Numerical Approximation of Highly Oscillatory Integrals

Sheehan Olver
Numerical Analysis Group, DAMTP, University of Cambridge

What are highly oscillatory integrals?

$$I[f] = \int_a^b f(x)\, e^{i\omega g(x)}\, dx$$

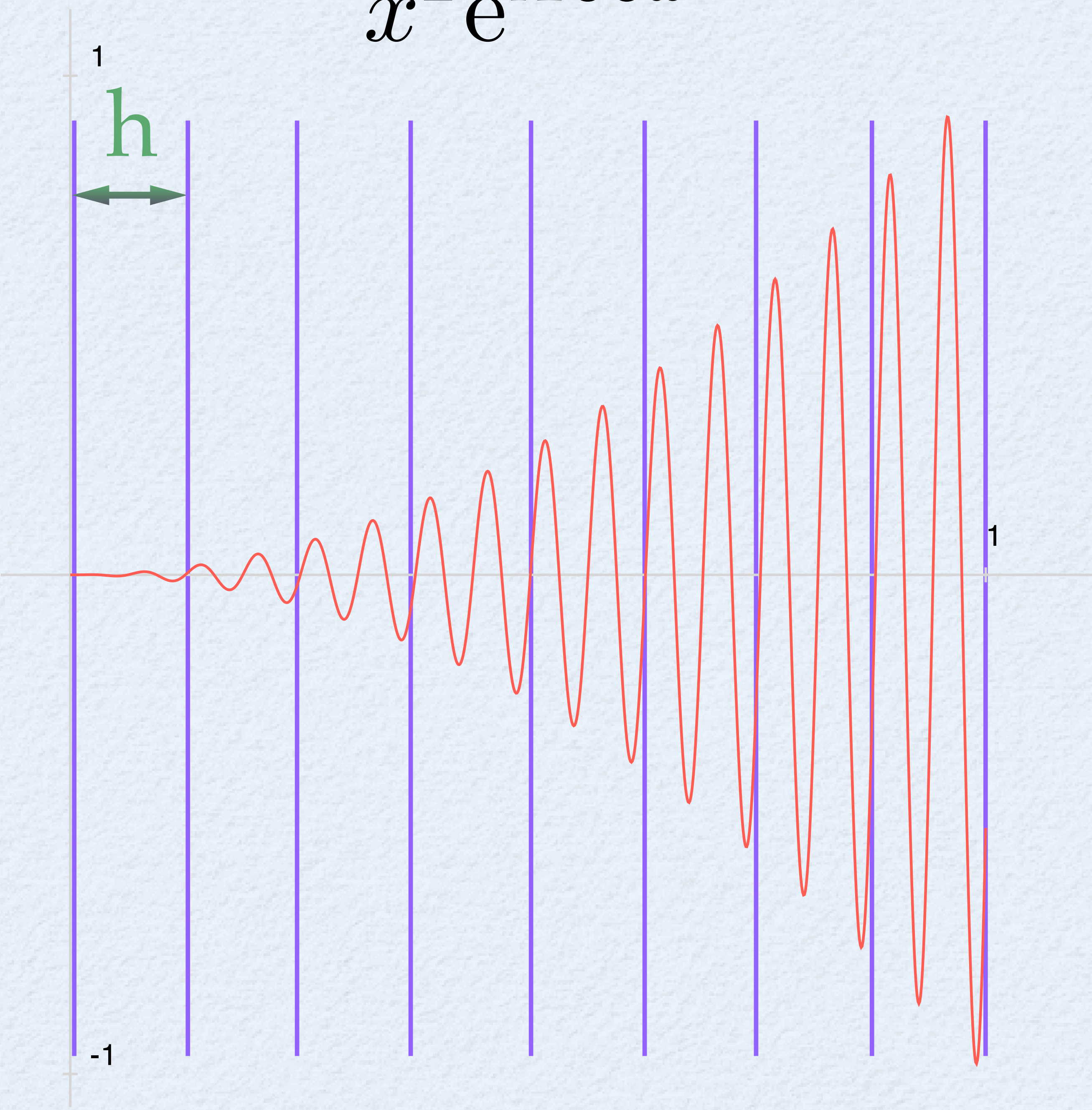
- The frequency of oscillations ω is large
- To begin with, no stationary points in interval:
  - $g'(x) \neq 0$ for $a \leq x \leq b$

