Numerical inverse scattering for Korteweg–de Vries and nonlinear Schrödinger equations

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joint work with
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- We numerically solve the Cauchy problem for the focusing and defocusing nonlinear Schrödinger (NLS) equations

$$\begin{aligned} \mathrm{i}u_t + u_{xx} \pm 2\,|u|^2\,u &= 0 \\ u(0,x) &= u_0(x) \end{aligned}$$

and the Korteweg–de Vries (KdV) equation

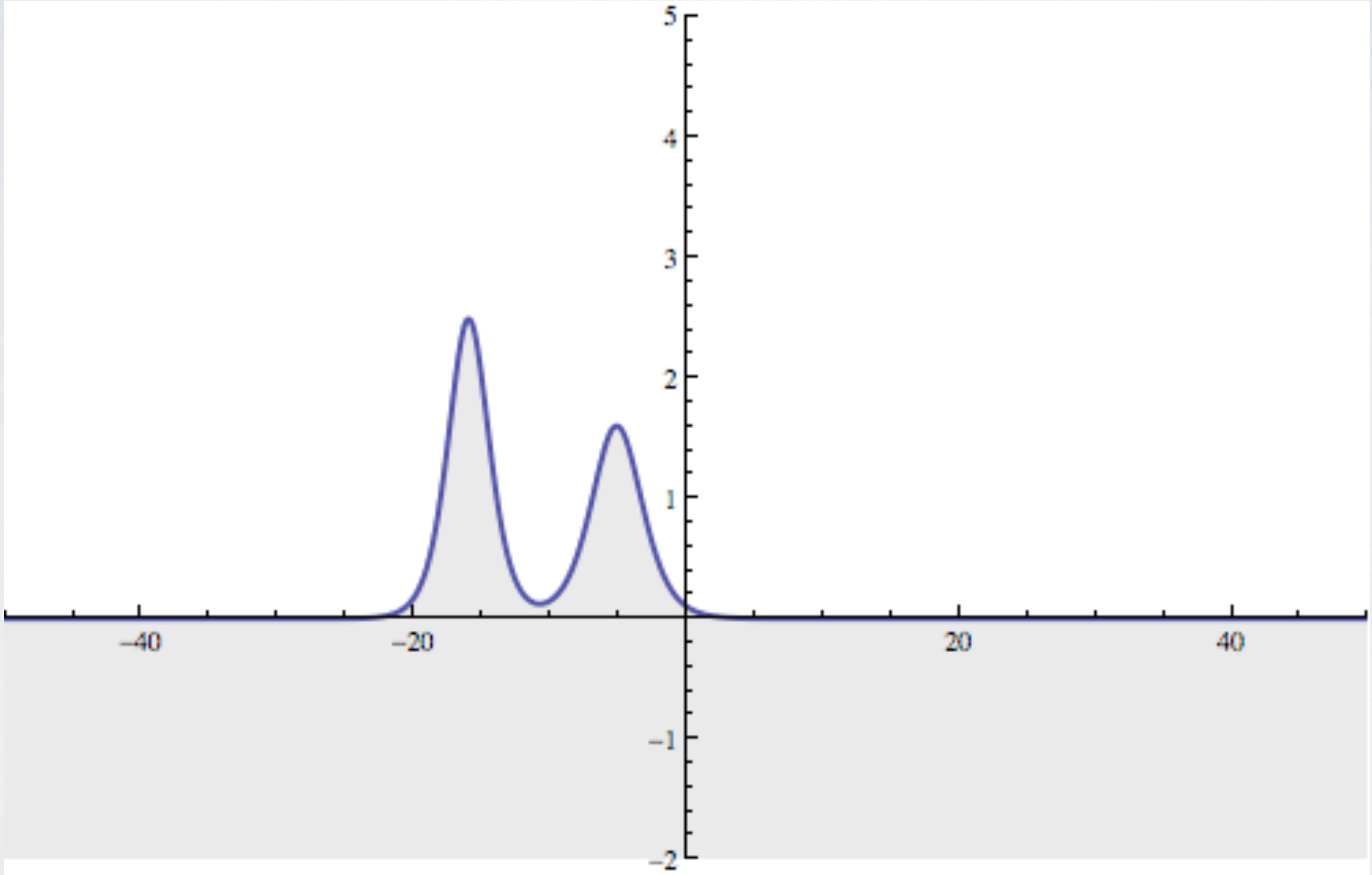
$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= 0 \\ u(0,x) &= u_0(x) \end{aligned}$$

- We assume the initial condition is smooth and exponentially decaying
- We use inverse scattering numerically:
  - We compute the forward transform by utilizing spectral methods
  - We compute the inverse transform by solving Riemann–Hilbert problems numerically
  - By deforming the Riemann–Hilbert problem in the complex plane, we get representations which are numerically stable and accurate uniformly for all space and time

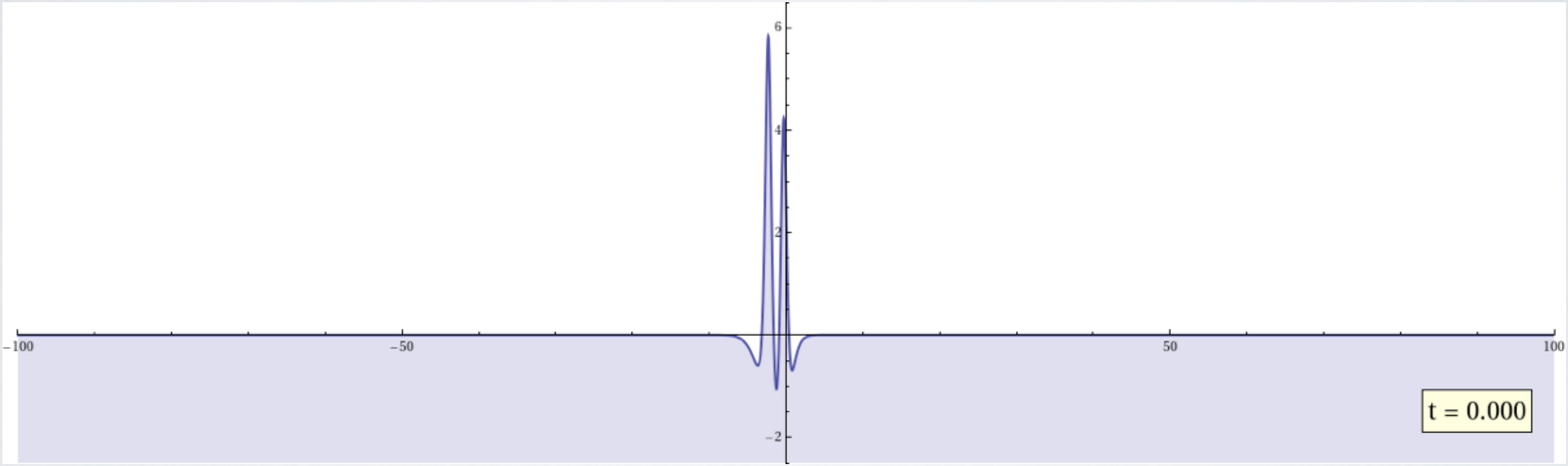
Why not use standard numerical methods?

- The standard numerical approach is to employ pseudo-spectral methods
  - Represent the solution by N evenly spaced points on, say, $[-L, L]$
  - Reduce the PDE to an ODE on how the solution evolves pointwise in time

KdV (Pure solitons)



A more generic solution to KdV



Why not use standard numerical methods?

- The standard numerical approach is to employ pseudo-spectral methods
  - Represent the solution by N evenly spaced points on, say, $[-L, L]$
  - Reduce the PDE to an ODE on how the solution evolves pointwise in time
- Dispersion causes this approach to quickly fail as time T increases
  - Dispersive waves travel linearly with time, so we must take $L = \mathcal{O}(T)$
  - Therefore, $N = \mathcal{O}(T)$
  - The CFL condition states that the time step must be $\Delta t = \mathcal{O}(N^{-1})$
  - The total work is at least

$$\mathcal{O}\left(N \log N \frac{T}{\Delta t}\right) = \mathcal{O}(T^3 \log T)$$

  - The constant in front is initial condition dependent, and typically very large!

Review: Fourier solution of linear KdV

$$u_t + u_{xxx} = 0$$
$$u(0, x) = u_0(x)$$

$$u_0(x) \xrightarrow{\frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(x) \mathrm{e}^{-\mathrm{i}kx} \,\mathrm{d}x} \hat{u}_0(k)$$

Time evolution

$$\hat{u}_0(k) \mathrm{e}^{\mathrm{i}tk^3}$$

$$u(t, x) \xleftarrow{\int_{-\infty}^{\infty} \hat{u}_0(k) \mathrm{e}^{\mathrm{i}(tk^3 + kx)} \,\mathrm{d}x} \hat{u}_0(k) \mathrm{e}^{\mathrm{i}tk^3}$$

Inverse scattering for KdV

Direct transform

$u_0(x)$ $\longrightarrow$ $r(k), \{\kappa_j\}, \{C_j\}$

Time evolution $\downarrow$

Inverse transform

$u(t,x)$ $\longleftarrow$ $r(k)\mathrm{e}^{\mathrm{i}tk^3}, \{\kappa_j\}, \left\{C_j \mathrm{e}^{\mathrm{i}tk_j^3}\right\}$

- Associated with an integrable PDE its Lax pair:

$$\mu_x = A(k,u,x)\mu \qquad \text{and} \qquad \mu_t = B(k,u,x)\mu$$

- The variable u must satisfy the relevant PDE for the pairs to be compatible
- For our purposes, we only need the $\mu_x = A\mu$ equations, which are:
  - KdV (self-adjoint):

$$\mu_{xx} + (k^2+u)\mu = 0$$

  - defocusing NLS (self-adjoint):

$$\mu_x = \begin{pmatrix} -\mathrm{i}k & u \\ \bar{u} & \mathrm{i}k \end{pmatrix}\mu$$

  - focusing NLS (not self-adjoint):

$$\mu_x = \begin{pmatrix} -\mathrm{i}k & u \\ -\bar{u} & \mathrm{i}k \end{pmatrix}\mu$$

  - The direct scattering transform consists of spectral analysis of these equations

KdV Direct Scattering

- The direct scattering transform is a map

$$\text{initial condition } u_0(x) \mapsto \text{scattering data } r(k), \{\kappa_1, \ldots, \kappa_N\}, \{C_1, \ldots, C_N\}$$

- The scattering data results from spectral analysis of

$$\mathcal{L} = \partial_x^2 + u_0(x)$$

- This operator has continuous spectra on $(-\infty, 0]$ and N discrete eigenvalues on the positive real axis at the points $\{\kappa_1, \ldots, \kappa_N\}$:

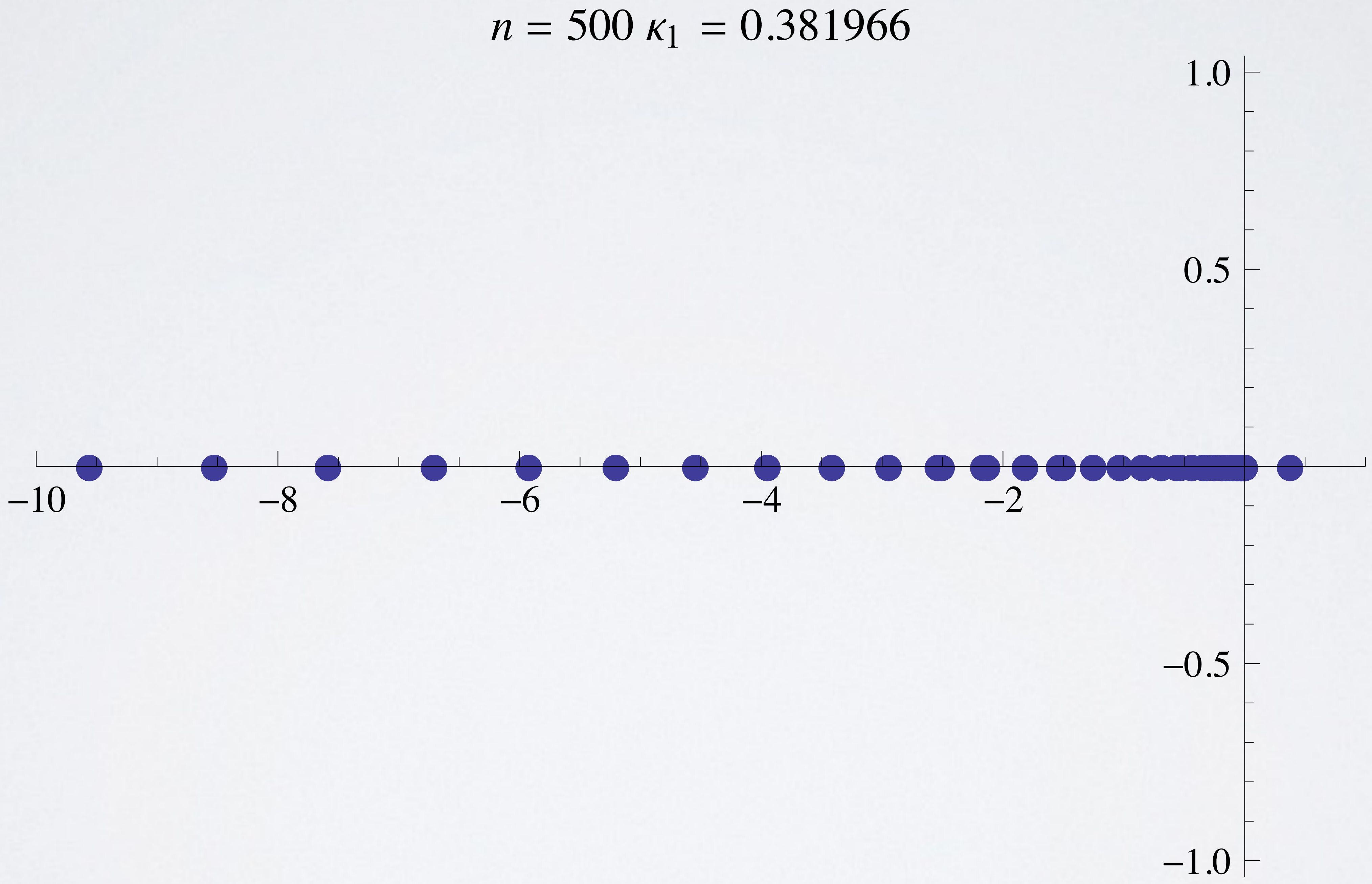
- The discrete eigenvalues $\kappa_1, \ldots, \kappa_N$ of

