Math 273 Lecture 1

Steven Sivek

January 25, 2012

Definition 1. Let M be a $(2n+1)$-dimensional Riemannian manifold. A *contact structure* on M is a hyperplane field $\xi^{2n} \subset TM$ which can be written locally as the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a positive volume form.

In this course we will focus on *co-orientable* contact structures, those for which the normal bundle $\xi^\perp$ is trivial. This is equivalent to the existence of a global choice of α: if v is a section of $\xi^\perp$, we can take $\alpha_p(x) = \langle v_p, x \rangle$. We will also restrict our attention to 3-manifolds after today.

Example 2. On $\mathbb{R}^{2n+1}$ we have the standard contact structure $\xi_{\mathrm{st}} = \ker(dz - \sum y_i dx_i)$.

Example 3. The overtwisted structure $(\mathbb{R}^3, \xi_{\mathrm{ot}})$, $\xi_{\mathrm{ot}} = \ker(\cos(r)dz + r\sin(r)d\theta)$.

Remark 4. The disk $D = \{r = \pi, z = 0\}$ is *overtwisted*: $\xi|_{\partial D} = TD|_{\partial D}$. Bennequin proved that no such disk exists in $(\mathbb{R}^3, \xi_{\mathrm{st}})$, so these two contact structures are distinct.

Example 5. (S^3, ξ_{st}) with $\alpha = \sum_{i=1}^2 x_i dy_i - y_i dx_i$, where S^3 is the unit sphere in $\mathbb{R}^4 = \mathbb{C}^2$.

Exercise 6. (S^3, ξ_{st}) minus a point is contactomorphic to $(\mathbb{R}^3, \xi_{\mathrm{st}})$.

The standard contact structure on S^3 is an example of a phenomenon that suggests one reason why we might study contact structures. Recall that a complex manifold (X, J) is a *Stein manifold* if it admits a strictly plurisubharmonic exhausting function, i.e. a function $\phi : X \to \mathbb{R}$ for which $\omega = -d(d\phi \circ J)$ defines a metric $g(v, w) = \omega(v, Jw)$ and for which the sets $\phi^{-1}((-\infty, c])$ are compact.

Proposition 7. *Suppose c is a regular value of ϕ and let $M = \phi^{-1}(c)$. Then the hyperplane field $TM \cap J(TM)$ defines a contact structure on M.*

Proof. We can check that $J(TM) = \ker(d\phi \circ J)$: if $v = Jx$ with $x \in T_pM$, then $d\phi(Jv) = d\phi(-x) = -d\phi(x) = 0$. Thus if $\alpha = -d\phi \circ J$ defines a 1-form on M by restriction and $\xi = \ker(\alpha)$ then we have $\xi = TM \cap J(TM)$. Given $x \in \xi$, we have

$$d\alpha_p(x, Jx) = d(-d\phi \circ J)(x, Jx) = \omega(x, Jx) > 0$$

and so $d\alpha$ is positive on ξ. It follows that $\alpha \wedge d\alpha$ is a volume form on M, i.e. that ξ is contact. □

Example 8. (T^3, ξ_n) where $\xi_n = \ker(\cos(2\pi nz)dx - \sin(2\pi nz)dy))$. Only ξ_1 is Stein fillable.

Just as in symplectic geometry, we have a theorem which says that locally ξ_{st} is the only possible contact structure; in general contact structures provide only global information.

Theorem 9 (Darboux's Theorem)**.** *Let ξ be a contact structure on M, and take a point $p \in M$. Then p has a neighborhood with local coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ such that $\xi = \ker(\alpha_0)$, where $\alpha_0 = dz - \sum y_i dx_i$.*

Proof. Assume without loss of generality that $p = (0, 0, \dots, 0) \in \mathbb{R}^{2n+1}$ and that $\xi = \ker(\alpha_1)$, where ξ_p has the desired form. Rescale α_1 so that $\alpha_0|_p = \alpha_1|_p$. The family of 1-forms

$$\alpha_t = (1-t)\alpha_0 + t\alpha_1$$

satisfy $\alpha_t \wedge (d\alpha_t)^n = \alpha_0 \wedge (d\alpha_0)^n > 0$ at p, hence α_t is a contact form on a neighborhood U of p. We now wish to find an isotopy ψ_t of U such that $\psi_t^*\alpha_t = \alpha_0$. For this we use Moser's trick.