Applications

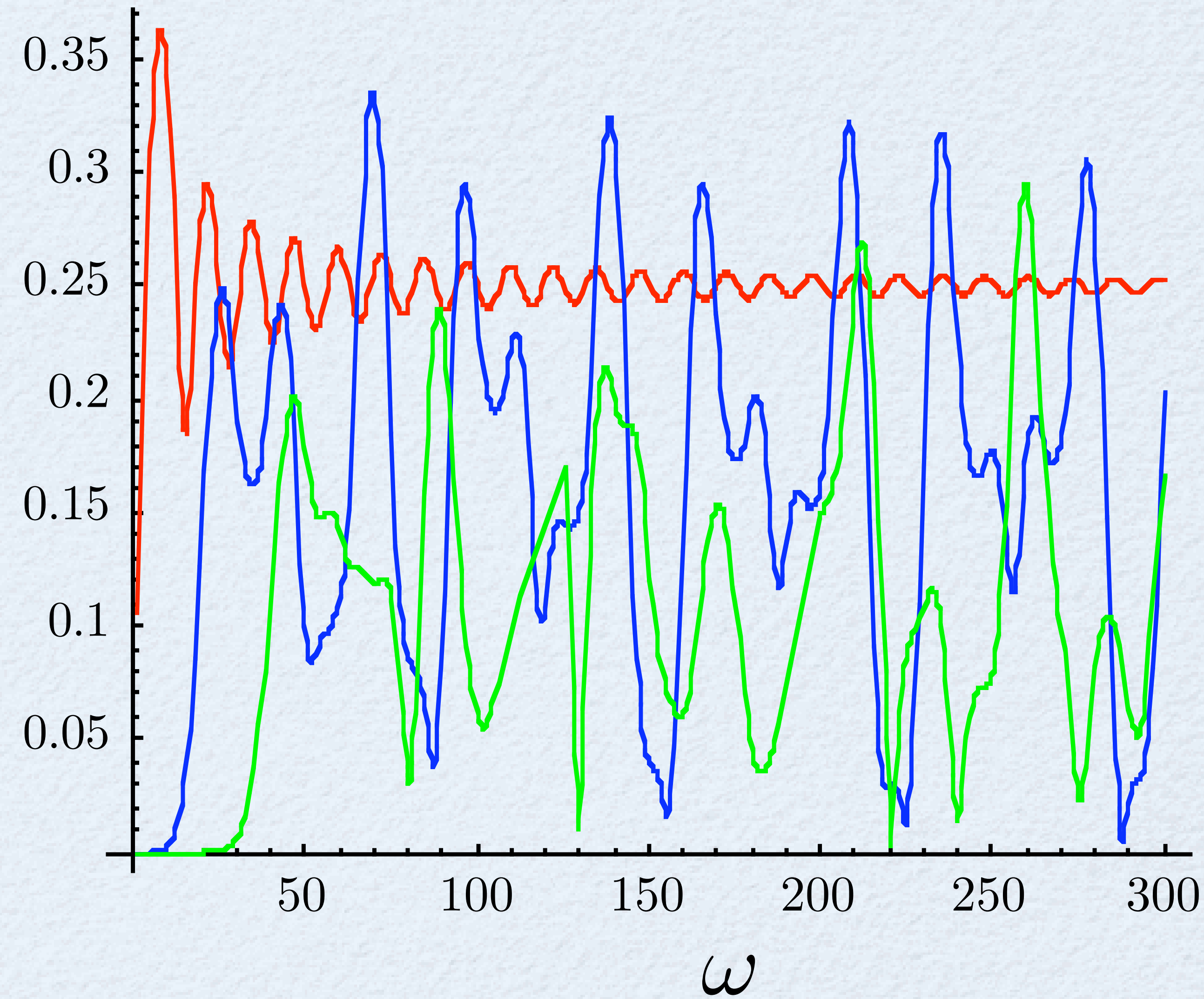
- Acoustic integral equations
- Function approximation
- Spectral methods
- Modified Magnus expansions
- Computing special functions

Why are these integrals "hard" to compute?

$x^2 e^{i100x}$



Gauss-Legendre quadrature error



$n = 1$, $n = 5$, $n = 10$

History

- Asymptotic theory (expansions, stationary phase, steepest descent)
- Filon method (1928)
    - Wrongly claimed to be inaccurate (Clendenin 1966)
- Levin collocation method (1982)
- Other methods (numerical steepest descent, Zamfirescu method, series transformations, Evans & Webster method)

Asymptotic expansion

- Rewrite the equation:

$$I[f] = \int_a^b f(x)\, e^{i\omega g(x)}\, dx = \frac{1}{i\omega} \int_a^b \frac{f(x)}{g'(x)} \frac{d}{dx} e^{i\omega g(x)}\, dx$$

- Integrate by parts:

$$= \frac{1}{i\omega} \left(\frac{f(b)}{g'(b)} e^{i\omega g(b)} - \frac{f(a)}{g'(a)} e^{i\omega g(a)} - \int_a^b \frac{d}{dx} \frac{f(x)}{g'(x)} e^{i\omega g(x)} dx \right)$$

- Error term is of order

$$-\frac{1}{i\omega} I\left[\frac{d}{dx} \frac{f(x)}{g'(x)}\right] = \mathcal{O}\left(\frac{1}{\omega^2}\right)$$

- Define σ_k by

$$\sigma_1 = \frac{f}{g'}$$
$$\sigma_{k+1} = \frac{\sigma_k'}{g'}$$

- The asymptotic expansion is

$$I[f] \sim -\sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \left[\sigma_k(b)\mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k(a)\mathrm{e}^{\mathrm{i}\omega g(a)}\right]$$

