Math 273 Lecture 10

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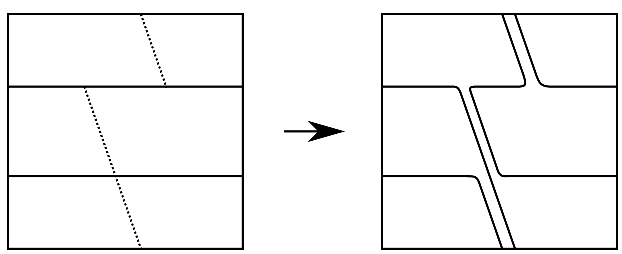
Having completed the study of tight contact structures on $S^2 \times I$, we now begin to investigate $T^2 \times I$. Our goal is to achieve enough of a classification for $T^2 \times I$ and solid tori so that we can describe all tight contact structures on lens spaces; we will do this by looking for bypasses to simplify a given contact structure.

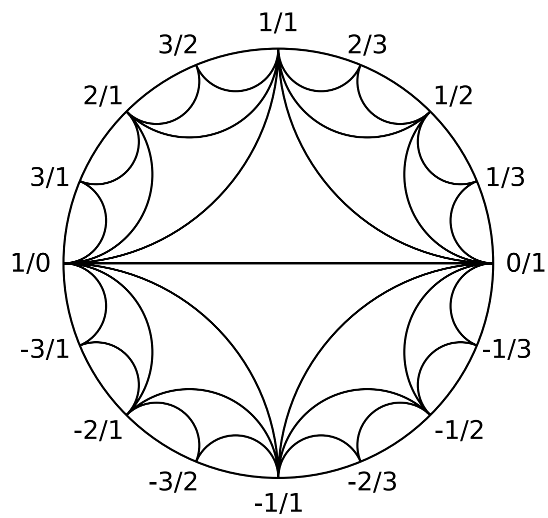
Lemma 1. *Let T be a convex torus with two parallel dividing curves of slope 0, and attach a bypass D along a linear arc D of slope r, $-\infty < r \leq -1$. Then there is a neighborhood $T^2 \times I$ of $T \cup D$, with $T = T^2 \times \{0\}$, so that $T^2 \times \{1\}$ is convex of slope -1.*

Proof. We have the following picture of the dividing set after isotoping T^2 across D:



The case $r = -1$ is degenerate, since the endpoints of the attaching arc coincide, but one can check that the end result is the same. □

In order to consider the effect of attaching bypasses to such tori in general, we must study the Farey tessellation of the hyperbolic plane $\mathbb{H}^2$. In the Poincaré disk model D of $\mathbb{H}^2$, this is achieved by placing numbers $\frac{0}{1}$ and $\frac{1}{0}$ at the points 1 and -1 in ∂D and joining them by a diameter, and then every time we see two points $\frac{p}{q}$ and $\frac{r}{s}$ connected by a geodesic in the upper half of ∂D we label the midpoint of the arc between them with $\frac{p+r}{q+s}$ and connect it to both $\frac{p}{q}$ and $\frac{r}{s}$ by geodesics. We then reflect this across the x–axis to label the bottom half of ∂D with the corresponding negative fractions:



Lemma 2. *The vectors* $\begin{pmatrix} p \\ q \end{pmatrix}$ *and* $\begin{pmatrix} r \\ s \end{pmatrix}$*, with all coordinates positive, are an integral basis of* $\mathbb{Z}^2$ *if and only if* $\frac{p}{q}$ *and* $\frac{r}{s}$ *are connected by an edge in the Farey tessellation.*

Proof. We will induct on $\max(q, s)$; note that it is true for $\max(q, s) = 1$ by inspection, so assume that $q < s$ and $s \geq 2$. There is an edge connecting $\frac{p}{q}$ to $\frac{r}{s}$ iff there is one connecting $\frac{p}{q}$ to $\frac{r-p}{s-q}$, which by hypothesis happens iff and $\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} r-p \\ s-q \end{pmatrix}$ is a basis, which is true iff $\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix}$ is. □

We remark that $SL_2(\mathbb{Z})$ acts on the Farey tessellation by fractional linear transformations; these preserve the ordering and whether or not two given points are connected by a geodesic.

Theorem 3 (Honda)**.** *Let T be a convex torus with two dividing curves of slope s, and attach a bypass D along an arc of slope $r \neq s$. Let s' be the first point we reach by traveling counterclockwise from r along ∂D which is connected to s by a geodesic. If $T \cup D$ has a neighborhood $T^2 \times [0, 1]$ with convex boundary as before, then $T^2 \times \{1\}$ has two dividing curves of slope s'.*

Example 4. If $r = \frac{1}{3}$ and $s = \frac{2}{1}$ then $s' = \frac{1}{1}$.

Proof. We have already shown this for $s = \frac{0}{1}$ and $r \leq -1$: then r lies in the lower left quadrant of the Farey tessellation, and since $\frac{0}{1}$ is only connected to points of the form $\pm\frac{1}{n}$, the first such point we will hit is $s' = -\frac{1}{1}$.