$$\mathcal{L} = \partial_x^2 + u_0(x)$$

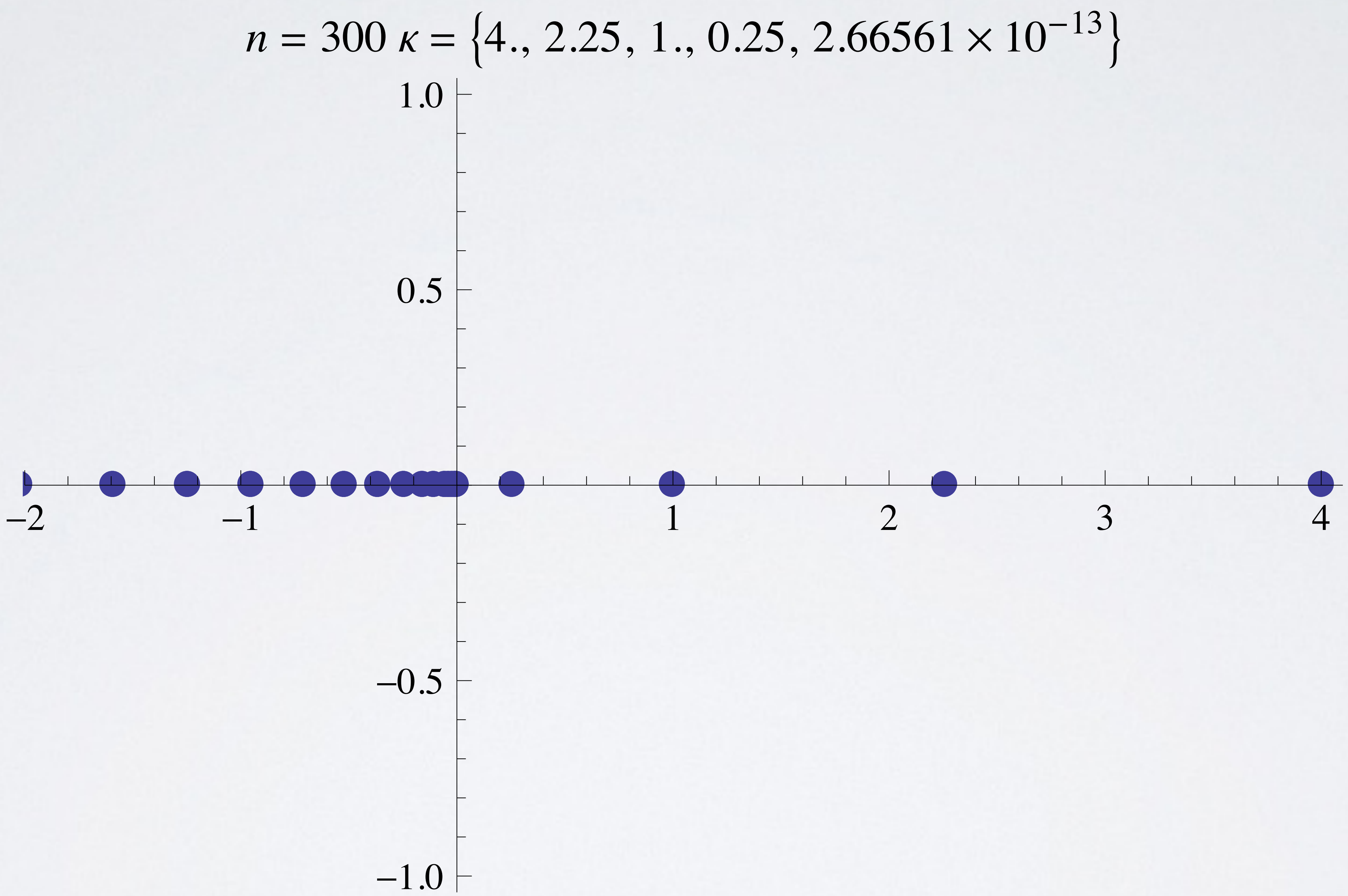
  correspond to the N-solitons of the solution

- These can be computed using Hill's method:
  - Map $(-\infty, \infty)$ to $[-\pi, \pi)$ using $\frac{2}{\pi} \arctan x$
  - Represent the mapped operator $\mathcal{L}$ by its action on n Fourier coefficients
  - Calculate the positive eigenvalues of this discretized matrix
  - Verify the eigenvectors correspond to $\mathrm{L}^2(-\infty, \infty)$ solutions

$$q_0(x) = \operatorname{sech}^2 x$$



Computed spectra for $u_0(x) = 5\,\mathrm{sech}^2 \frac{x}{2}$



- The *reflection coefficient* $r(k)$ is defined for real k and corresponds to the continuous spectrum of the previous Schrödinger operator

- For each k, we can solve

$$\mu_{xx} + (u_0(x) + k^2)\mu = 0$$

with three initial conditions:

$$\phi(x) \sim \mathrm{e}^{\mathrm{i}kx}, \quad x \to -\infty$$
$$\psi^+(x) \sim \mathrm{e}^{\mathrm{i}kx} \quad \text{and} \quad \psi^-(x) \sim \mathrm{e}^{-\mathrm{i}kx}, \quad x \to \infty$$

- This is a second order equation: therefore there exists a and b so that

$$\phi(x) = a(k)\psi^+(x) + b(k)\psi^-(x)$$

- Then $r(k) = b(k)/a(k)$

- We need to solve

$$\phi''(x) + (u_0(x) + k^2)\phi(x) = 0, \qquad \phi(x) \sim \mathrm{e}^{\mathrm{i}kx}, \quad x \to -\infty$$

- We do a change of variables to factor out oscillations: $\phi = (1+v)\mathrm{e}^{\mathrm{i}kx}$
- Plugging this in gives us an equation for v:

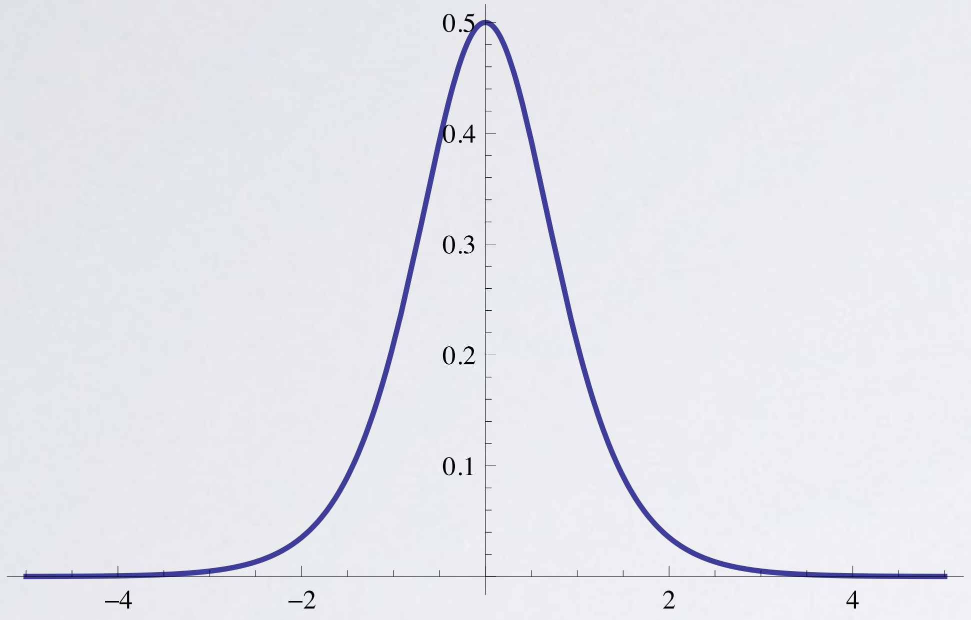
$$v'' + 2\mathrm{i}kv' + u_0 v = -u_0, \quad v(-\infty) = 0,\ v'(-\infty) = 0$$

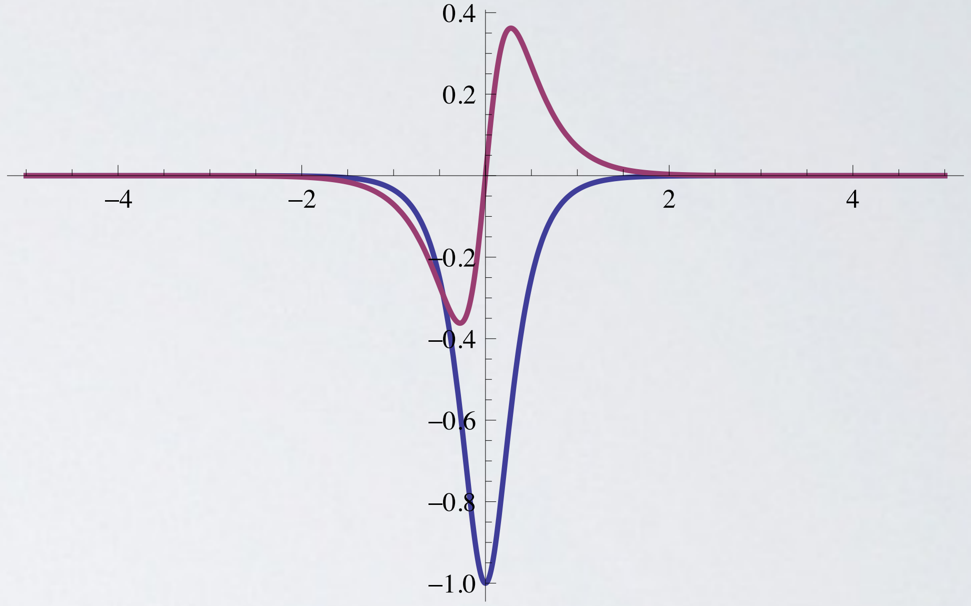
- We can easily solve this by mapping $(-\infty, 0)$ to the unit interval and using a spectral method
- Using the same approach, we can compute $\psi^{\pm}$ on $(0, \infty)$
- We then solve

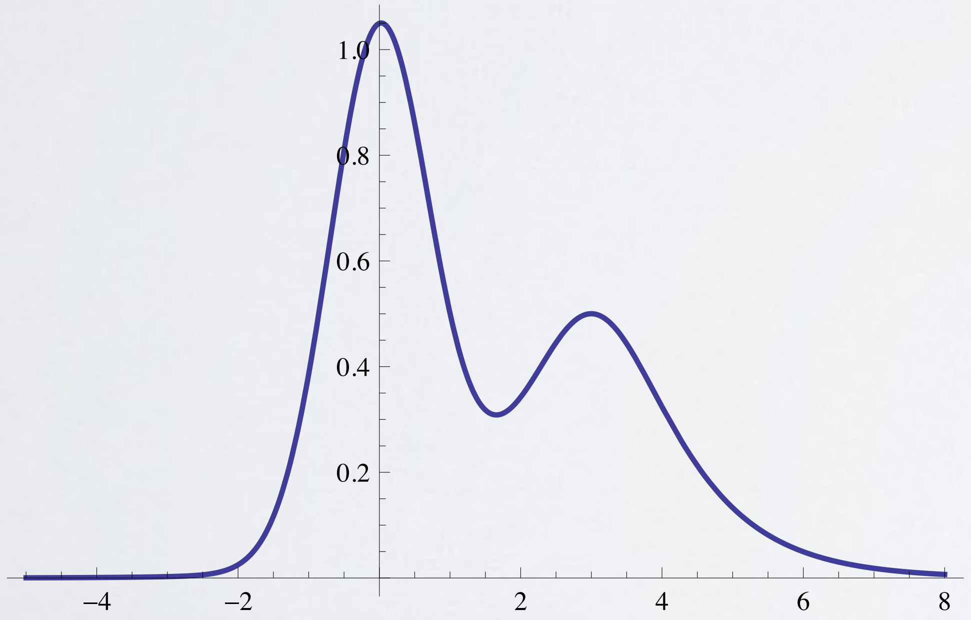
$$\begin{pmatrix} \psi^+(0) & \psi^-(0) \\ \psi_x^+(0) & \psi_x^-(0) \end{pmatrix} \begin{pmatrix} a(k) \\ b(k) \end{pmatrix} = \begin{pmatrix} \phi(0) \\ \phi_x(0) \end{pmatrix}$$