- For increasing frequency, the s-step partial sum has an error of order

$$Q_s^A[f] - I[f] \sim \mathcal{O}\left(\omega^{-s-1}\right)$$

COROLLARY

- Suppose

$$0 = f(a) = f'(a) = \cdots = f^{(s-1)}(a)$$

$$0 = f(b) = f'(b) = \cdots = f^{(s-1)}(b)$$

If f and its derivatives are bounded as ω increases, then

$$I[f] \sim \mathcal{O}\left(\frac{1}{\omega^{s+1}}\right)$$

The Filon-Type Method

- Interpolate f by a polynomial v such that the function values and the first $s - 1$ derivatives match at the boundary (Hermite interpolation)
- $I[v]$ is a linear combination of moments
- We can compute $I[v]$ if we can compute moments
- Use corollary to determine the order of the error

(From Iserles & Nørsett, 2005a)

- For nodes $a = x_0 < \cdots < x_\nu = b$ and multiplicities $\{m_k\}$, let $v(x) = \sum c_k x^k$ satisfy the system

$$\begin{aligned} v(x_k) &= f(x_k) \\ &\vdots \\ v^{(m_k-1)}(x_k) &= f^{(m_k-1)}(x_k) \end{aligned} \qquad k = 0, 1, \ldots, \nu$$

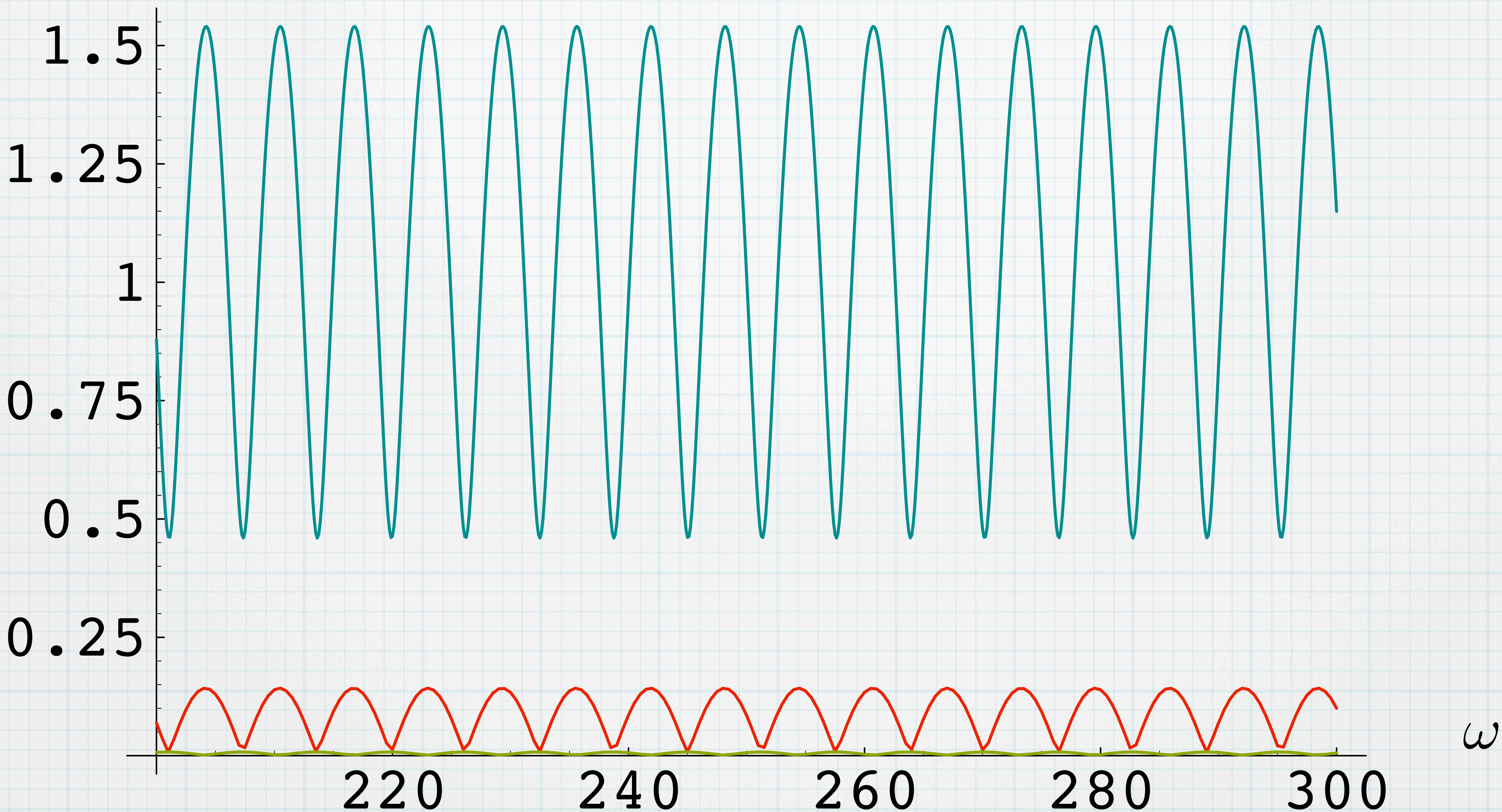
- Approximate $I[f]$ by $I[v]$

- If $m_0, m_\nu \geq s$ then the corollary implies

$$I[f] - I[v] = I[f - v] \sim \mathcal{O}\left(\frac{1}{\omega^{s+1}}\right)$$

(Iserles & Nørsett, 2005a)

$$\int_0^1 \cos x \, \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}x$$



Two-term asymptotic expansion, Filon-type method with endpoints and multiplicities equal to 2, and Filon-type method with nodes $\{0, \frac{1}{2}, 1\}$ and multiplicities $\{2, 1, 2\}$