Suppose that ψ_t exists and is the flow of a vector field v_t such that $v_t \in \xi_t$. By differentiating $\psi_t^*\alpha_t = \lambda_t\alpha_0$, we get

$$\psi_t^*(\dot{\alpha}_t + \mathcal{L}_{v_t}\alpha_t) = \dot{\lambda}_t\alpha_0 = \frac{\dot{\lambda}_t}{\lambda_t}\psi_t^*\alpha_t$$

and Cartan's formula gives us $\mathcal{L}_{v_t}\alpha_t = \iota_{v_t}d\alpha_t + d(\iota_{v_t}\alpha_t) = \iota_{v_t}d\alpha_t$, so after applying the diffeomorphism $(\psi_t^*)^{-1}$ we wish to solve the equation

$$\dot{\alpha}_t + \iota_{v_t}d\alpha_t = \mu_t\alpha_t$$

where $\mu_t = (\psi_t^*)^{-1}\frac{d}{dt}(\log\lambda_t)$. On the *Reeb vector field* R_t of α_t, which is defined by the equations $\alpha_t(R_t) = 1$ and $\iota_{R_t}d\alpha_t = 0$, this equation becomes $\mu_t = \dot{\alpha}_t(R_t)$, so we may solve for λ_t. Now that this equation is automatically satisfied on the line through R_t, it remains to find $v_t \in \xi_t$ so that

$$d\alpha_t(v_t, \cdot)|_{\xi_t} = (\mu_t\alpha_t - \dot{\alpha}_t)\,|_{\xi_t}.$$

But $d\alpha_t$ is a symplectic form on ξ_t, and in particular it is nondegenerate, so the desired v_t exists. □

Exercise 10 (Contact neighborhood theorem)**.** Generalize this proof to show the following: if $L_1 \subset (M_1, \xi_1)$ and $L_2 \subset (M_2, \xi_2)$ are closed submanifolds and there is a diffeomorphism $\phi : L_1 \to L_2$ such that $\phi^*(\xi_2|_{L_2}) = \xi_1|_{L_1}$, then ϕ extends to a contactomorphism on open neighborhoods of the L_i.

Moser's trick can also be used to show that deformations of a contact structure can be realized by isotopies of the ambient manifold.

Theorem 11 (Gray's Theorem)**.** *Let $\{\xi_t\}_{t\in[0,1]}$ be a family of contact structures on the closed manifold M. There is a family ψ_t of diffeomorphisms with $\psi_0 = \mathrm{id}_M$ such that $\psi_t^*\xi_t = \xi_0$.*

Proof. Write $\xi_t = \ker(\alpha_t)$ for α_t a smooth family of 1-forms, so we want to solve the equation $\psi_t^*\alpha_t = \lambda_t\alpha_0$ for some $\lambda_t : M \to \mathbb{R}_{>0}$. Again we suppose that ψ_t is the flow of a vector field $v_t \in \xi_t$, and the equation becomes

$$\dot{\alpha}_t + \iota_{v_t} d\alpha_t = \mu_t \alpha_t$$

where $\mu_t = \frac{d}{dt}(\log \lambda_t) \circ \psi_t^{-1}$. Plugging in R_t determines μ_t uniquely, and then v_t is determined just as before. $\square$

So contact structures all look the same locally – namely, they look like $(\mathbb{R}^{2n+1}, \xi_{\text{st}})$ – and the moduli space of contact structures on a manifold M is discrete. The fact that it is also nonempty for closed, oriented 3-manifolds is due to Lutz and Martinet, and there are many proofs of this fact. We will give a proof due to Thurston and Winkelnkemper which hints at the Giroux correspondence, a classification of contact structures which we will discuss later in the course.