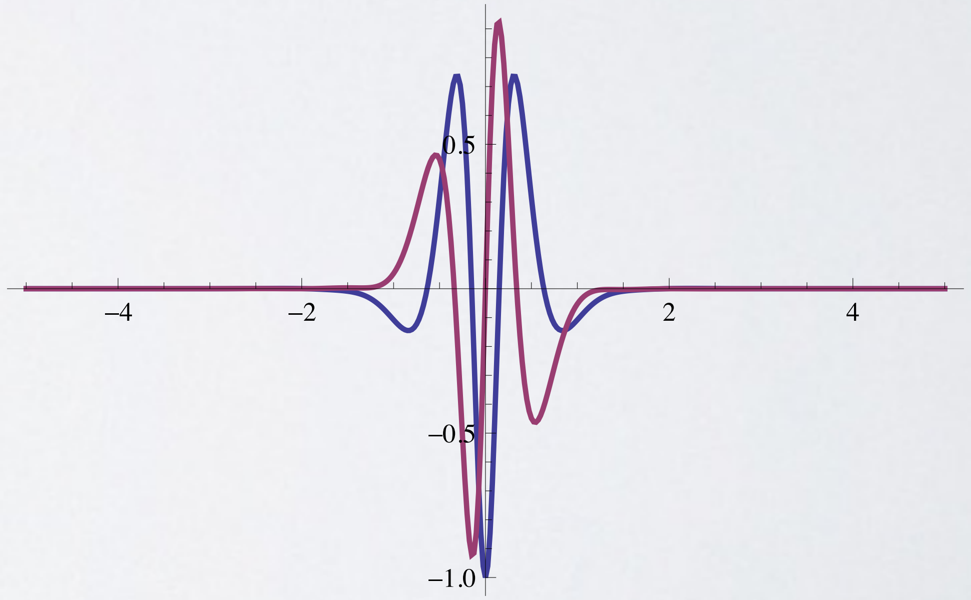
Initial condition

Reflection coefficient

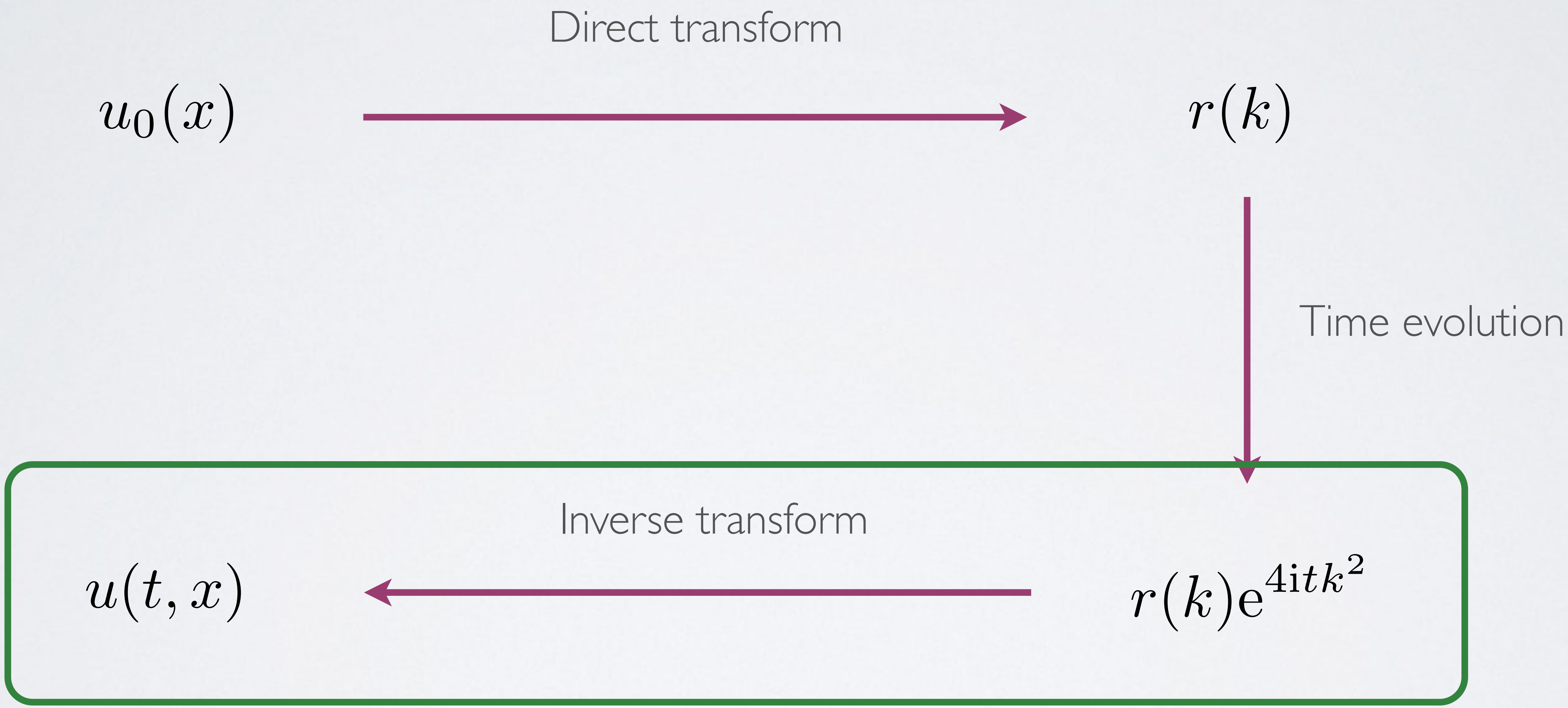








Inverse scattering for defocusing NLS



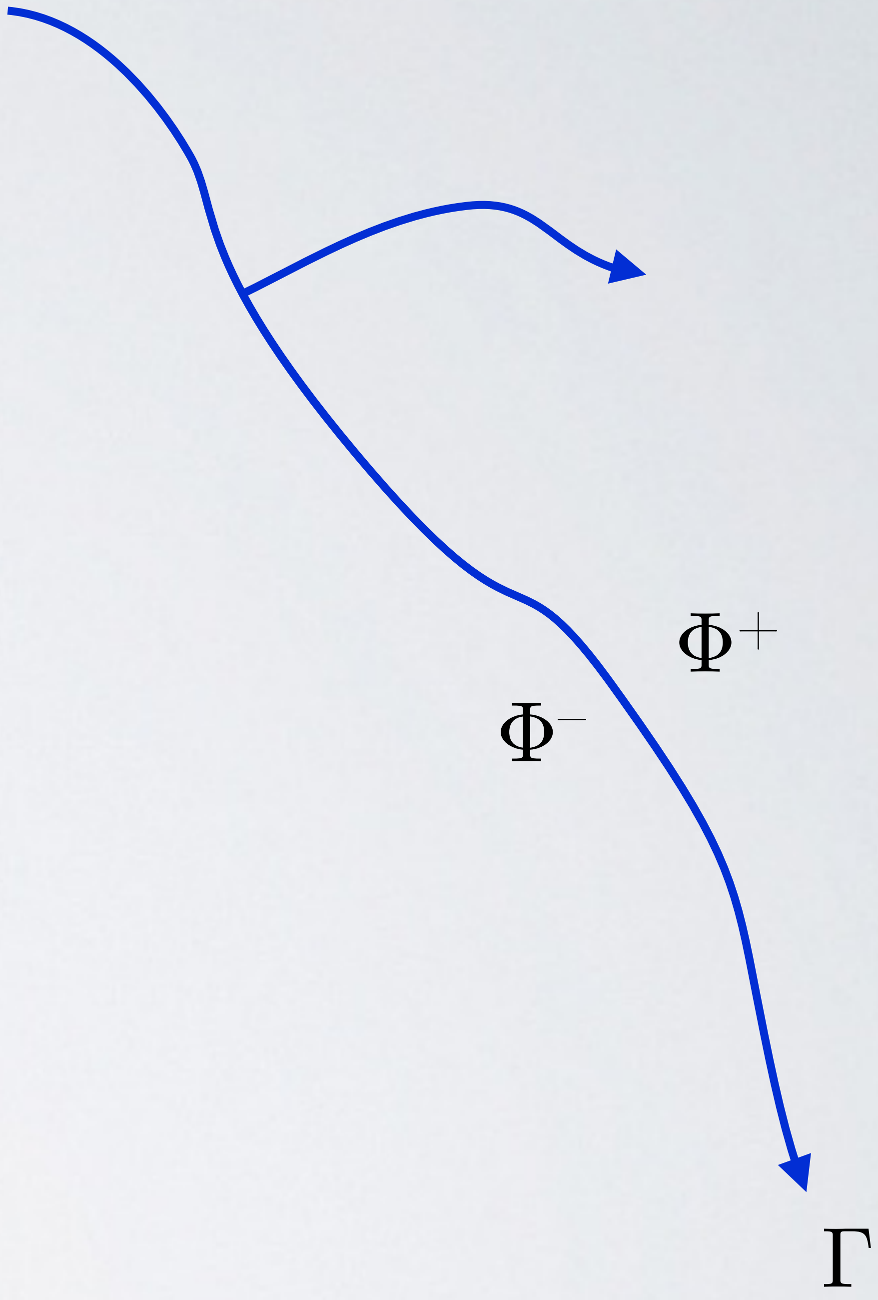
- A matrix-valued *Riemann–Hilbert problem* is the following:
  - Given an oriented contour Γ in the complex plane and a matrix-valued function G defined on Γ (here, all functions on Γ are analytic along each piece of Γ);
  - Find a matrix-valued function Φ that is analytic everywhere in the complex plane off of Γ such that

$\Phi^+(z) = \Phi^-(z)G(z)$ for $z \in \Gamma$ and $\Phi(\infty) = I$

where

$$\Phi^+(z) = \lim_{\substack{x \to z \\ \text{where } x \text{ is left of } \Gamma}} \Phi(x)$$

$$\Phi^-(z) = \lim_{\substack{x \to z \\ \text{where } x \text{ is right of } \Gamma}} \Phi(x)$$



(see eg. Muskhelishvili 1953)

- Many linear differential equations have well-known integral representations
  - e.g., Airy equation, Bessel equation, Hypergeometric equation and heat and wave equations (via Fourier transform)
- Matrix-valued RH problems can be (loosely) viewed as an analogy of integral representations for *nonlinear equations*
- Importantly, RH problems can be used to determine asymptotics of solutions
  - This works similar to integral representations: the contour is deformed along the *path of steepest descent*
- Using a new approach, RH problems can now be used as a numerical tool
- Previous method: the Sine kernel RH problem (on the unit interval) and a special solution to Painlevé V were computed in (Dienstfrey 1998), by adapting standard singular integral equation (SIE) methods
  - Required exponentially clustered collocation points near the endpoints
  - Not applicable to more complicated domains

Inverse transform

- The inverse transform is a Riemann–Hilbert problem on the real line:

$$\Phi^+(z) = \Phi^-(z) \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z)\mathrm{e}^{-2\mathrm{i}(2tz^2+xz)} \\ r(z)\mathrm{e}^{2\mathrm{i}(2tz^2+xz)} & 1 \end{pmatrix}$$

- Then the solution to dNLS is

$$u(t,x) = 2\mathrm{i} \lim_{z\to\infty} z\Phi(z)_{12}$$

Construction of a collocation method

- Consider the *Cauchy transform*

$$\mathcal{C}_\Gamma f(z) = \frac{1}{2\mathrm{i}\pi} \int_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t.$$

This map defines a one-to-one correspondence between a function defined on Γ and a function which is analytic everywhere off Γ which decays at ∞

- Let

$$\Phi = I + \mathcal{C}V$$

- The RH problem $\Phi^+ = \Phi^- G$ becomes

$$\mathcal{C}^+ V(x) - \mathcal{C}^- V(x) G(x) = G(x) - I \qquad \text{for} \qquad x \in \Gamma$$

- Having a method to compute the Cauchy transform and its left and right limits allows us to apply the linear operator

$$\mathcal{M}V = \mathcal{C}^+ V - (\mathcal{C}^- V)G$$

(similar to Dienstfrey 1998)

- We want to construct an approximation to V which satisfies

$$\mathcal{M}V = G - I$$

at a sequence of points; i.e., we construct a collocation method:

- For some basis $\{\psi_1, \ldots, \psi_n\}$ of functions defined on Γ and set of nodes $\{z_1, \ldots, z_m\}$ on Γ
- Write

$$V_n = \sum_{k=1}^{n} \boldsymbol{c}_k \psi_k$$

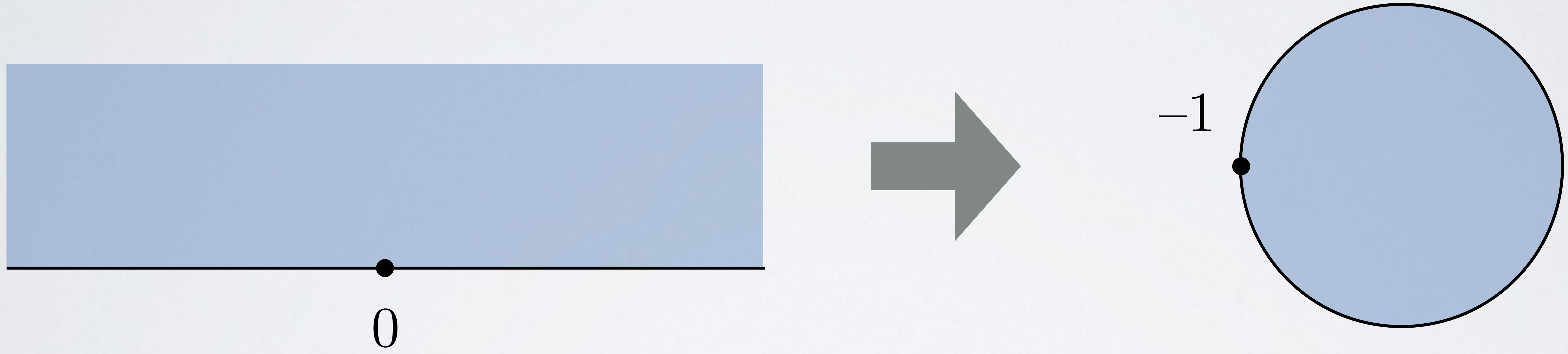
- Solve the linear system

$$\begin{aligned} \boldsymbol{c}_1 \mathcal{M}\psi_1(z_1) + \cdots + \boldsymbol{c}_n \mathcal{M}\psi_n(z_1) &= G(z_1) - I \\ &\vdots \\ \boldsymbol{c}_1 \mathcal{M}\psi_1(z_m) + \cdots + \boldsymbol{c}_n \mathcal{M}\psi_n(z_m) &= G(z_m) - I \end{aligned}$$

- Consider the map

$$M^{-1}(z) = \frac{\mathrm{i} - z}{\mathrm{i} + z}$$

- This maps the real line to the unit circle
  - More precisely: it conformally maps the upper half plane to the interior of the circle



- It also conformally maps the lower half plane to the exterior of the circle

- The inverse

$$M(z) = \mathrm{i}\frac{1-z}{z+1}$$

conformally maps the unit circle to the real line

- The Cauchy transform is (due to Plemelj's lemma)

$$\mathcal{C}_{(-\infty,\infty)}f(z) = \mathcal{C}_{\bigcirc}[f \circ M](M^{-1}(z)) - \mathcal{C}_{\bigcirc}[f \circ M](-1)$$

- Thus we have reduced the construction of our collocation method to one problem: the computation of the Cauchy transform over the unit circle

- We know

$$\mathcal{C}_{\bigcirc}[\diamond^k](z) = \begin{cases} z^k & |z| < 1 \text{ and } k \geq 0 \\ -z^k & |z| > 1 \text{ and } k < 0 \\ 0 & \text{otherwise} \end{cases}$$

- Thus we know the Cauchy transform exactly on the real line:

$$\mathcal{C}_{(-\infty,\infty)}[M^{-1}(\diamond)^k](z) = \begin{cases} M^{-1}(z)^k - (-1)^k & \Im z > 0 < 1 \text{ and } k \geq 0 \\ (-1)^k - M^{-1}(z)^k & \Im z < 0 \text{ and } k < 0 \\ 0 & \text{otherwise} \end{cases}$$

- We use this basis in the collocation system, with mapped evenly spaced points

(Based on Weideman 1995)

$$\psi_1(z), \ldots, \psi_n(z) = M^{-1}(z)^{-n/2}, \ldots, M^{-1}(z)^{n/2}$$

$$\boldsymbol{c}_1 \mathcal{M}\psi_1(z_1) + \cdots + \boldsymbol{c}_n \mathcal{M}\psi_n(z_1) = G(z_1) - I$$

$$\vdots$$

$$\boldsymbol{c}_1 \mathcal{M}\psi_1(z_m) + \cdots + \boldsymbol{c}_n \mathcal{M}\psi_n(z_m) = G(z_m) - I$$

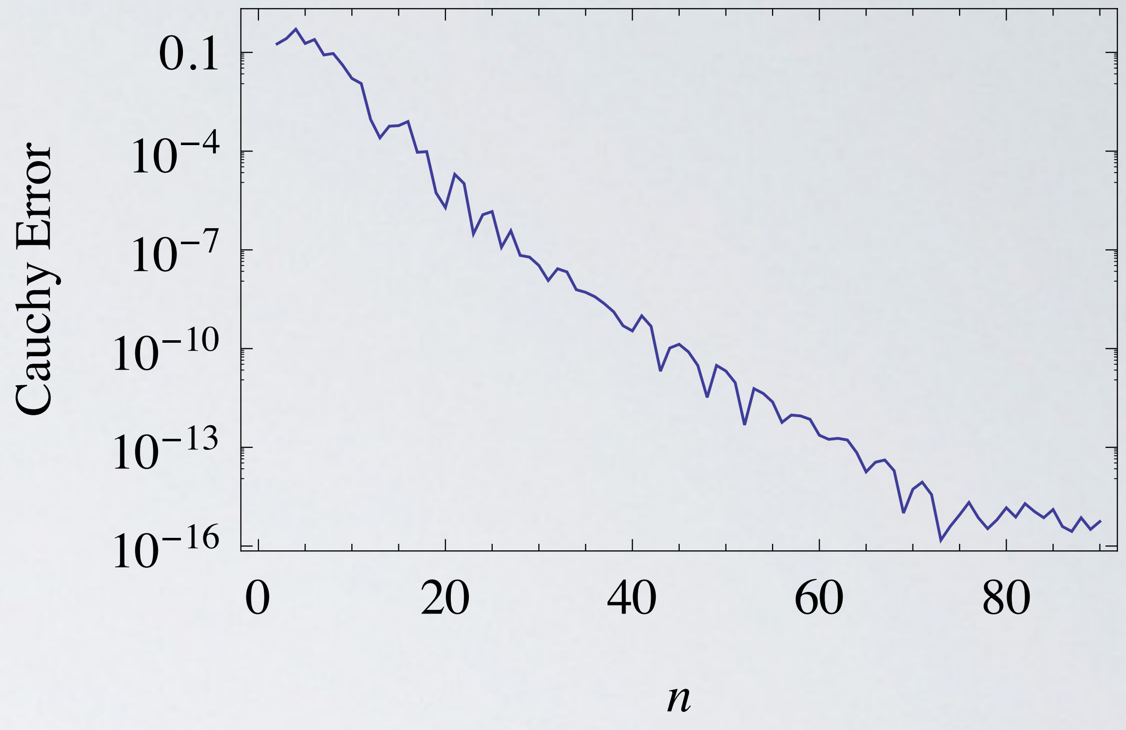
$$z_1, \ldots, z_m = M(-1), M(\mathrm{e}^{(\frac{2}{m}-1)\mathrm{i}\pi}), \ldots, M(\mathrm{e}^{(1-\frac{2}{m})\mathrm{i}\pi})$$