The Original Levin Collocation Method

- Suppose F is a function such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\right] = f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}$$

- Rewrite preceding equation as

$$L[F] \equiv F'(x) + \mathrm{i}\omega g'(x)\,F(x) = f(x)$$

- Collocate F by $v = \sum c_k \psi_k$ using the system

$$L[v](x_0) = f(x_0)\,, \cdots\,, L[v](x_\nu) = f(x_\nu)$$

- Approximate $I[f]$ by

$$Q^L[f] \equiv I[L[v]] = v(b)\mathrm{e}^{\mathrm{i}\omega g(b)} - v(a)\mathrm{e}^{\mathrm{i}\omega g(a)}$$

(From Levin, 1982)

Levin-Type Method

- For nodes $\{x_k\}$ and multiplicities $\{m_k\}$ suppose

$$\begin{aligned} L[v](x_k) &= f(x_k) \\ &\vdots \qquad\qquad\qquad k = 0, 1, \ldots, \nu \\ L[v]^{(m_k-1)}(x_k) &= f^{(m_k-1)}(x_k) \end{aligned}$$

- Regularity condition: $\{g'\psi_k\}$ can interpolate the nodes and multiplicities (always satisfied with polynomial basis)

- Then, for $m_0, m_\nu \geq s$:

$$I[f] - Q^L[f] = I[f - L[v]] \sim \mathcal{O}\left(\frac{1}{\omega^{s+1}}\right)$$

Sketch of Proof

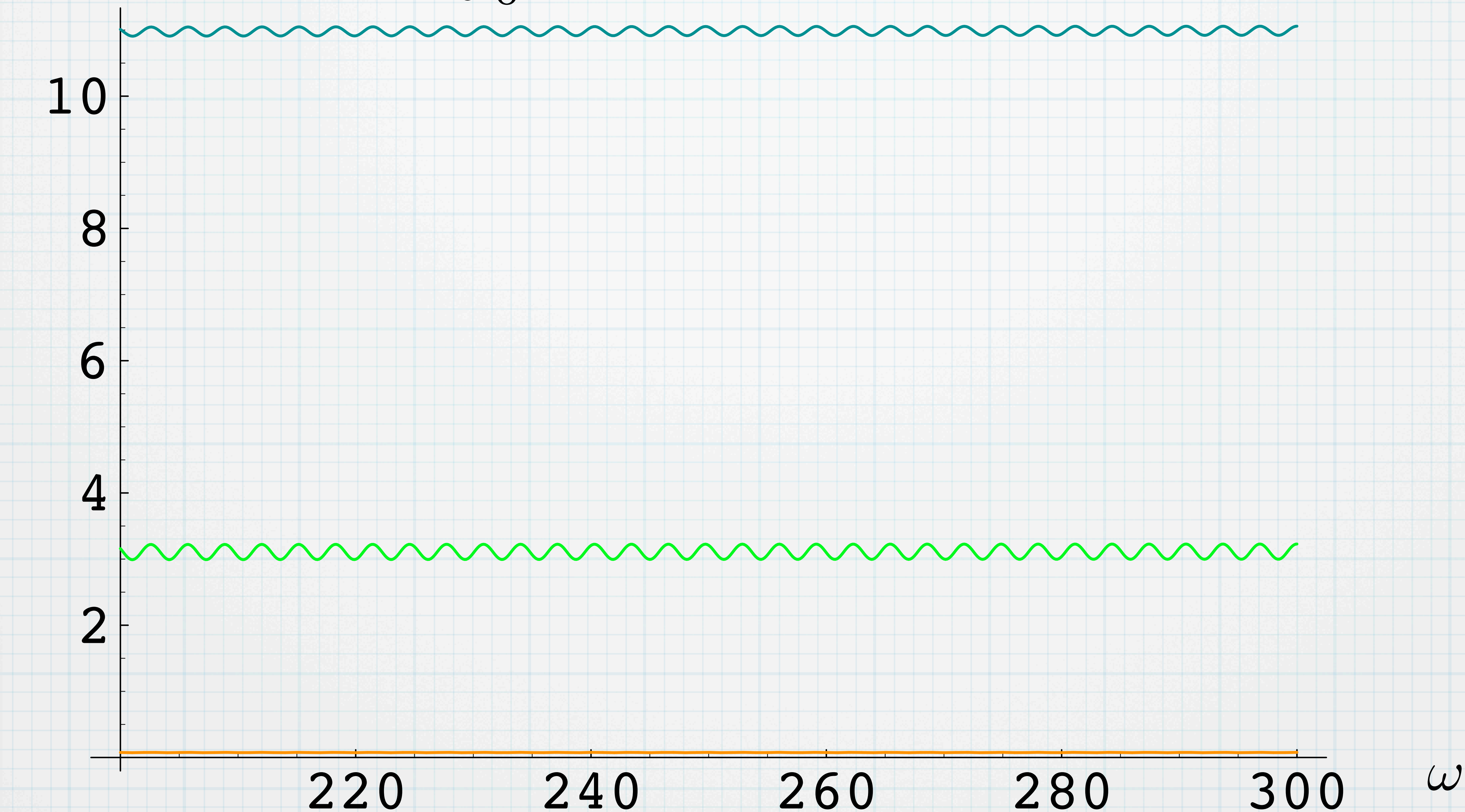
- The order follows from the corollary if $f - L[v]$ and all its derivatives are bounded for increasing ω
- Collocation matrix can be written as $A = P + \mathrm{i}\omega G$
- Regularity condition ensures G is non-singular
- From Cramer's rule