Theorem 12. *Every closed, oriented* 3*-manifold admits a contact structure.*

Proof. Let M be such a manifold. Alexander showed in the 1920s that M has an open book decomposition: there is an oriented link $L \subset M$ called the "binding" and a fibration $\pi : M \backslash L \to S^1$ for which the "pages", i.e. the fibers $\Sigma_\theta = \pi^{-1}(\theta)$, are surfaces with $\partial\Sigma_\theta = L$. We will construct a contact form on $M \backslash \nu(L)$ and then extend it to each component of the binding.

Write $M \backslash \nu(L)$ as the mapping torus of $h : \Sigma \to \Sigma$, i.e. as

$$M_h = \frac{\Sigma \times [0,1]}{(x,1) \sim (h(x),0)},$$

where Σ is compact and h is the identity on a neighborhood of $\partial\Sigma$. Let ω be an area form on Σ with total area $2\pi|\partial\Sigma|$ and $\omega = dt \wedge d\theta$ on a neighborhood $[-1,0]_t \times S^1_\theta$ of each component of $\partial\Sigma$. If α is a 1-form on Σ which equals $(1+t)d\theta$ near $\partial\Sigma$,then $\omega - d\alpha$ vanishes near $\partial\Sigma$ and

$$\int_\Sigma \omega - d\alpha = 2\pi|\partial\Sigma| - \int_{\partial\Sigma} \alpha = 0.$$

Thus $\omega - d\alpha = d\beta$ for some 1-form β supported away from $\partial\Sigma$, and so $\lambda = \alpha + \beta$ is a 1-form with $d\lambda$ an area form and $\lambda = (1+t)d\theta$ near $\partial\Sigma$. The set of all such λ is convex, and if λ is in this set then so is $h^*\lambda$. Define a 1-form on M_h by $\lambda_\phi = \phi\lambda + (1-\phi)h^*\lambda$, and then set

$$\alpha_h = \lambda_\phi + K d\phi.$$

Then $\alpha_h \wedge d\alpha_h = \lambda_\phi \wedge d\lambda_\phi + K d\lambda_\phi \wedge d\phi$, and since λ_ϕ is an area form on each page Σ_ϕ this is a volume form for large K. Furthermore, we have $\alpha_h = (1+t)d\theta + K d\phi$ near each component of $\partial\Sigma \times S^1 = \partial M_h$.

To finish the construction, we need to find a contact structure on the solid torus $D^2_{(r,\phi)} \times S^1_\theta$ which is equal to $-rd\theta + Kd\phi$ near its boundary (here $r = 1+t$). Near the center of D^2 we take the contact form

$$\alpha = d\theta + r^2 d\phi$$

and in between we interpolate by taking

$$\alpha = f(r)d\theta + g(r)d\phi.$$

The condition $\alpha \wedge d\alpha > 0$ becomes $fg' - gf' > 0$, with $(f,g) = (1, r^2)$ near $r = 0$ and $(f,g) = (-r, K)$ near $r = 1$. It is an easy exercise to show that such f, g exist. ☐

We are also interested in the study of various knots in contact 3-manifolds.

Definition 13. A smoothly embedded knot $K \subset (M, \xi)$ is called *Legendrian* if $T_x K \subset \xi_x$ for all $x \in K$, i.e. if it is tangent to the contact planes. A knot for which $T_x K \pitchfork \xi_x$ instead is called *transverse*; if K is oriented and parametrized by $\gamma : S^1 \to M$, and $\xi = \ker(\alpha)$, then K is positively or negatively transverse depending on the sign of $\alpha(\gamma'(t))$.

For example, a Legendrian knot $K \subset (\mathbb{R}^3, \xi_{\mathrm{st}})$ satisfies the condition $y = \frac{dz}{dx}$. Thus we can recover the knot completely from its projection to the xz-plane, known as the *front projection.*

Any Legendrian knot K has a standard neighborhood. Indeed, since we can split $\xi|_K = TK \oplus \nu$ where ν is the $d\alpha$-orthogonal complement of TK, a section of ν canonically determines the *Thurston-Bennequin* framing of K. Given two Legendrian knots, we can find a diffeomorphism carrying one knot with this framing to the other and thus apply the contact neighborhood theorem.