$=$ 

- Finally, we recover the solution to dNLS:

$$
\begin{aligned}
u(t,x) &= 2\mathrm{i} \lim_{z\to\infty} z\Phi(z)_{12} = 2\mathrm{i} \lim_{z\to\infty} z\mathcal{C}V(z)_{12} \\
&= \frac{1}{\pi} \lim_{z\to\infty} \int_{-\infty}^{\infty} \frac{zV(x)_{12}}{x-z}\,\mathrm{d}x \\
&= -\frac{1}{\pi} \int_{-\infty}^{\infty} V(x)_{12}\,\mathrm{d}x \\
&\approx -\frac{1}{\pi} \int_{-\infty}^{\infty} V_n(x)_{12}\,\mathrm{d}x
\end{aligned}
$$

Convergence



- The error in approximation is bounded by

$$\|V - V_n\| \leq \left[1 + C \log n \left\|M_n^{-1}\right\| \|\mathcal{M}\|\right] \|V - \mathcal{P}_n V\|$$

Number of collocation points n

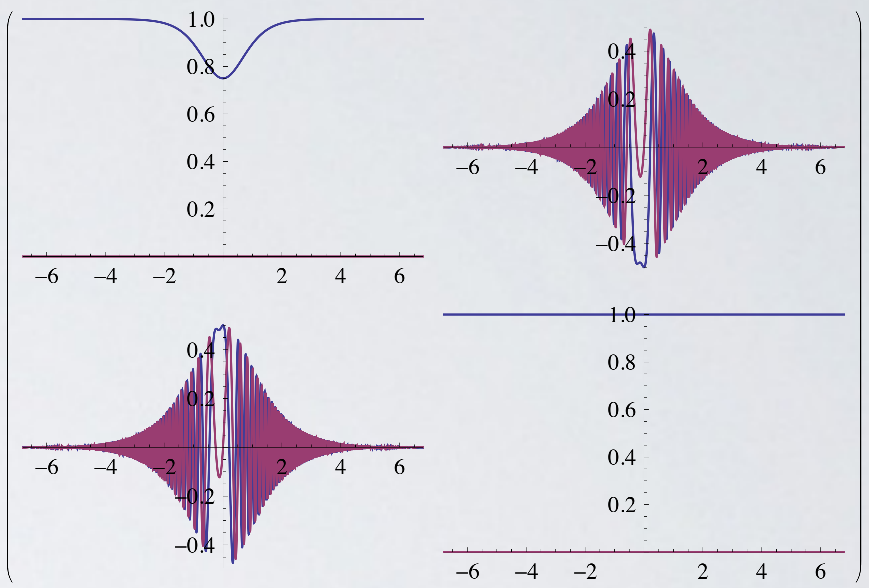
Collocation matrix (Grows logarithmically)

Interpolation of solution by basis (Converges spectrally)

Nonlinear steepest descent

- The previous method works well for small $\boldsymbol{x}$ and $\boldsymbol{t}$
- But as $\boldsymbol{x}$ and $\boldsymbol{t}$ become large the jump matrix

$$\begin{pmatrix} 1 - r(z)r(-z) & -\bar{r}(z)\mathrm{e}^{-2\mathrm{i}(2tz^2+xz)} \\ r(z)\mathrm{e}^{2\mathrm{i}(2tz^2+xz)} & 1 \end{pmatrix} =$$



  becomes oscillatory
- We overcome this by deforming the Riemann–Hilbert problem into the complex plane
  - This converts oscillations into exponential decay
  - We must deform through the stationary points:

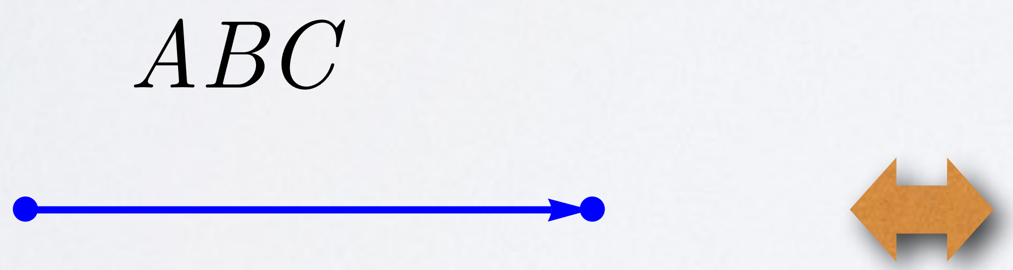
$$k_0 = -\frac{x}{4t}$$

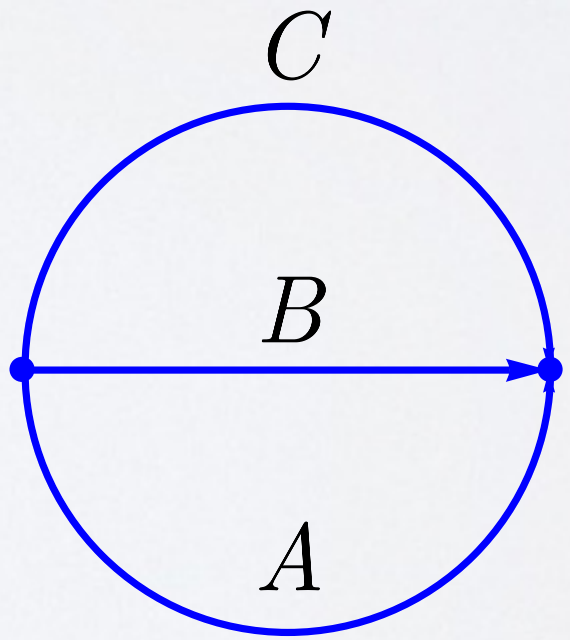
- There are two ways we use to factor the jump matrix:

$$\begin{pmatrix} \tau & \bar{r}\mathrm{e}^{-\theta} \\ r\mathrm{e}^{\theta} & 1 \end{pmatrix} = LDU = \begin{pmatrix} 1 & \\ \frac{r}{\tau}\mathrm{e}^{\theta} & 1 \end{pmatrix} \begin{pmatrix} \tau & \\ & \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}}{\tau}\mathrm{e}^{-\theta} \\ & 1 \end{pmatrix}$$

$$= MP = \begin{pmatrix} 1 & \bar{r}\mathrm{e}^{-\theta} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ r\mathrm{e}^{\theta} & 1 \end{pmatrix}$$

- The key now is that we can lens the jump contours without altering behaviour at infinity:

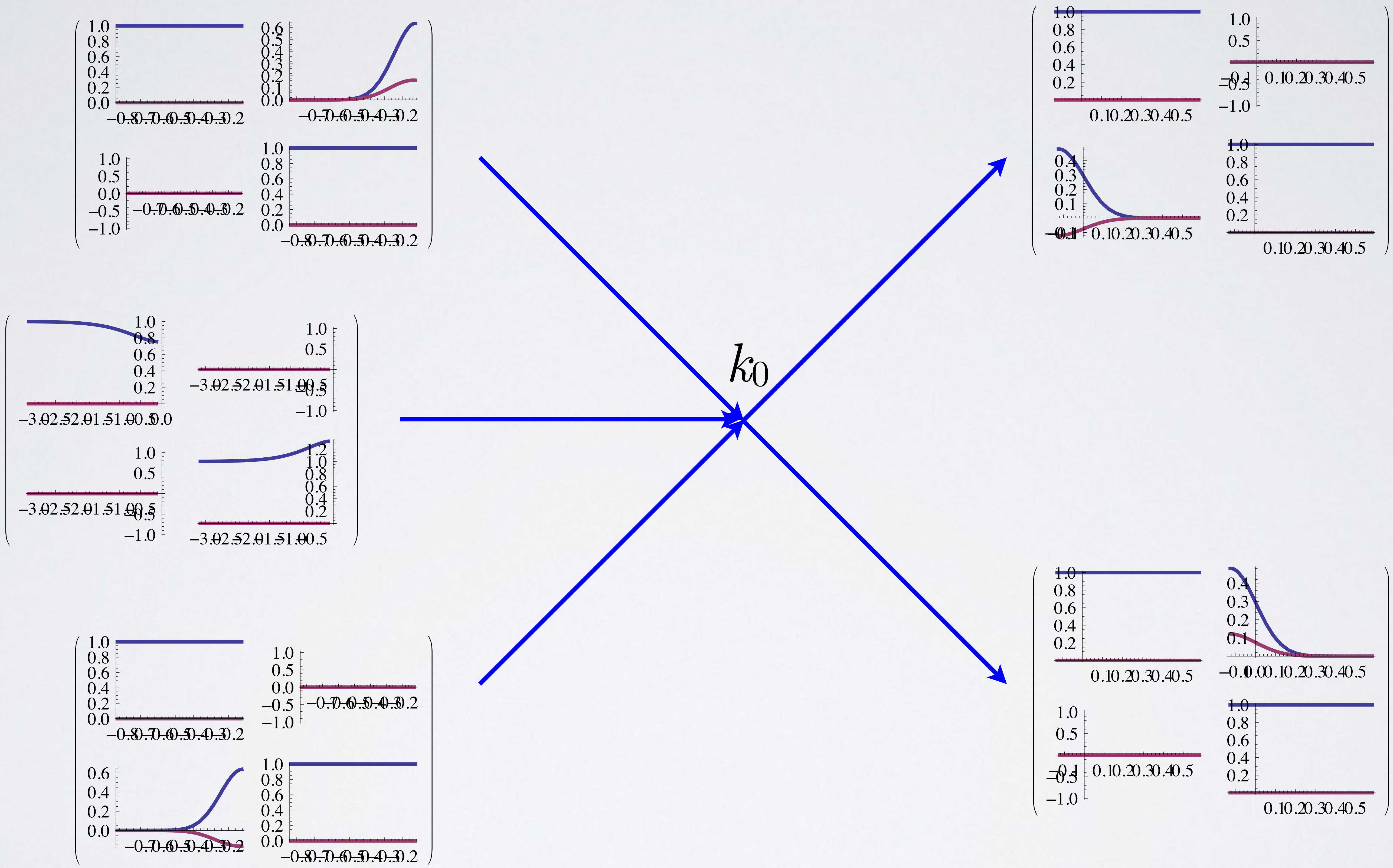




(based on Deift & Zhou 2003)


G

Jump matrices after truncation



Algorithm for computation

- We must choose a basis and compute the Cauchy transform over Γ
  - By splitting the domain and using conformal maps, this can be reduced to computing the Cauchy transform over the unit interval
  - The Cauchy transform for Chebyshev polynomials over the unit interval can be found in closed form!
- We must include the junction points of Γ in the collocation system
  - This is needed to ensure that the approximation is bounded
  - The Cauchy transform of our basis explodes there; therefore, we assign it a special value corresponding to the value if the singularities were to cancel

- We use the fact that the Cauchy transform of Chebyshev polynomials

$$\mathcal{C}_{(-1,1)}T_k(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{T_k(x)}{x - z} \, \mathrm{d}x$$

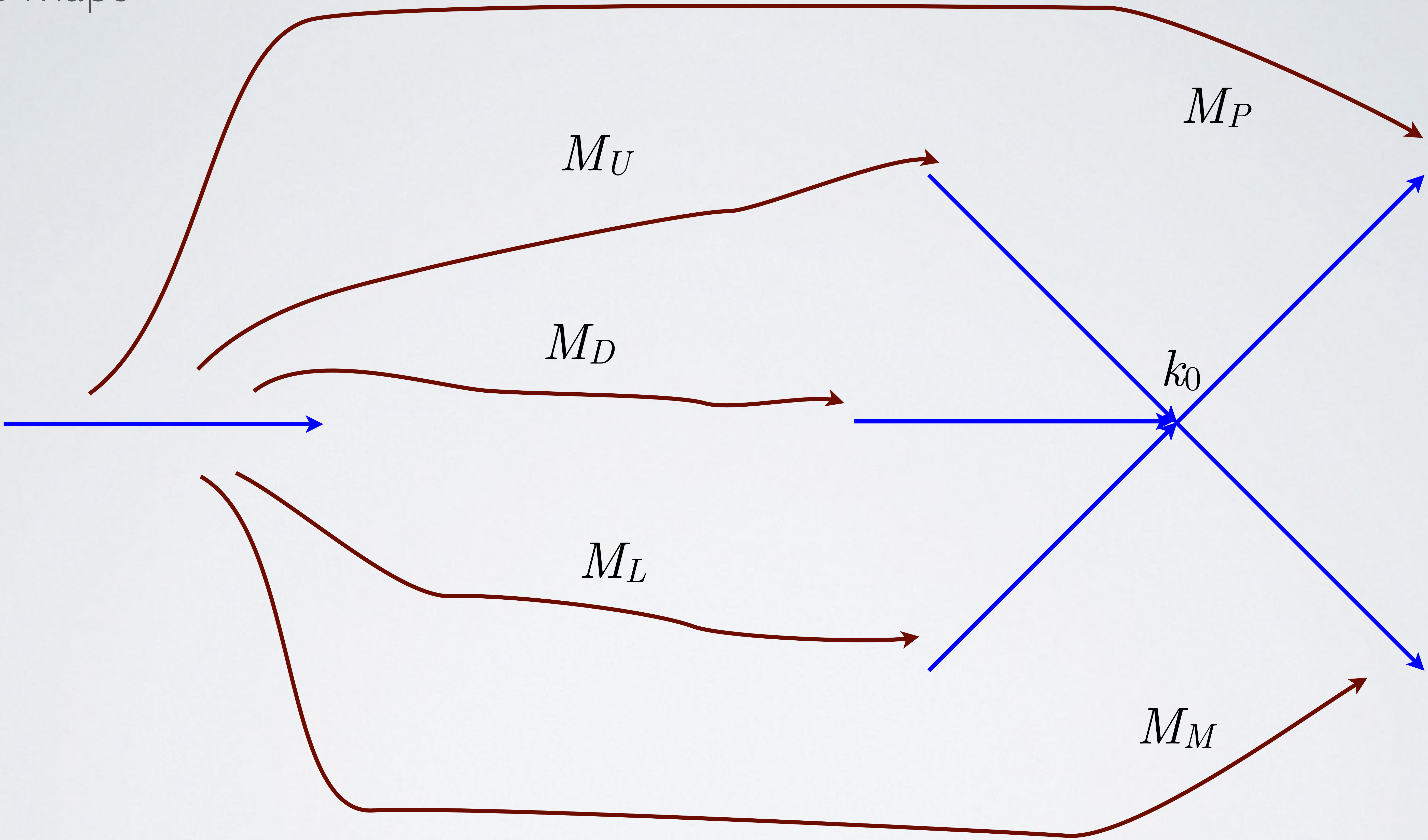
can be expressed exactly in terms of hypergeometric functions

  - And calculated rapidly, uniformly accurate in z using a one-term recurrence relationship