In the general case, let s'' be defined the same way as s' but by traveling clockwise from r. Then s, s' specify vectors $v_s, v_{s'}$ which form an integral basis of $\mathbb{Z}^2$, so there is a matrix $A \in SL_2(\mathbb{Z})$ which sends s to $\frac{0}{1}$ and s' to $\frac{1}{0}$, and r and s'' to something in the upper half plane. Since As'' is connected to $As = \frac{0}{1}$, As'' must have the form $\frac{1}{n}$ for some $n > 0$. Similarly, since $As' = \frac{1}{0}$ is the first point we reach of the form $\frac{1}{m}$ traveling counterclockwise from Ar, we must have $Ar > 1$; but then this implies that $As'' = \frac{1}{1}$. Now s' and s'' are connected by an edge, since $As' = \frac{1}{0}$ and $As'' = \frac{1}{1}$ are.

Since s, s', s'' are all connected by edges, we can find an element $B \in SL_2(\mathbb{Z})$ which takes s to $\frac{0}{1}$, s'' to $\frac{1}{0}$, and s' to $-\frac{1}{1}$. Then Br lies on the arc traveling clockwise from $Bs'' = \frac{1}{0}$ to $Bs' = -\frac{1}{1}$, so $Br \leq -1$. From the first case we know that if the dividing curves on T have slope Bs and we attach a bypass along an arc of slope Br, then the result will have dividing curves of slope $Bs' = -1$. Applying B^{-1} to each of these slopes yields the desired result. □

Definition 5. Let ξ be a tight contact structure on $T^2 \times [0,1]$ with convex boundary, and let s_i be the slopes of the dividing curves Γ_i on $T^2 \times \{i\}$ for $i = 0, 1$. Then ξ is called *minimally twisting* if for any convex torus parallel to the boundary, the slopes of the dividing curves are between s_0 and s_1. (This means that the slopes all lie on the arc which travels clockwise around the unit circle from s_0 to s_1.)

Our goal will be to factor tight contact structures on $T^2 \times I$ into a union of minimally twisting contact structures, and then to understand the minimally twisting ones completely. We will eventually see that the simplest minimally twisting structures correspond to bypass attachments.

Definition 6. Let $(T^2 \times [0,1], \xi)$ be a tight contact structure for which each $T^2 \times \{i\}$ $(i = 0, 1)$ is convex with two dividing curves of slope s_i. We say that ξ is a *basic slice* if it is minimally twisting and s_0 is connected to s_1 by a geodesic in the Farey tesselation.

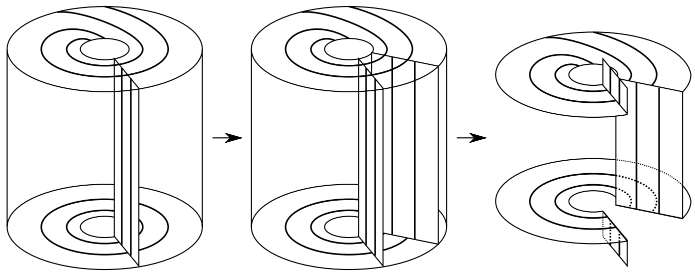
Lemma 7. *There are at most two basic slices for a given s_0 and s_1.*

Proof. Assume by a change of basis that $s_0 = 0$ and $s_1 = -1$, and by Giroux flexibility we take each $T^2 \times \{i\}$ $(i = 0, 1)$ to have a characteristic foliation consisting of lines of infinite slope. (When we have such a foliation, these lines are called *ruling curves*, and by definition they are Legendrian.) Take an annulus A with boundary consisting of one ruling curve c_i in each $T^2 \times \{i\}$, and perturb A to be convex; then each c_i intersects Γ_A in $|c_i \cap \Gamma_i| = 2$ points, and so $tw(c_i, A) = -1$.

By Giroux's criterion, Γ_A has no closed components, so it must consist of a pair of arcs. If each arc connects two points on the same component of ∂A, then at least one of them is boundary-parallel and so gives rise to a bypass along c_i; pushing c_i along this bypass yields a Legendrian curve $\gamma \subset A$ with $tw(\gamma, A) = 0$. But then we can find a convex torus T parallel to $T^2 \times \{i\}$ and containing γ, and since $tw(\gamma, T) = 0$, Γ_T does not intersect γ.This means that the dividing curves on Γ_T have slope ∞, contradicting the fact that ξ is minimally twisting. Therefore each arc of Γ_A connects the two components of ∂A.

Given an arc α of Γ_A, we can now define the *holonomy* $h(A) \in \mathbb{Z}$ to be the number of times α wraps around $A = S^1 \times I$ in the S^1-direction, where we take a vertical arc $\{p\} \times I$ to have zero holonomy. In an I-invariant neighborhood of

A, we can take a parallel copy A' of A and construct a new convex annulus by gluing pieces of A and A' to pieces of parallel copies of $T^2 \times \{i\}$ as follows:



(In this picture, each horizontal slice is a T^2 in which the inner circle and outer circle are identified, and A is drawn vertically.) After edge rounding, the new annulus A'' has holonomy $h(A'') = h(A) \pm 1$ depending on the direction in which we pushed A' off of A. But clearly A'' is isotopic to A rel boundary, so we can choose A to have any holonomy we want. Thus any basic slice has a convex annulus A of this form with holonomy 0, and we can choose it to have a standard characteristic foliation.