$$c_k = \frac{\det A_k}{\det A} = \frac{\mathcal{O}(\omega^n)}{(\mathrm{i}\omega)^{n+1} \det G + \mathcal{O}(\omega^n)} = \mathcal{O}(\omega^{-1})$$

- Hence v and its derivatives are $\mathcal{O}(\omega^{-1})$ and

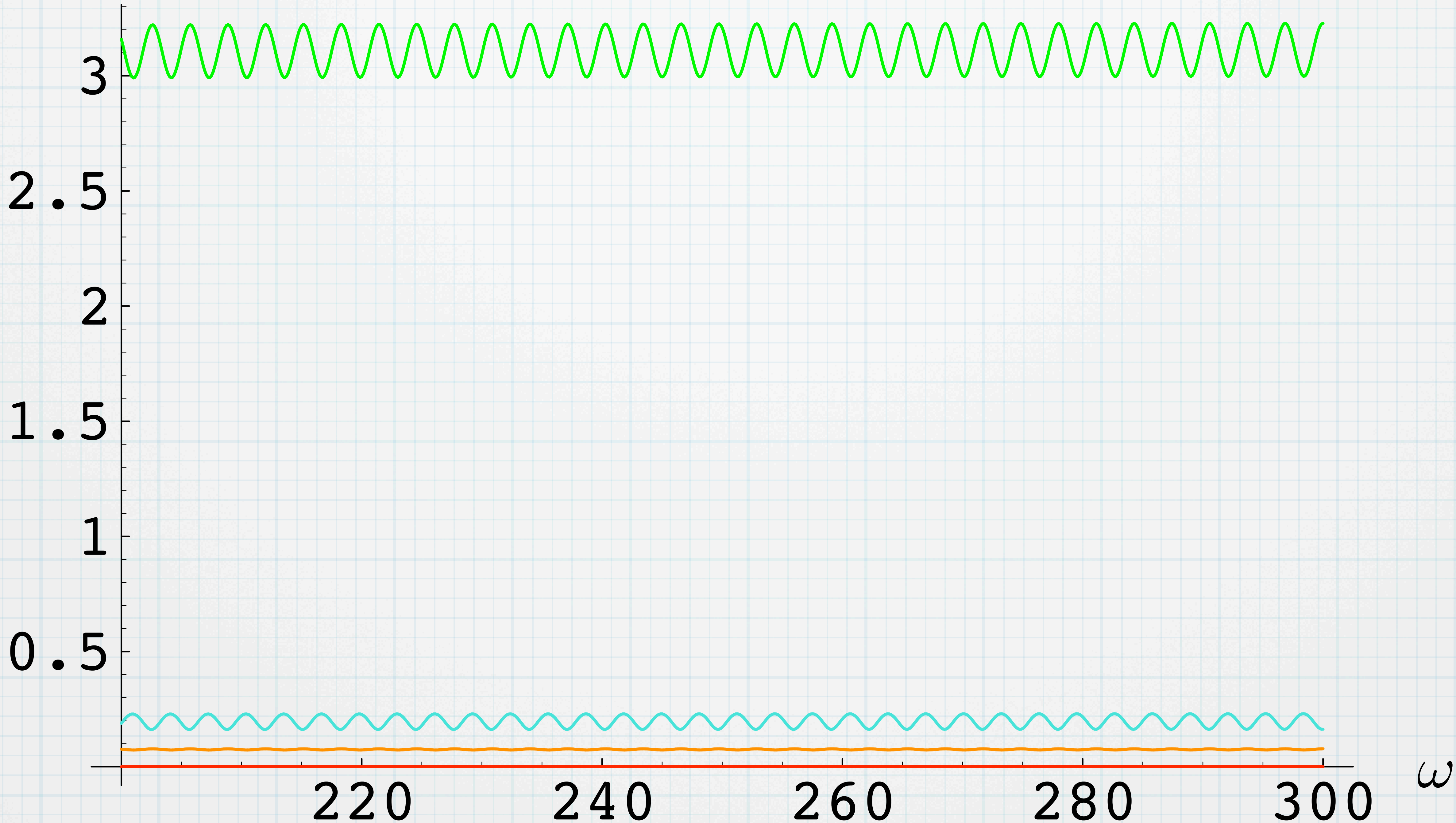
$$L[v] = v' + \mathrm{i}\omega g' v = \mathcal{O}(1)$$

$$\int_0^1 \cos x \, \mathrm{e}^{\mathrm{i}\omega(x^2+x)} \, \mathrm{d}x$$



Asymptotic expansion, Filon-type method with only endpoints and multiplicities equal to 3, and Levin-type method with same nodes and multiplicities