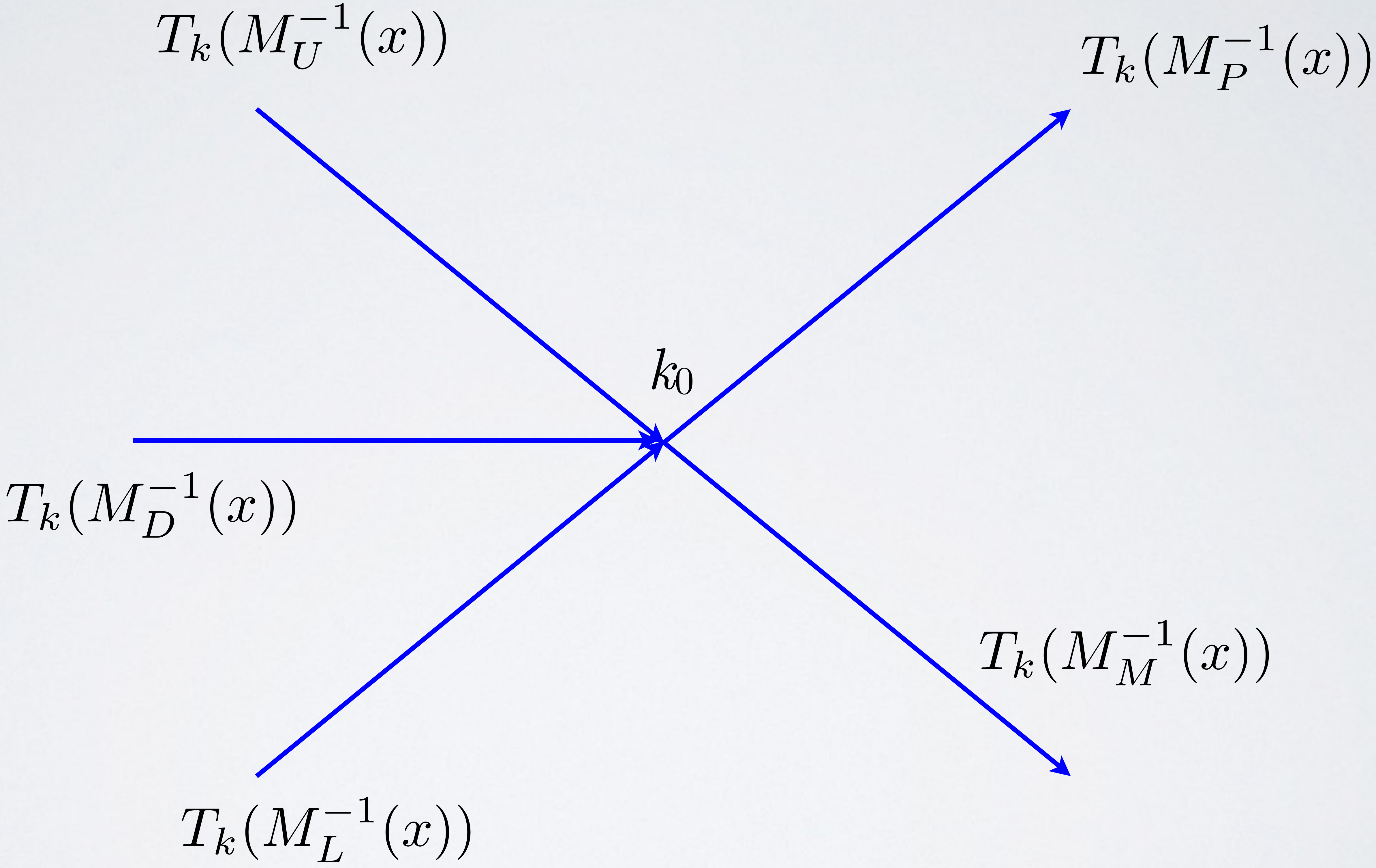
- If M is an affine transformation, then we have (from Plemelj's lemma)

$$\mathcal{C}_{M((-1,1))}f(z) = \mathcal{C}_{(-1,1)}[f \circ M](M^{-1}(z))$$

Affine maps



Piecewise collocation basis



- For our choice of basis and each affine transformation M, we can calculate

$$\mathcal{C}_{M((-1,1))}[T_k \circ M^{-1}](z) = \mathcal{C}_{(-1,1)} T_k(M^{-1}(z))$$

exactly

- The problem:

$$\mathcal{C}_{M((-1,1))}[T_k \circ M^{-1}](k_0) = \mathcal{C}_{(-1,1)} T_k(1) = \infty$$

- However, the solution V lies in a special space $\mathcal{Z}$ that implies its Cauchy transform is bounded at k_0
- We therefore assume that our numerical approximation $V_n \in \mathcal{Z}$, allowing us to assign a value

$$\mathcal{C}_{M((-1,1))}[T_k \circ M^{-1}](k_0) = \mathcal{C}_{(-1,1)} T_k(1) \approx \alpha_k$$

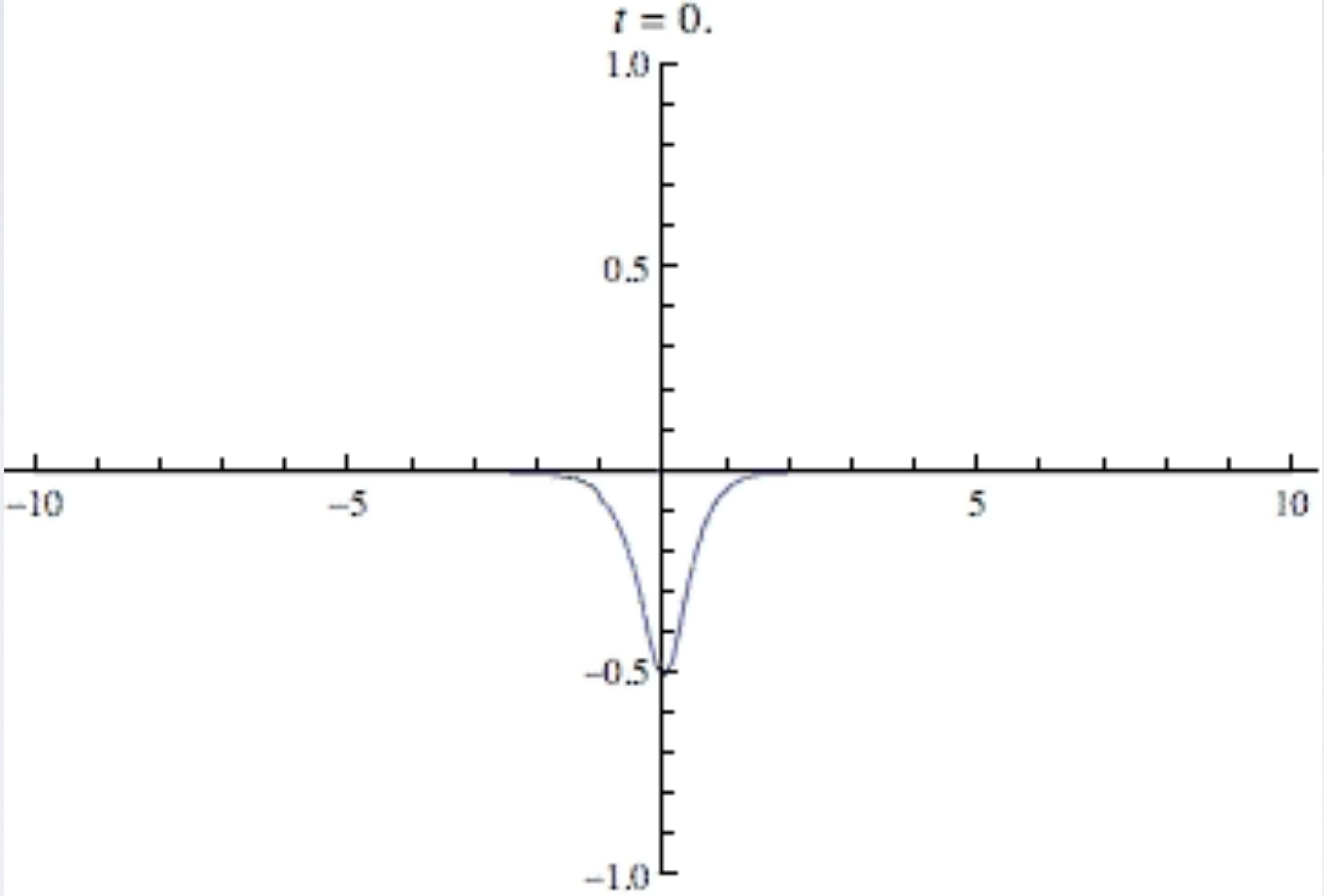
- The resulting linear system imposes that the approximation V_n is indeed in $\mathcal{Z}$

$$\psi_1(z), \ldots, \psi_n(z) = T_0(M_L^{-1}(x)), \cdots, T_k(M_D^{-1}(x)), \cdots, \cdots, T_p(M_P^{-1}(x))$$

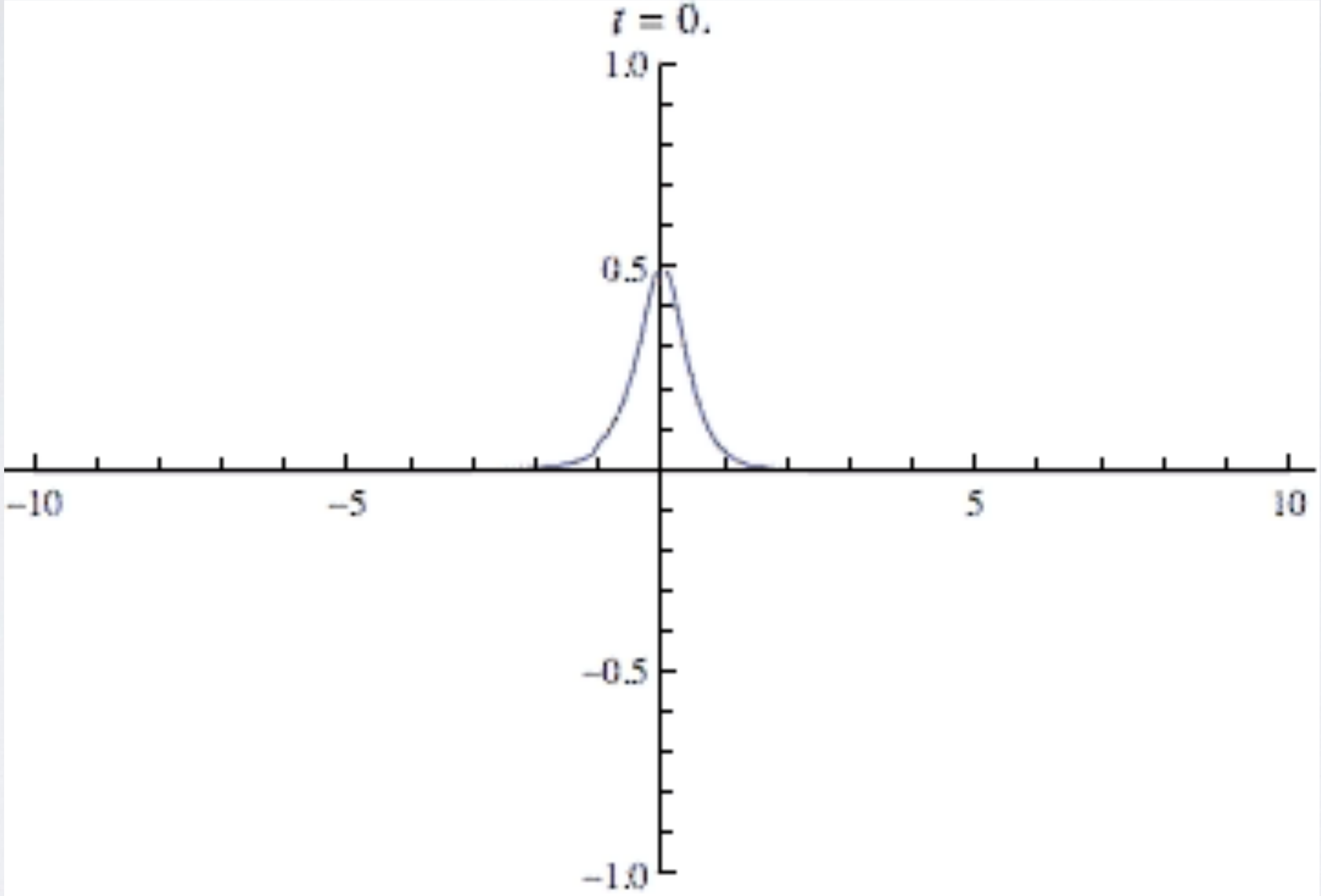
$$\begin{aligned}
\boldsymbol{c}_1 \mathcal{M} \psi_1(z_1) + \cdots + \boldsymbol{c}_n \mathcal{M} \psi_n(z_1) &= G(z_1) - I \\
&\vdots \\
\boldsymbol{c}_1 \mathcal{M} \psi_1(z_m) + \cdots + \boldsymbol{c}_n \mathcal{M} \psi_n(z_m) &= G(z_m) - I
\end{aligned}$$

$$z_1, \ldots, z_m =$$

Evolution of defocusing NLS



Evolution of defocusing NLS (Absolute value)



KdV

Inverse scattering for KdV

Direct transform

$$u_0(x) \longrightarrow r(k), \{\kappa_j\}, \{C_j\}$$

Time evolution

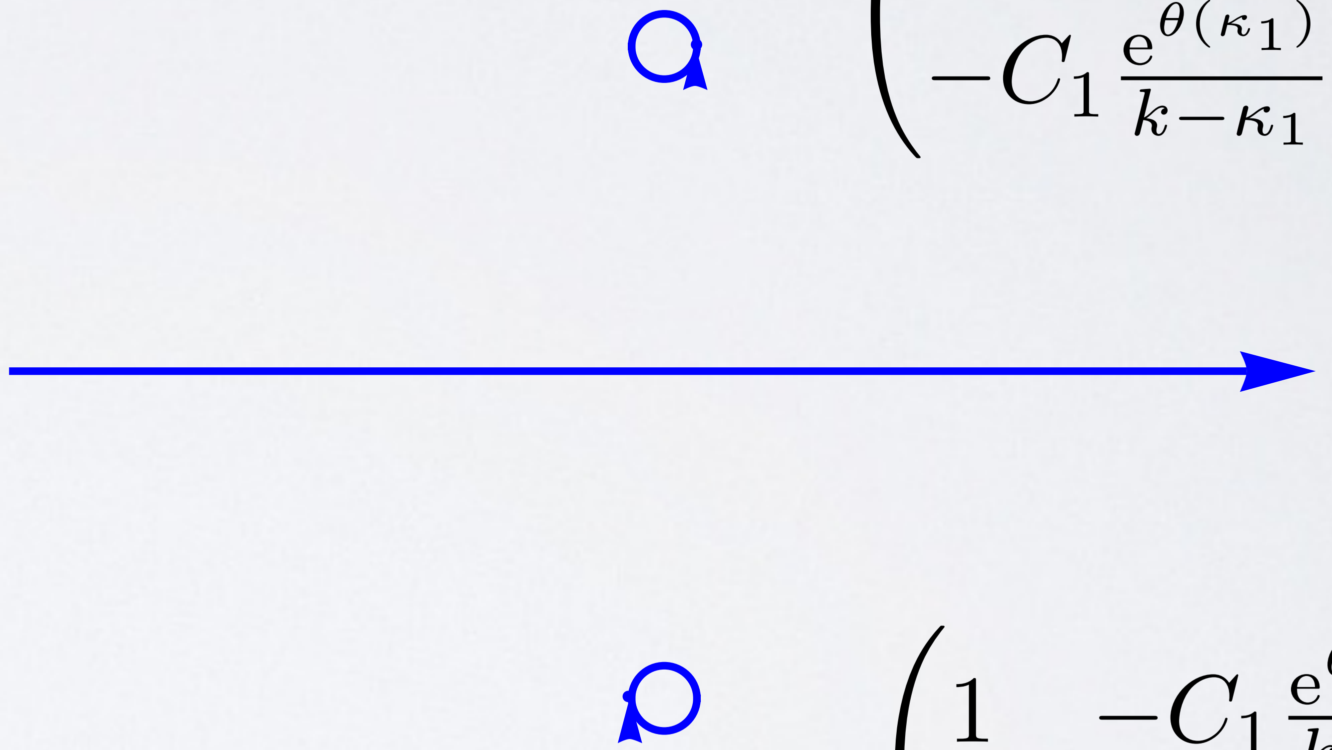
$$u(t,x) \longleftarrow r(k)e^{itk^3}, \{\kappa_j\}, \left\{C_j e^{itk_j^3}\right\}$$

