Exercises for V5D3: Advanced topics in geometry

Steven Sivek

Wintersemester 2016/17

This document will be updated regularly; the exercises will not be graded, but are highly recommended anyway.

Week of October 17:

1. Prove that $S^2 \times S^4$ does not admit a symplectic structure.

2. Let $\phi_t : M \to M$ be the family of diffeomorphisms generated by a *time-dependent* vector field X_t on M. Prove that $\frac{d}{dt}\big(\phi_t^* \alpha\big) = \phi_t^*\big(\mathcal{L}_{X_t}\alpha\big)$ for any k-form α. Conclude that if M is symplectic and X_t is a Hamiltonian vector field, then ϕ_1^* is a symplectomorphism.

3. Prove that $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}(M, \omega)$.

Week of October 24:

4. If X is a smooth submanifold of M, we define its *conormal bundle* as
$$N^*X = \{(x, \xi) \in T^*M \mid x \in X, \xi(v) = 0 \text{ for all } v \in T_xS\}.$$
Prove that N^*X is an exact Lagrangian in T^*M with its usual contact form, and conclude the same for the zero section and any cotangent fiber as special cases.

5. Let $(M, \omega = d\alpha)$ be an exact symplectic manifold, and $L \subset M$ an exact Lagrangian. Prove that if $\Sigma \subset M$ is a nonempty compact surface with boundary $\partial\Sigma \subset L$, then $(\Sigma, \omega|_\Sigma)$ is not symplectic.

6. Let $i_0 : M \hookrightarrow T^*M$ be the inclusion of the zero section into T^*M. Prove that any map $i : M \hookrightarrow T^*M$ which is sufficiently C^1-close to i_0 is also a section of $\pi : T^*M \to M$.

November 3–10:

7. Let Ω_0 and J_0 be the standard symplectic and complex structures on the vector space $\mathbb{R}^{2n}$, and let $Sp(2n), GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ be the automorphisms which preserve Ω_0 and commute with J_0 respectively. Prove that the intersection of any two of $Sp(2n)$, $GL(n, \mathbb{C})$, and $O(2n)$ is the same group, namely $U(n)$.

8. Prove that $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit an almost complex structure.

9. Show that the Nijenhuis tensor $\mathcal{N}(u, v) = [Ju, Jv] - J[u, Jv] - J[Ju, v] - [u, v]$ vanishes for any almost complex structure on a surface.

November 15–22:

10. Recall that the Fubini-Study form ω_{FS} on $\mathbb{CP}^1$ is defined on each coordinate chart $\varphi_i : \{z_i \neq 0\} \to \mathbb{C}$ as $\omega_{FS} = \varphi_i^* \omega$, where $\omega = \frac{i}{2}\partial\bar{\partial}\log(|z|^2 + 1)$ is a Kähler form on $\mathbb{C}$. Compute ω in terms of the real coordinates x and y, where $z = x + iy$, and use this to show that $\int_{\mathbb{CP}^1} \omega_{FS} = \pi$.

11. Show that the Kodaira-Thurston manifold is diffeomorphic to $S^1 \times Y$, where if we write $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates $x, y \in \mathbb{R}/\mathbb{Z}$ then Y is the quotient
$$Y = [0,1] \times T^2 / \big((0,(x,y)) \sim (1,(x+y,y)\big),$$
i.e. the *mapping torus* of the diffeomorphism $h : T^2 \to T^2$, $h(x,y) = (x+y,y)$. Show that if $h : X \to X$ is an orientation-preserving diffeomorphism of a closed, oriented manifold inducing an action $h_* : H_1(X) \to H_1(X)$, then the mapping torus of h has first Betti number $1 + \dim\ker(h_* - \mathrm{Id})$. Conclude that $H_1(S^1 \times Y)$ has rank 3 as claimed.

12. Verify that the standard symplectic form $dx \wedge dy$ on T^2 is h-invariant and use this to construct a symplectic form on $S^1 \times Y$.