$\omega^3 \, |Error|$ $$\int_0^1 \cos x \, e^{i\omega(x^2+x)} \, dx$$



Levin-type method and **Filon-type method** with endpoints and multiplicities 2, **Levin-type method** and **Filon-type method** with nodes $\{0, \frac{1}{4}, \frac{2}{3}, 1\}$ and multiplicities $\{2,2,1,2\}$

Multivariate highly oscillatory integrals

$$\int_\Omega f(x,y) \mathrm{e}^{\mathrm{i}\omega g(x,y)} \, \mathrm{d}V$$

- The boundary of Ω is piecewise smooth
- Nonresonance condition is satisfied:
  - ∇g is never orthogonal to the boundary
- No critical points in domain:
  - $\nabla g(x,y) \neq 0$ for $(x,y) \in \Omega$

- There exists an asymptotic expansion that depends on f and its derivatives at the vertices
- The s-step approximation, which uses the order $s-1$ partial derivatives of f at the boundary, has an error

$$\mathcal{O}\left(\frac{1}{\omega^{s+d}}\right)$$

- The multivariate Filon-type method consists of interpolating f and its derivatives at the vertices

- For a function $\boldsymbol{F}$, we write the integral as

$$\oint_{\partial\Omega} \mathrm{e}^{\mathrm{i}\omega g(x,y)} \boldsymbol{F} \cdot \, \mathrm{d}\boldsymbol{s} = \oint_{\partial\Omega} \mathrm{e}^{\mathrm{i}\omega g(x,y)} (F_1(x,y)\, \mathrm{d}y - F_2(x,y)\, \mathrm{d}x)$$

- Green's theorem states that the above integral is the same as

$$\iint_{\Omega} \left[F_{1,x} + F_{2,y} + \mathrm{i}\omega(g_x F_1 + g_y F_2)\right] \mathrm{e}^{\mathrm{i}\omega g}\, \mathrm{d}V$$

- Thus collocate f by $\boldsymbol{v}$ using the operator

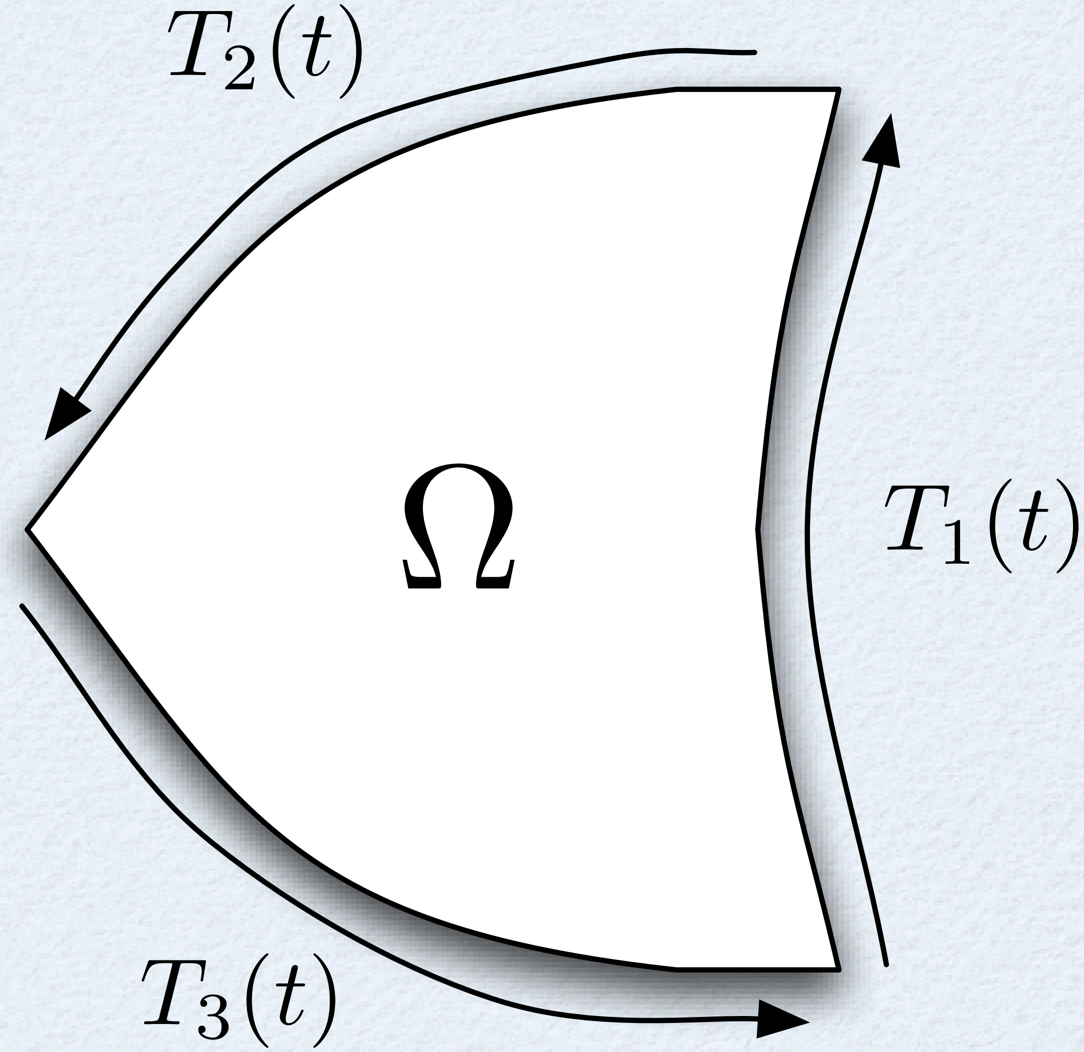
$$L[\boldsymbol{v}] = v_{1,x} + v_{2,y} + \mathrm{i}\omega(g_x v_1 + g_y v_2) = \nabla \cdot \boldsymbol{v} + \mathrm{i}\omega \nabla g \cdot \boldsymbol{v}$$