Inverse transform

Reflection data

$$r(k), \{\kappa_j\}, \{C_j\}$$

Undeformed RH Problem



$$\begin{pmatrix} 1 \\ -C_1 \frac{e^{\theta(\kappa_1)}}{k-\kappa_1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 - r(z)r(-z) & -r(-z)e^{-2i(4tz^3+xz)} \\ r(z)e^{2i(4tz^3+xz)} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -C_1 \frac{e^{\theta(\kappa_1)}}{k+\kappa_1} \\ & 1 \end{pmatrix}$$

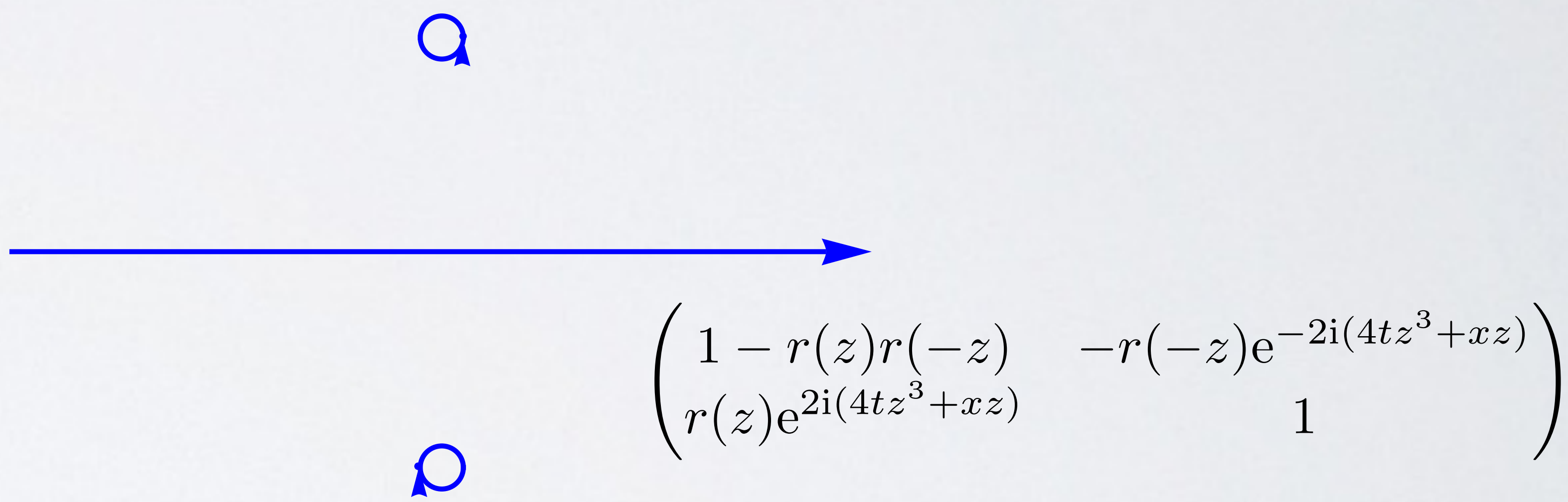
Deformations

- We have two stationary points at

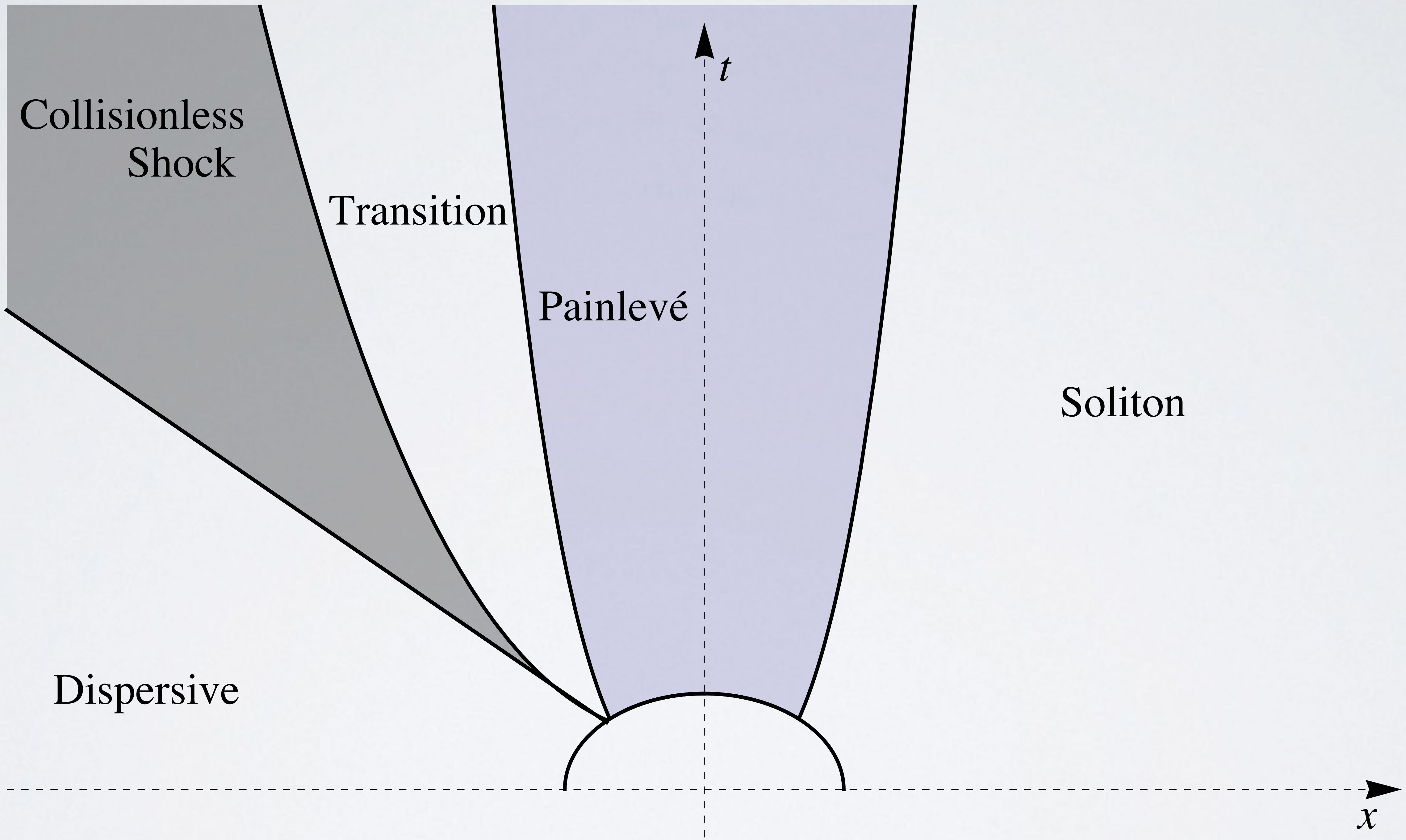
$$\pm k_0 = \pm\sqrt{-\frac{x}{12t}}$$

- We will deform the contour through these stationary points along the paths of steepest descent
- Different regimes of x and t require different *lensings*
  - Added difficulty: the lensing introduces a pole
- Deformations based on Deift & Zhou 1993, Deift, Zhou & Venakides 1994, Grunert & Teschl 2008