As a model of this neighborhood, we can consider $S^1 \times \mathbb{R}^2$ with contact form

$$\alpha = \cos\theta dx - \sin\theta dy$$

and Legendrian knot $K = S^1 \times \{(0,0)\}$.

Remark 14. If K is null-homologous, one can consider its Thurston-Bennequin framing with respect to the framing induced by a Seifert surface. For Legendrian knots in S^3, for example, this Seifert framing is unique, and so we have a Legendrian isotopy invariant

$$tb(K) \in \mathbb{Z}$$

measuring the twisting of the Thurston-Bennequin framing with respect to the Seifert framing.

Proposition 15. *Any smooth knot in a contact manifold can be C^0-approximated by a Legendrian knot.*

Proof. First, assume we are considering a smooth knot or arc in $(\mathbb{R}^3, \xi_{\text{st}})$. We can approximate it arbitrarily well by a Legendrian knot or arc by drawing lots of zigzags along its front projection, connected by cusps, so that the slope of each segment approximates the y-coordinate of the smooth curve. If we are approximating an arc, we require the segments on either end to have slope equal to the y-coordinate of the corresponding endpoint.

Now assume we have a knot K in an arbitrary contact manifold, and cover it with finitely many Darboux balls $B_1, \dots, B_r, B_{r+1} = B_1$. We pick points $p_i \in B_i \cap B_{i+1}$ to divide the knot into segments, each of which lies in a single Darboux ball, and thus reduce to the case of arcs in $(\mathbb{R}^3, \xi_{\text{st}})$. □

Proposition 16. *Any smooth knot in a contact manifold can be C^0-approximated by a transverse knot.*

Proof. Given $K \subset (M, \xi)$, we first choose a Legendrian approximation. Once K is Legendrian, we take a slight push-off in the normal direction along $\xi|_K$; the result is called the positive or negative push-off depending on which normal direction we take (recall that ξ is oriented by $d\alpha$).

In order to check that the push-off is transverse, we use the model neighborhood $S^1 \times \{(0,0)\} \subset (S^1 \times \mathbb{R}^2, \ker(\cos\theta dx - \sin\theta dy))$: there the contact planes at $p = (\theta, 0, 0)$ are spanned by dz and

$$\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}$$

and so the push-off is parametrized by

$$\gamma(\theta) = (\theta, \pm\epsilon\sin\theta, \pm\epsilon\cos\theta)$$

with tangent vector $\gamma'(\theta) = (1, \pm\epsilon\cos\theta, \mp\epsilon\sin\theta)$ satisfying $\alpha(\gamma'(\theta)) = \pm\epsilon \neq 0$. □

Theorem 17. *A car of length L can be parallel parked in any space of length $L + \epsilon$, $\epsilon > 0$.*

Proof. Let us assume that the car is on the plane $\mathbb{R}^2$. Its position can be described by a single coordinate (x, y) and the angle $\theta \in S^1$ its tires are facing, or equivalently a point in the configuration space $\mathbb{R}^2 \times S^1$, which has contact form

$$\alpha = \sin\theta dx - \cos\theta dy.$$

(Note that $\alpha \wedge d\alpha = -dx \wedge dy \wedge d\theta$, so we will reverse the usual orientation of S^1.) The car's path $\gamma(t) = (x(t), y(t), \theta(t))$ will satisfy $\frac{dy}{dx} = \tan\theta$, or equivalently $\frac{dx}{dt}\sin\theta - \frac{dy}{dt}\cos\theta = 0$: thus $\gamma(t)$ must be Legendrian. We now take a path through configuration space which pulls the car up parallel to the parking spot and then slides it horizontally into place without turning the wheel; this is physically impossible, but an arbitrarily close Legendrian approximation will successfully park the car. □