November 24–December 1:

13. Fix $d \geq 1$ and define the *Veronese embedding* $i_d : \mathbb{CP}^2 \hookrightarrow \mathbb{CP}^{\binom{d+2}{2}}$ by
$$i_d([z_0 : z_1 : z_2]) = [z_0^d : z_0^{d-1} z_1 : \dots : z_2^d]$$
where the coordinates of $i_d(z)$ range over all degree-d monomials in z_0, z_1, z_2. Prove that i_d is an embedding, and that if f and g are two generic, homogeneous degree-d polynomials in z_0, z_1, z_2 and $B = \{z \mid f(z) = g(z) = 0\}$, then the Lefschetz pencil $\pi(\mathbb{CP}^2 \setminus B) \to \mathbb{CP}^1$ defined by $\pi(z) = [f(z) : g(z)]$ can also be constructed by intersecting $i_d(\mathbb{CP}^2)$ with a pencil of hyperplanes on $\mathbb{CP}^{\binom{d+2}{2}}$.

14. Let (Σ, ω) be a closed surface with symplectic embeddings into two closed 4-manifolds (M_1, ω_1) and (M_2, ω_2), where Σ has self-intersection zero in each M_i. Prove that the fiber sum along Σ satisfies
$$\chi(M_1 \#_\Sigma M_2) = \chi(M_1) + \chi(M_2) - 2\chi(\Sigma),$$
$$\sigma(M_1 \#_\Sigma M_2) = \sigma(M_1) + \sigma(M_2)$$
where $\sigma(X)$ denotes the signature of the intersection form on $H_2(X)$. (Hint: *Novikov additivity* says that if $\partial X_1 = Y$ and $\partial X_2 = -Y$, then $\sigma(X_1 \cup_Y X_2) = \sigma(X_1)\#\sigma(X_2)$.) Construct symplectic manifolds $E(n)$ with elliptic fibrations $E(n) \to \mathbb{CP}^1$, where $\chi(E(n)) = 12n$ and $\sigma(E(n)) = -8n$, for all $n \geq 1$.

15. Let A be the annulus $B^2(\epsilon) \setminus \mathrm{int}(B^2(\delta))$ for some $0 < \delta < \epsilon$. Prove that there is an area-preserving self-diffeomorphism $\phi : A \to A$ which exchanges the two boundary components.

December 6–15:

16. Let X be a smooth 4-manifold and $\pi : X \to S^2$ a Lefschetz fibration with a smooth fiber F and vanishing cycles $c_1, \dots, c_k \subset F$. Prove that $\pi_1(X) \cong \pi_1(F)/N$, where $N \subset \pi_1(F)$ is the normal subgroup generated by $c_1, \dots, c_k$.

17. Define a function $f : \mathbb{CP}^n \to \mathbb{R}$ by the formula

$$f\big([z_0 : z_1 : \cdots : z_n]\big) = \frac{\sum a_i |z_i|^2}{\sum |z_i|^2},$$

where $a_0 < a_1 < \cdots < a_n$. Show that f is Morse by computing the critical points of f and their indices, and determine the corresponding Morse homology.

18. Let M be a connected, open n-manifold, and $f : M \to \mathbb{R}$ a Morse function with finitely many critical points. Prove that f can be modified to give a Morse function with no critical points of index n.

Week of January 16:

19. Let (X, ω) be a symplectic $2n$-manifold, and $Y \subset X$ a closed $(2n-1)$-dimensional submanifold. Suppose that there is a vector field v defined on a neighborhood of Y such that v is transverse to Y and $\mathcal{L}_v \omega = \omega$. Show that:

    - the 1-form $\alpha = \iota_v \omega|_Y$ is a *contact form* on Y, meaning that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on Y;
    - $d\alpha$ is a symplectic form on the *contact structure* $\xi = \ker(\alpha) \subset TY$;
    - some neighborhood of Y in X is symplectomorphic to a neighborhood of $\{0\} \times Y$ in the *symplectization*
    $$(\mathbb{R} \times Y, d(e^t \alpha)).$$

    (Y is called a *contact-type* hypersurface in X.)

20. Suppose that Y embeds into two symplectic manifolds (X, ω) and (X', ω') as a separating contact-type hypersurface, and write

$$X = X_1 \cup_Y X_2, \qquad X' = X_1' \cup_Y X_2'$$

with the Liouville vector field pointing from X_1 and X_2 into X_1' and X_2'. If there is a diffeomorphism $\varphi : Y \to Y$ such that $\varphi^* \xi' = \xi$, where ξ and ξ' are the induced contact structures, show that the manifold $X_1 \cup_\varphi X_2'$ admits a symplectic structure.

January 24:

21. Which homology classes in $H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$ can be represented by a pseudo-holomorphic sphere $u : (S^2, j) \to (\mathbb{CP}^2, J)$ for some J? Are there homology classes in which any two such spheres with distinct images must intersect transversally?