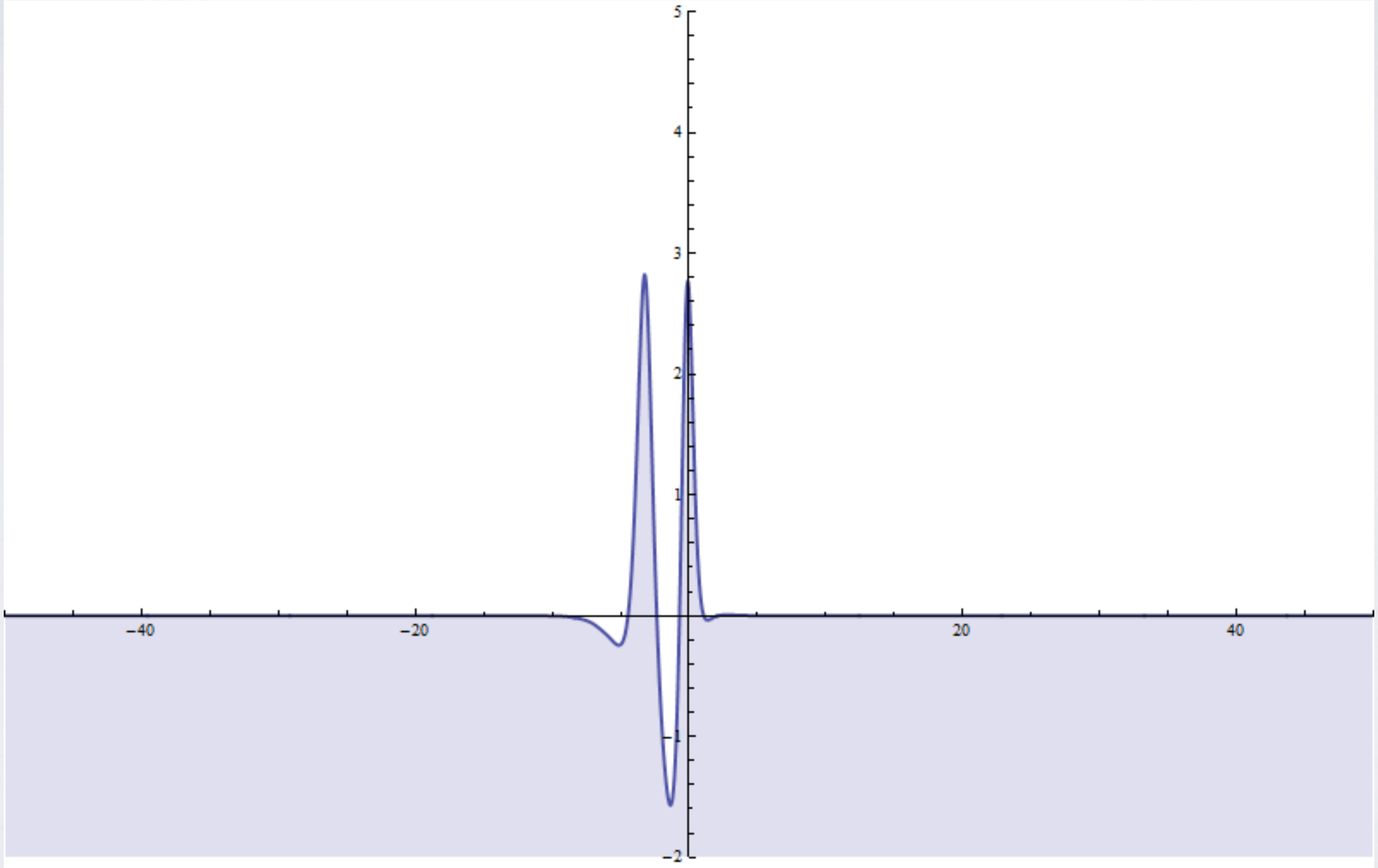
Undeformed



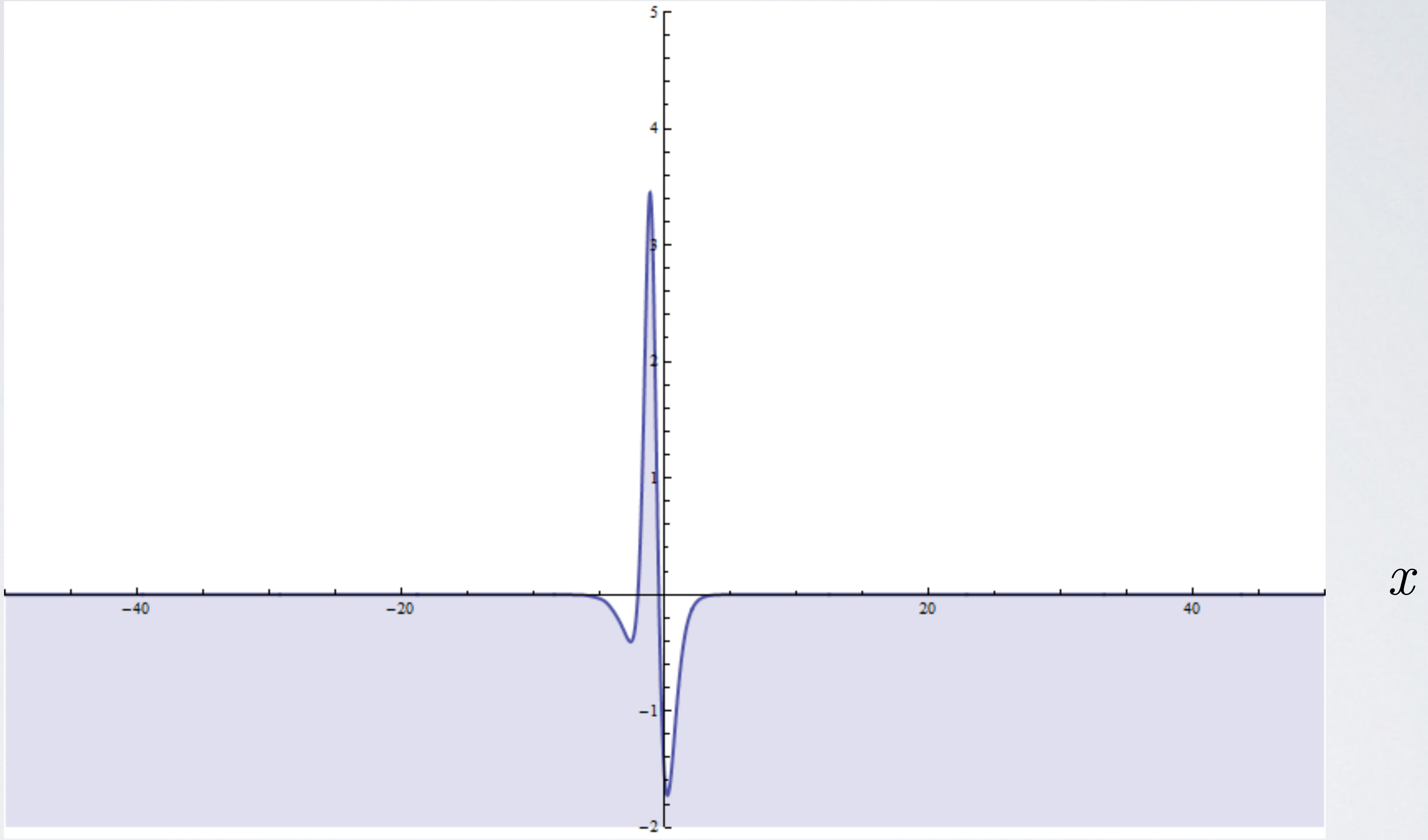
Different regions for deformations



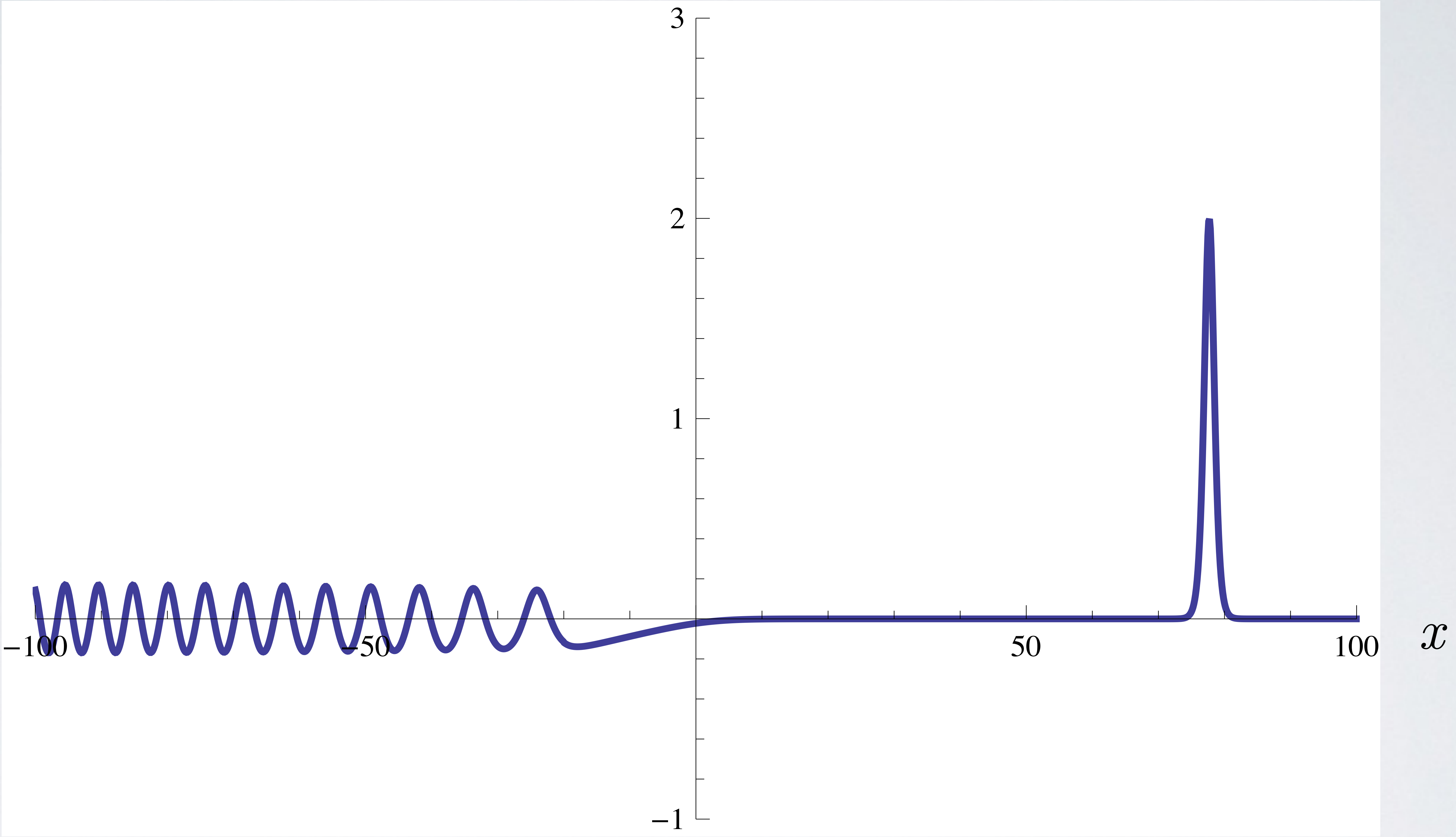
KdV (Generic)



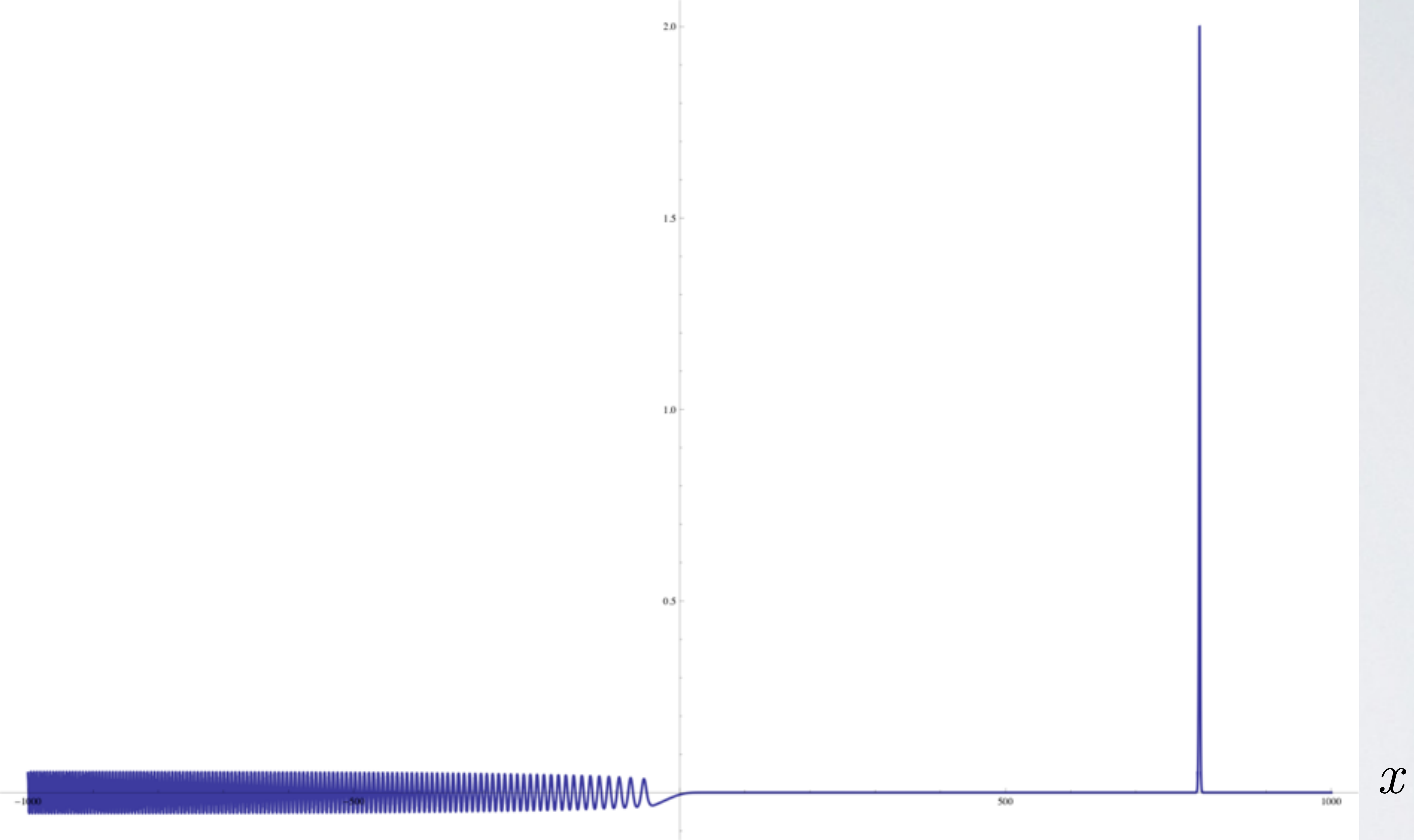
KdV One soliton



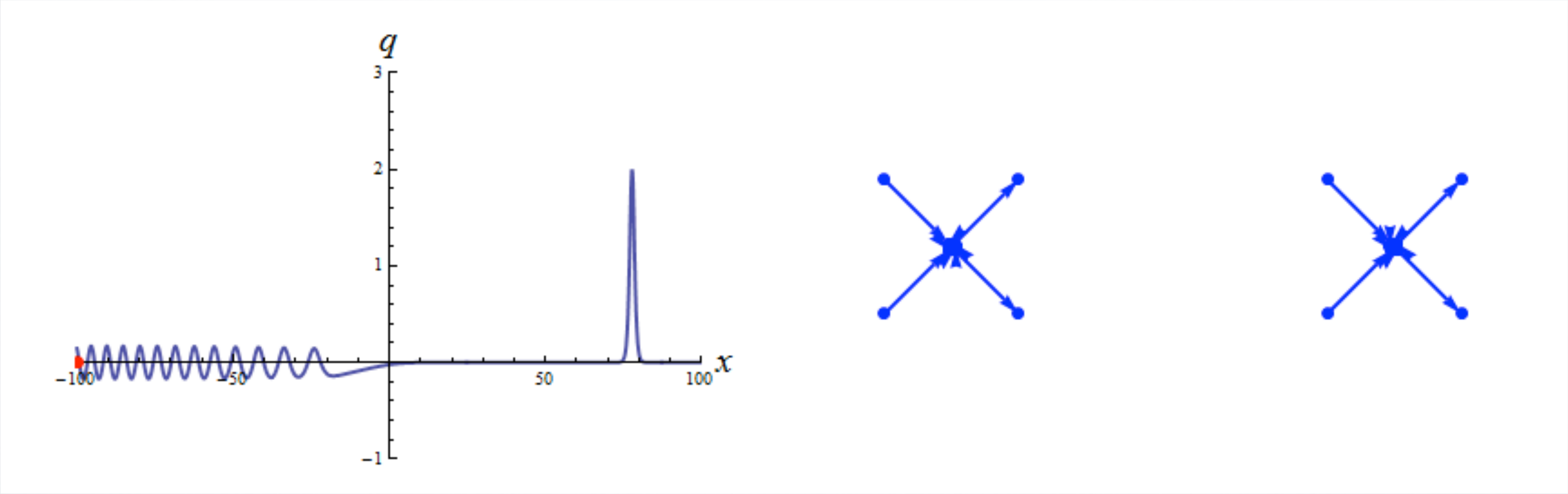
Plot for $t = 20$



Plot for $t = 200, \ -1000 < x < 1000$



Change of contour



Focusing NLS

Focusing NLS

- The undeformed jump contour is very similar to defocusing NLS:

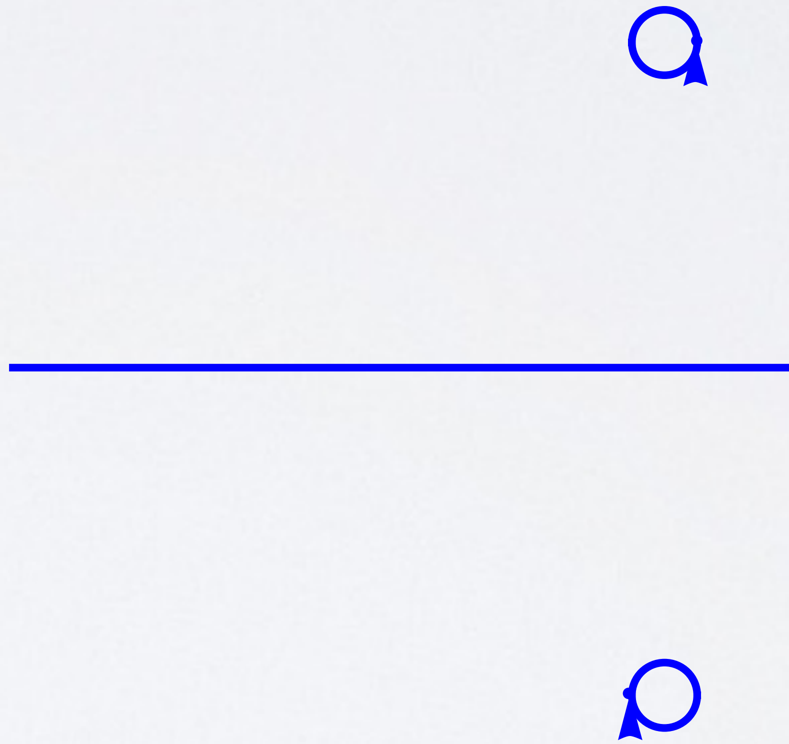
$$\begin{pmatrix} 1-|r(z)|^2 & -\bar{r}(z)e^{-2i(2tz^2+xz)} \\ r(z)e^{2i(2tz^2+xz)} & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1+|r(z)|^2 & \bar{r}(z)e^{-2i(2tz^2+xz)} \\ r(z)e^{2i(2tz^2+xz)} & 1 \end{pmatrix}$$

- The deformations are the exact same as well!
- Except now we have discrete spectra, which may be anywhere in the complex plane
- For numerical examples, we just specify simple discrete spectra

Reflection data

$$r(k), \{\kappa_j\}, \{C_j\}$$

Undeformed RH Problem

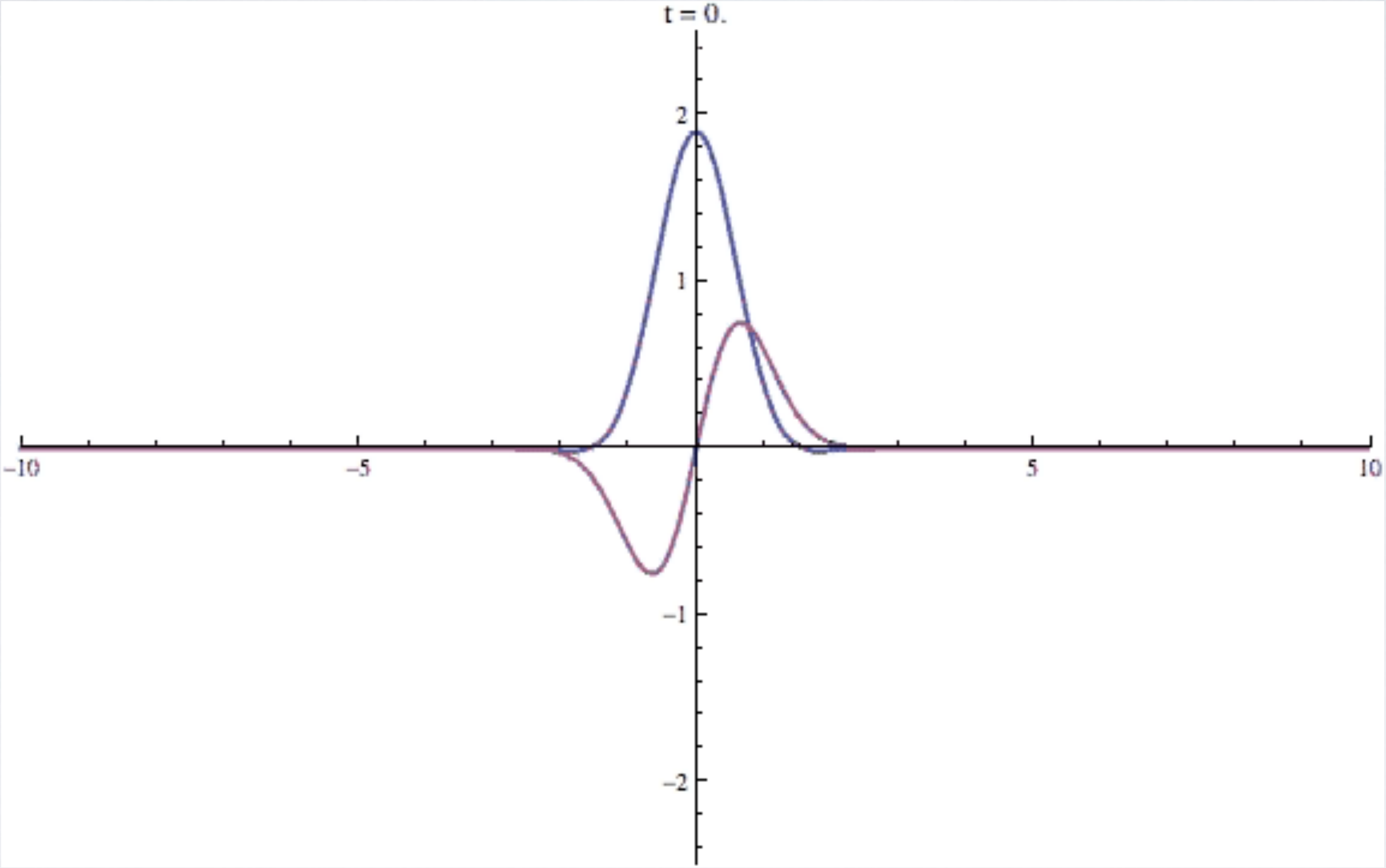


$$\begin{pmatrix} 1 & \\ -C_1 \frac{\mathrm{e}^{\theta(\kappa_1)}}{k-\kappa_1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 + |r(z)|^2 & \bar{r}(z)\mathrm{e}^{-2\mathrm{i}(2tz^2+xz)} \\ r(z)\mathrm{e}^{2\mathrm{i}(2tz^2+xz)} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \overline{-C_1 \frac{\mathrm{e}^{\theta(\kappa_1)}}{\bar{k}-\kappa_1}} \\ & 1 \end{pmatrix}$$