Levin-type Method

- For nodes $\{\boldsymbol{x}_k\}$ and multiplicities $\{m_k\}$ collocate $\boldsymbol{v} = [v_1, v_2]^\top = \sum c_k \boldsymbol{\psi}_k$ using the system

$$\frac{\partial^{|\boldsymbol{m}|}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} L[\boldsymbol{v}](\boldsymbol{x}_k) = \frac{\partial^{|\boldsymbol{m}|}}{\partial \boldsymbol{x}^{\boldsymbol{m}}} f(\boldsymbol{x}_k) \qquad \begin{array}{c} k = 0, 1, \ldots, \nu \\ |\boldsymbol{m}| \leq m_k - 1 \end{array}$$

- Regularity condition: $\{\nabla g \cdot \boldsymbol{\psi}_k\}$ can interpolate at the given nodes, plus regularity condition satisfied in lower dimensions
- Method has asymptotic order $\mathcal{O}\left(\omega^{-s-2}\right)$



$$\iint_\Omega f \mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}V \approx \iint_\Omega \mathcal{L}[\boldsymbol{v}]\mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}V = \oint_{\partial\Omega} \mathrm{e}^{\mathrm{i}\omega g}\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{s} = \sum_\ell \int_{T_\ell} \mathrm{e}^{\mathrm{i}\omega g}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s}$$

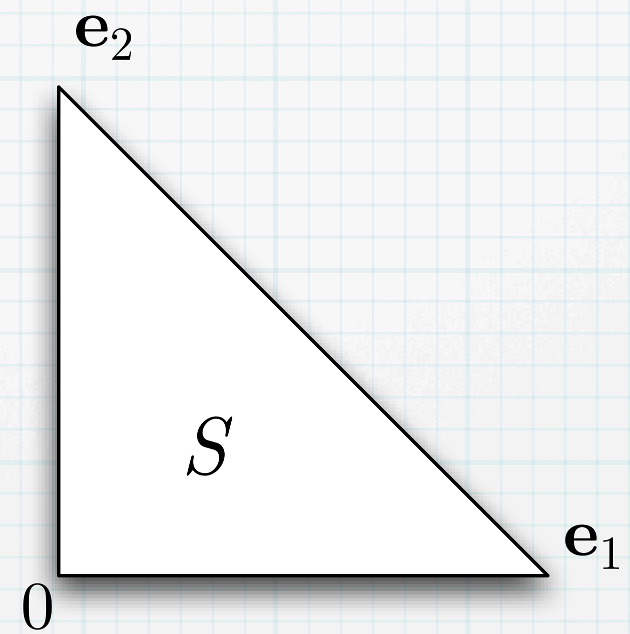
$$= \sum_\ell \int_0^1 \mathrm{e}^{\mathrm{i}\omega g(T_\ell(t))}\boldsymbol{v}(T_\ell(t))\cdot \boldsymbol{J}_{T_\ell}(t)\,\mathrm{d}t \approx Q_g^L[f]$$

