$, we flip this picture upside down and thus reverse the signs of all the slopes: this is the same as attaching a bypass on top of a torus with slope $\frac{p}{q}$ along an arc of slope $\frac{0}{1}$, so $\frac{a}{b}$ is the first point we reach by traveling counterclockwise along the Farey tessellation from $\frac{0}{1}$ which is connected to $\frac{p}{q}$ by a geodesic. But we have already seen that such $\frac{a}{b}$ must satisfy $pb - aq = 1$, $a < p$, and $b \leq q$, so $\frac{a}{b}$ is exactly the point $\frac{p'}{q'}$ described above. □

Corollary 7. *Let $\xi \in \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1)$ with $p > q > 0$. and*

$$-\frac{p}{q} = [r_0, \dots, r_k].$$

Then ξ may be factored into a union of

$$(-r_k - 1) + (-r_{k-1} - 2) + \dots + (-r_0 - 2)$$

basic slices with predetermined boundary slopes. In particular, $\mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1)$ is finite.

Proposition 8. *Let $\xi \in \mathrm{Tight}(T^2 \times I, -\frac{p}{q}, -1)$ be a tight contact structure with $p > q > 0$. Then given any slope s with $-\frac{p}{q} < s < -1$, there is a convex torus parallel to $T^2 \times \{0\}$ with two dividing curves of slope s.*

Proof. If ξ is minimally twisting then we can factor ξ into a union of basic slices as above; on one of them, the interval between its boundary slopes must contain s, and then we know that this basic slice must contain the desired torus.

If instead ξ is not minimally twisting, we can find a torus T parallel to $T^2 \times \{0\}$ with slope $r \notin [-\frac{p}{q}, -1]$ and use the above argument to factor out a sequence of basic slices with boundary slopes between $-\frac{p}{q}$ and r; again, the interval determined by the boundary slopes on one of these slices must contain s. □

Proposition 9. *There is an injective map*

$$\pi_0 \mathrm{Tight}(S^1 \times D^2, -\frac{p}{q}) \to \pi_0 \mathrm{Tight}^{\min}(T^2 \times I, -\frac{p}{q}, -1).$$

Proof. Given a tight contact structure on $S^1 \times D^2$, let K be a Legendrian knot isotopic to $S^1 \times \{0\}$, stabilized sufficiently many times to ensure $tw(K) < -1$. Let $N \subset \mathrm{int}(S^1 \times D^2)$ be a standard neighborhood of K, so that ∂N has two dividing curves of slope $\frac{1}{tw(K)}$, and let $M = (S^1 \times D^2) \backslash N$. Then $\xi|_M$ is a tight contact structure on $T^2 \times I$, and $-\frac{p}{q} < -1 < \frac{1}{tw(K)}$, so we can find a convex torus $T \subset M$ parallel to ∂N with two dividing curves of slope -1. Then T bounds a solid torus N' on which ξ is unique up to isotopy rel boundary, so if $M' = (S^1 \times D^2) \backslash N'$ then it just remains to be seen that (M', ξ) is minimally twisting.

If (M', ξ) is not minimally twisting, then M' contains a convex boundary-parallel torus with dividing slope s not between $-\frac{p}{q}$ and -1. This splits M' into two $T^2 \times I$ with boundary slopes $(-\frac{p}{q}, s)$ and $(s, -1)$; if $s > -1$ then the second $T^2 \times I$ contains a convex torus with slope 0, and if $s < -\frac{p}{q}$ then the first one does. Either way, M' contains such a torus T' and we can Legendrian realize a curve $\gamma \subset T'$ of slope 0 with $tw(\gamma, T') = 0$. But then γ is isotopic to ∂D^2, i.e. it bounds a disk in $S^1 \times D^2$, and so γ is a topological unknot with $tb(\gamma) = 0$, contradicting the tightness of ξ. We conclude that (M', ξ) is minimally twisting after all. □