fNLS Soliton + Dispersion



Focusing NLS with boundary conditions

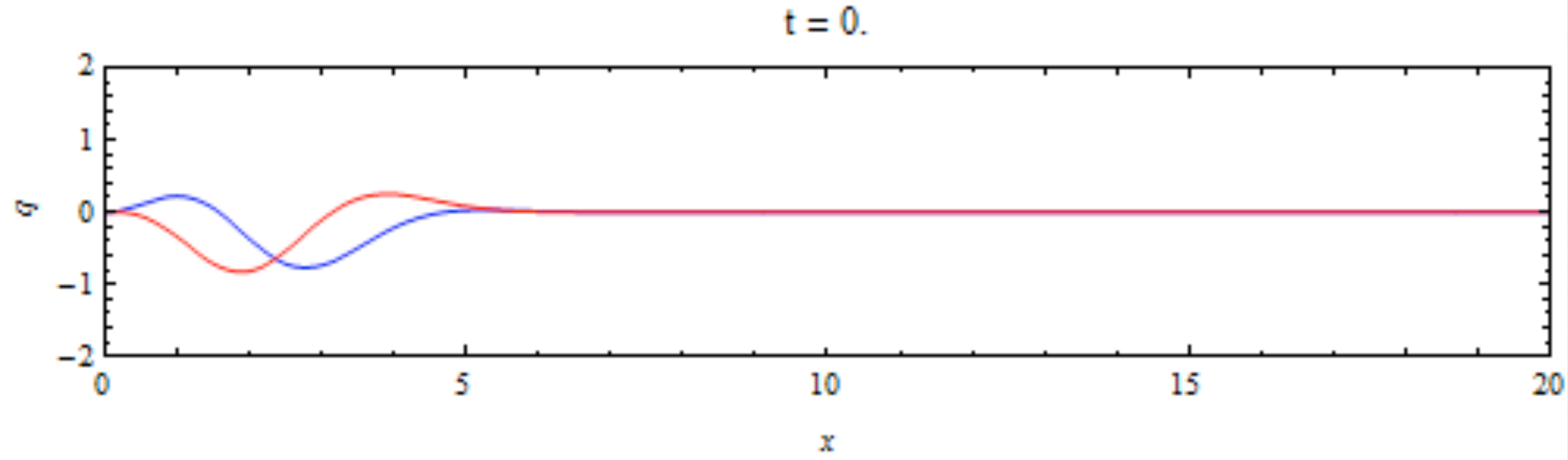
- We can use a similar approach to solve NLS on the half line with Robin boundary conditions:

$$\begin{aligned} \mathrm{i}u_t + u_{xx} \pm 2\left|u\right|^2 u &= 0, \\ \alpha u(0,t) + u_x(0,t) &= 0 \qquad \text{and} \\ u(x,0) &= u_0(x) \end{aligned}$$

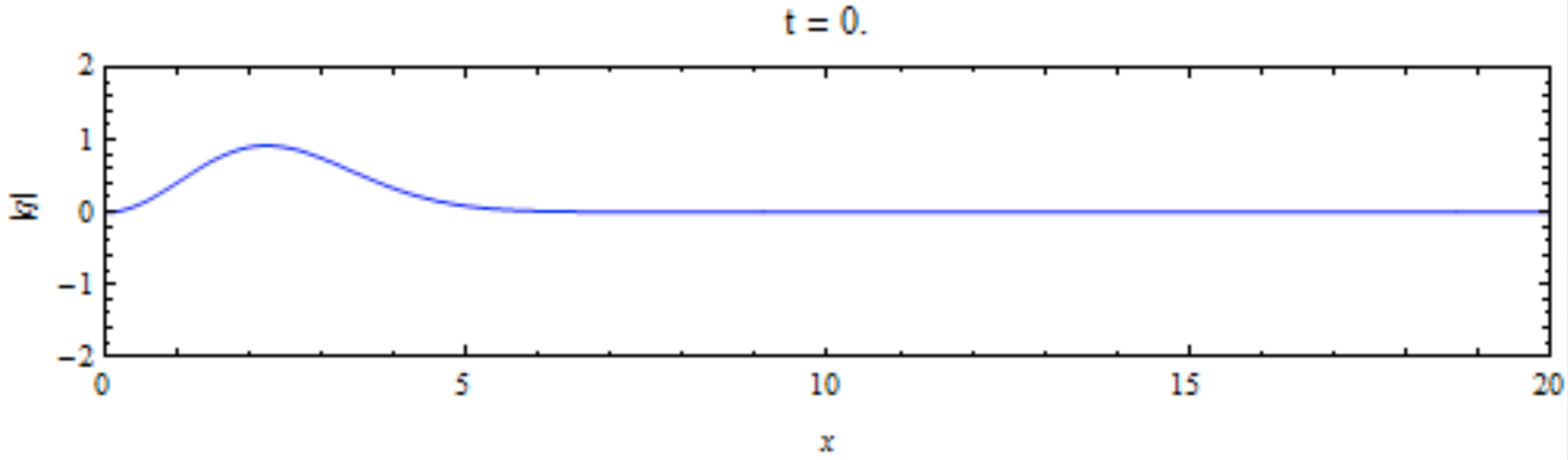
- The basic idea: use a Darboux transformation to extend $u_0(x)$ to the entire real line
  - For Neumann $\alpha = 0$ it is the even extension $u_0(-x) = u_0(x)$
  - For Dirichlet $\alpha = \infty$, it is the odd extension $u_0(-x) = -u_0(x)$
  - In general, the extension results from solving a linear ODE.
- The extension can have a discontinuity in derivatives at zero. However, our method for computing the direct scattering transform is not affected!

(based on Biondini & Bui 2012)

fNLS with Neumann conditions



fNLS with Neumann conditions (Absolute value)

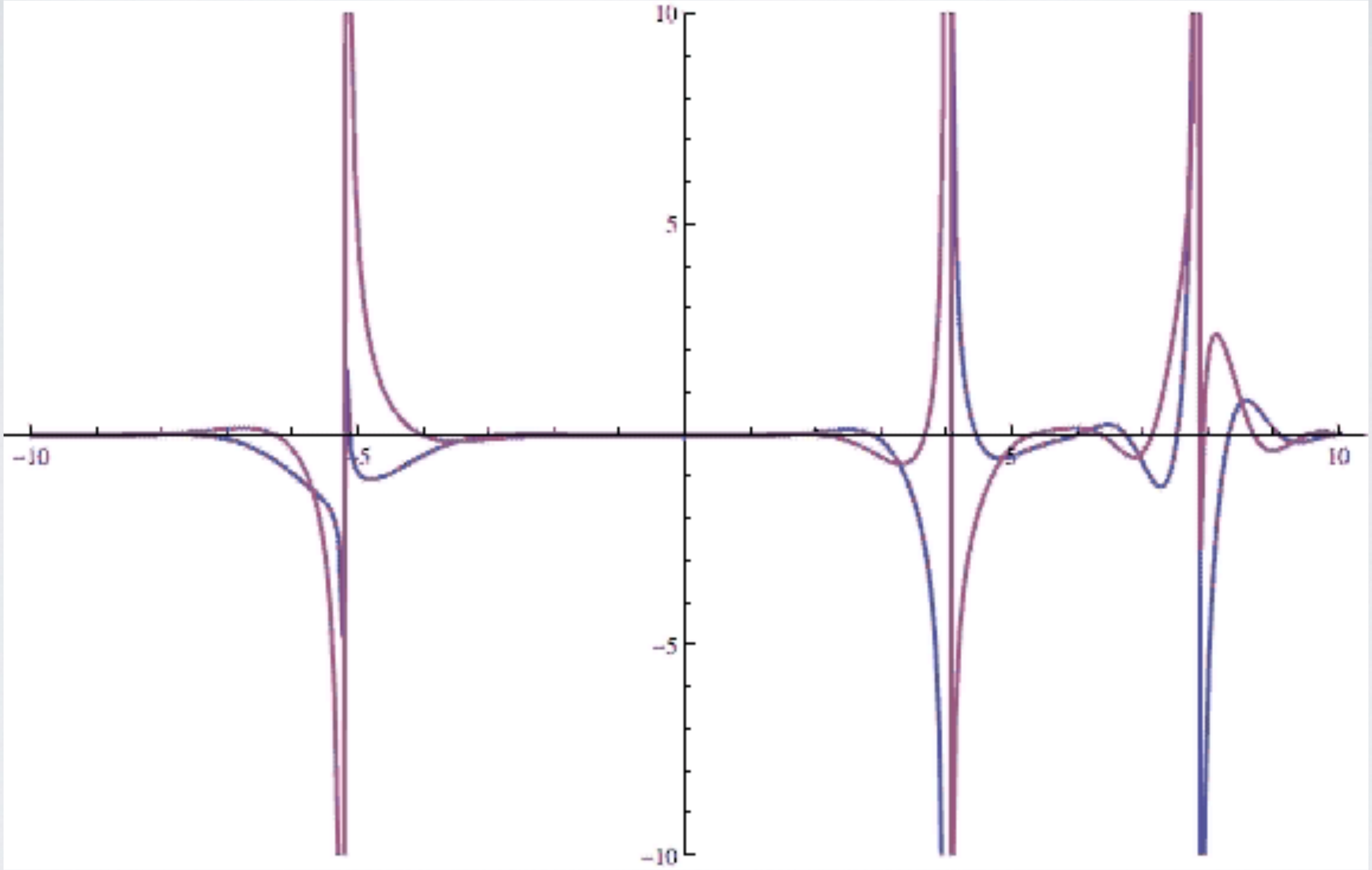


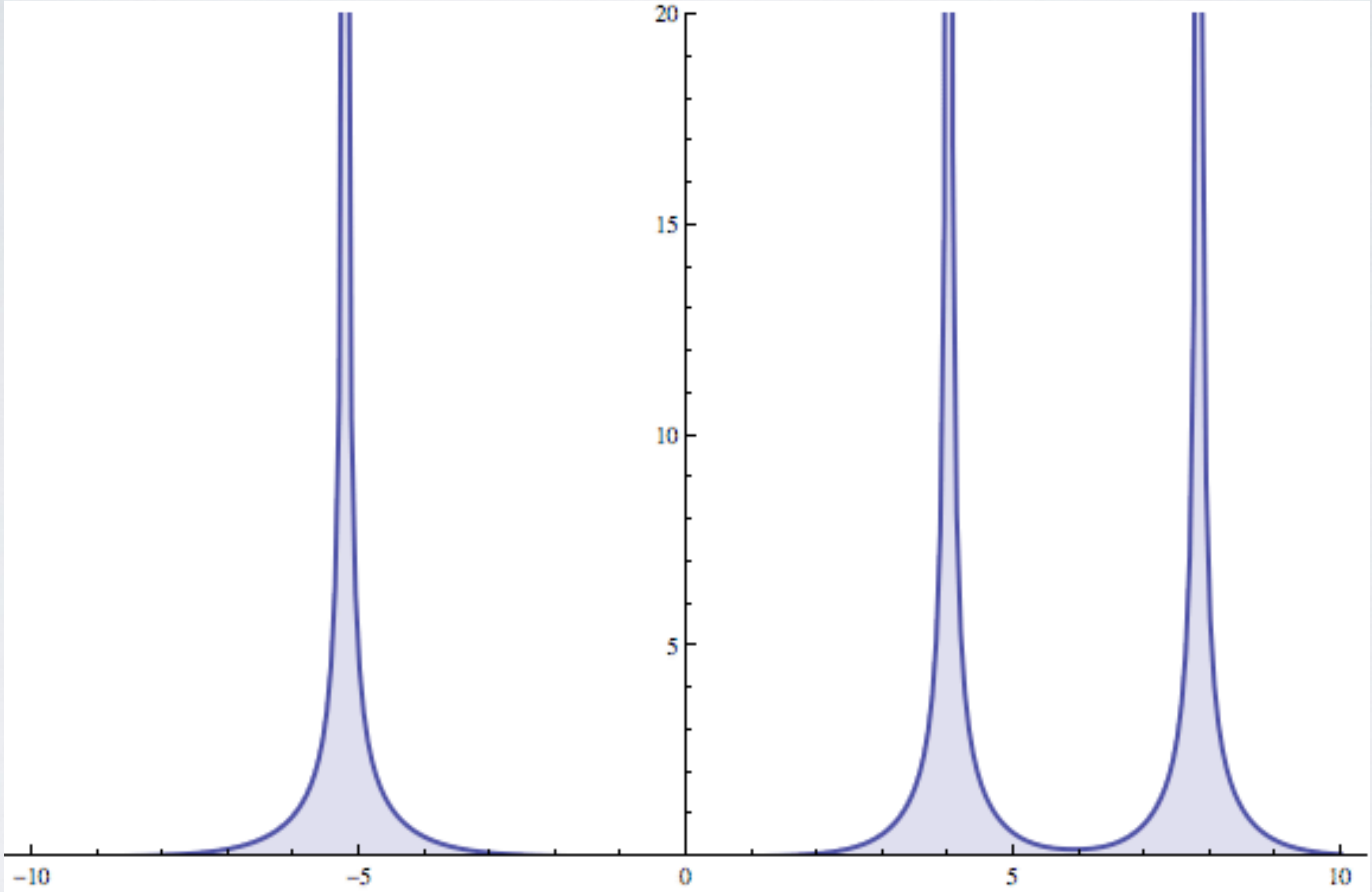
Defocusing solitons

- Suppose we switch the sign in the jump matrix for the focusing solitons

$$\begin{pmatrix} 1 & \overline{-C_1 \frac{e^{\theta(\kappa_1)}}{\bar{k}-\kappa_1}} \\ & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & \overline{C_1 \frac{e^{\theta(\kappa_1)}}{\bar{k}-\kappa_1}} \\ & 1 \end{pmatrix}$$

- Then the solution satisfies defocusing NLS
- Thus we get “solitons”
- How can this be? Doesn’t defocusing NLS have no solitons (except dark solitons)?
- The answer: these solitons have poles!


10
5
−5
−10
−10
−5
5
10


20
15
10
5
−10
−5
0
5
10

Happy ‘‘Birthday’’ Dad!