Next, we cut $T^2 \times I$ open along A. The result is a tight contact structure on a solid torus $D^2 \times S^1$ whose boundary has two dividing curves in the homology class $[\partial D^2] + 2[S^1]$. This contact structure is *not* unique up to isotopy, but it is close: we can Legendrian realize a meridian γ, isotopic to $\partial D^2 \times \{*\}$, which intersects the dividing set in four points, and find a convex disk Δ with boundary γ. Then Γ_Δ consists of two arcs, but there are two possible choices for Γ_Δ; for either one we fix a characteristic foliation and cut along Δ to get a tight B^3, which must be unique up to isotopy. Gluing back together along whichever Δ we had and then along the uniquely determined A, we see that there are at most two possibilities for ξ up to isotopy. □

We claim that there are exactly two basic slices with $s_0 = 0$ and $s_1 = -1$, and hence for any other pair of slopes connected by a geodesic in the Farey tessellation. We get these by examining the contact structure

$$(T^2 \times \mathbb{R}, \xi = \ker(\sin(2\pi z)dx + \cos(2\pi z)dy)),$$

which is tight: if there were an overtwisted disk contained in some $T^2 \times [-n, n]$, then it would exist in the quotient under the map $z \mapsto z + 2n$, which is the tight contact structure ξ_{2n} on T^3. Each level $T^2 \times \{z\}$ has characteristic foliation directed by $\cos(2\pi z)\partial_x - \sin(2\pi z)\partial_y$, which has slope $-\tan(2\pi z)$. In particular, we can take

$$T_1 = T^2 \times [0, \frac{1}{8}]$$

and perturb the boundary so that the dividing curves have slopes 0 on $z = 0$ and -1 on $z = \frac{1}{8}$. Similarly, we take

$$T_2 = T^2 \times [\frac{1}{2}, \frac{5}{8}].$$

First, we need to distinguish between the contact structures (T_i, ξ_i), where $\xi_i = \xi|_{T_i}$. To do this we introduce an invariant of contact structures on manifolds with boundary.

Definition 8. Let (M, ξ) have convex boundary, assume that $\xi|_{\partial M}$ is trivializable, and let s be a nonzero section of $\xi|_{\partial M}$. The *relative Euler class* $e(\xi, s) \in H^2(M, \partial M)$ is defined as the Poincaré dual of the zero section of any generic extension of s to a global section of ξ.

Lemma 9. *If $\Sigma \subset (M, \xi)$ is a properly embedded, oriented convex surface with boundary, and s is a nonzero section of $\xi_{\partial M}$ which is tangent to $\partial\Sigma$ with the correct orientation, then*

$$\langle e(\xi, s), \Sigma\rangle = \chi(\Sigma_+) - \chi(\Sigma_-).$$

Proof. Perturb Σ so that Σ_ξ is Morse-Smale, and then describe each side as a count of singularities of Σ_ξ just as in the closed case. □

Lemma 10. *The contact structures (T_1, ξ_1) and (T_2, ξ_2) are not isotopic rel boundary.*

Proof. We compute their relative Euler classes in $H^2(T^2 \times I, T^2 \times \partial I) \cong H_1(T^2 \times I)$. Fix the characteristic foliation of each $T^2 \times \{i\}$ to consist of lines of infinite slope; then we can find a convex annulus A with boundary consisting of one such curve in each $T^2 \times \{i\}$. We have already argued that A has two dividing arcs, each of which connects one component of ∂A to the other, and so $\langle e(\xi_1, s), A\rangle = \chi(A_+) - \chi(A_-) = 0$ and likewise for ξ_2.

On the other hand, we can arrange the characterisic foliation so that there are Legendrian curves of slope 0 in each $T^2 \times \{i\}$, and let A' be a convex annulus each of whose boundary components is one such curve. Then $\partial A'$ doesn't intersect the dividing set of $T^2 \times \{0\}$, but it intersects the dividing set of $T^2 \times \{1\}$ twice, so $\Gamma_{A'}$ consists of a single boundary-parallel arc. This means that A'_+ and A'_- are a disk and an annulus in some order, so $\langle e(\xi_i, s), [A']\rangle = \pm 1$.

Now we observe that we can obtain T_2 from T_1 by the map $(x, y, z) \mapsto (-x, -y, z)$, and this map changes the sign of $\langle e(\xi_i, s), [A']\rangle$. In particular, the relative Euler classes of ξ_1 and ξ_2 have opposite signs, and so these contact structures are not isotopic. □

If we can show that (T_1, ξ_1) and (T_2, ξ_2) are basic slices, then, this will complete the proof that there are exactly two basic slices with boundary slopes $s_0 = 0$ and $s_1 = -1$. It remains to be seen that they are minimally twisting; we will postpone the proof of this fact until next time.

Proposition 11. *T_1 and T_2 are minimally twisting.*

Corollary 12. *There are exactly two basic slices with $s_0 = 0$ and $s_1 = -1$.*