for

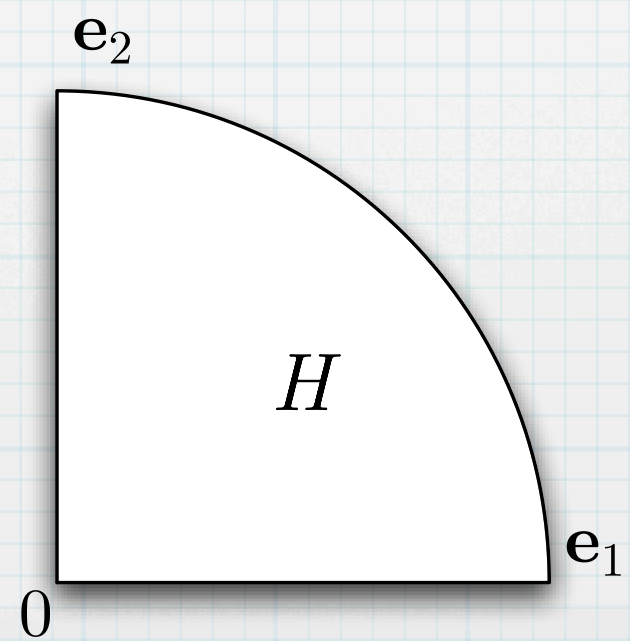
$$Q_g^L[f] = \sum_\ell Q^L_{g(T_\ell(t))}\left[\boldsymbol{v}(T_\ell(t))\cdot\boldsymbol{J}_{T_\ell}(t)\right] \qquad \boldsymbol{J}_T(t) = \begin{pmatrix} T_2'(t) \\ -T_1'(t)\end{pmatrix}$$

Domains

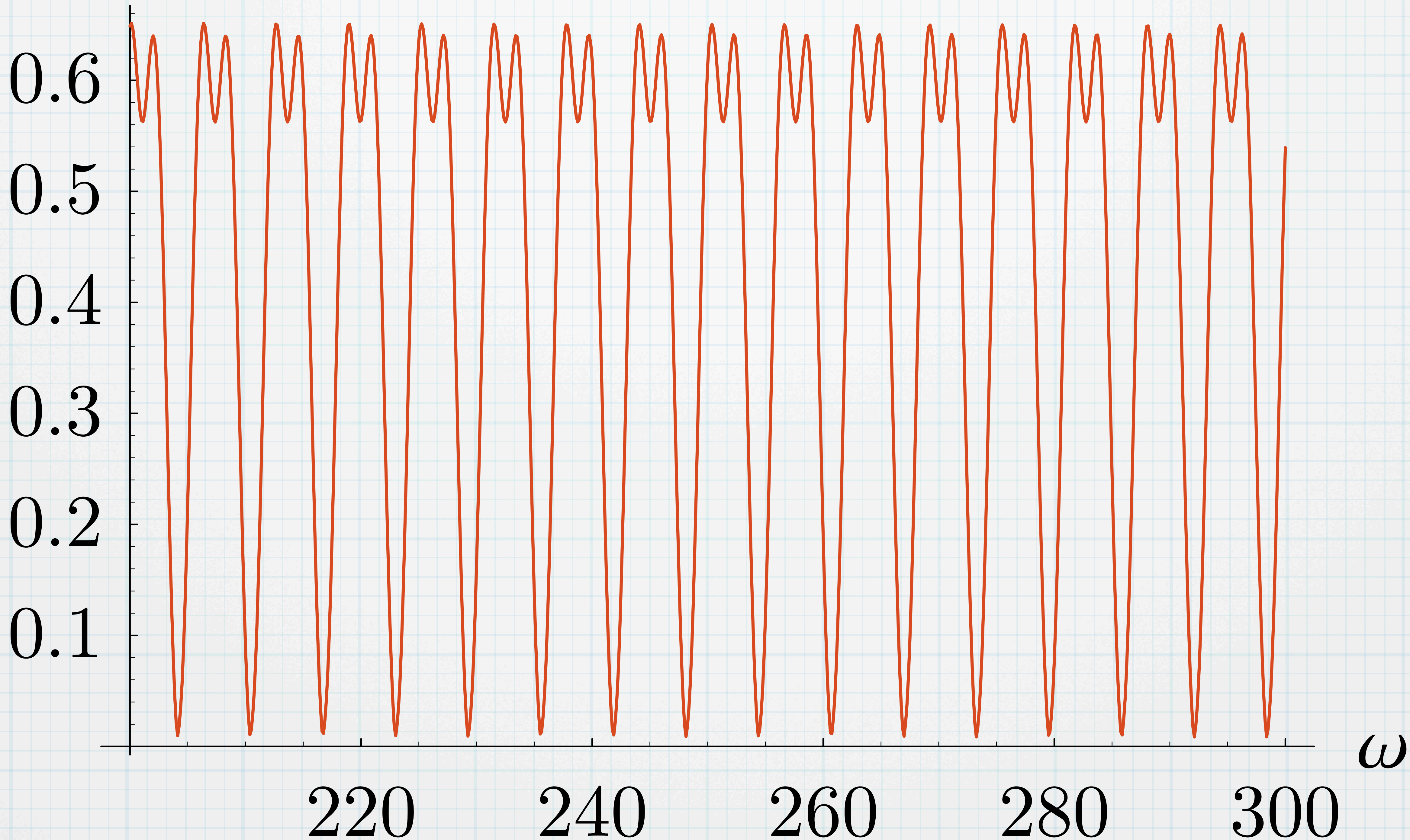
* Simplex



* Quarter Circle

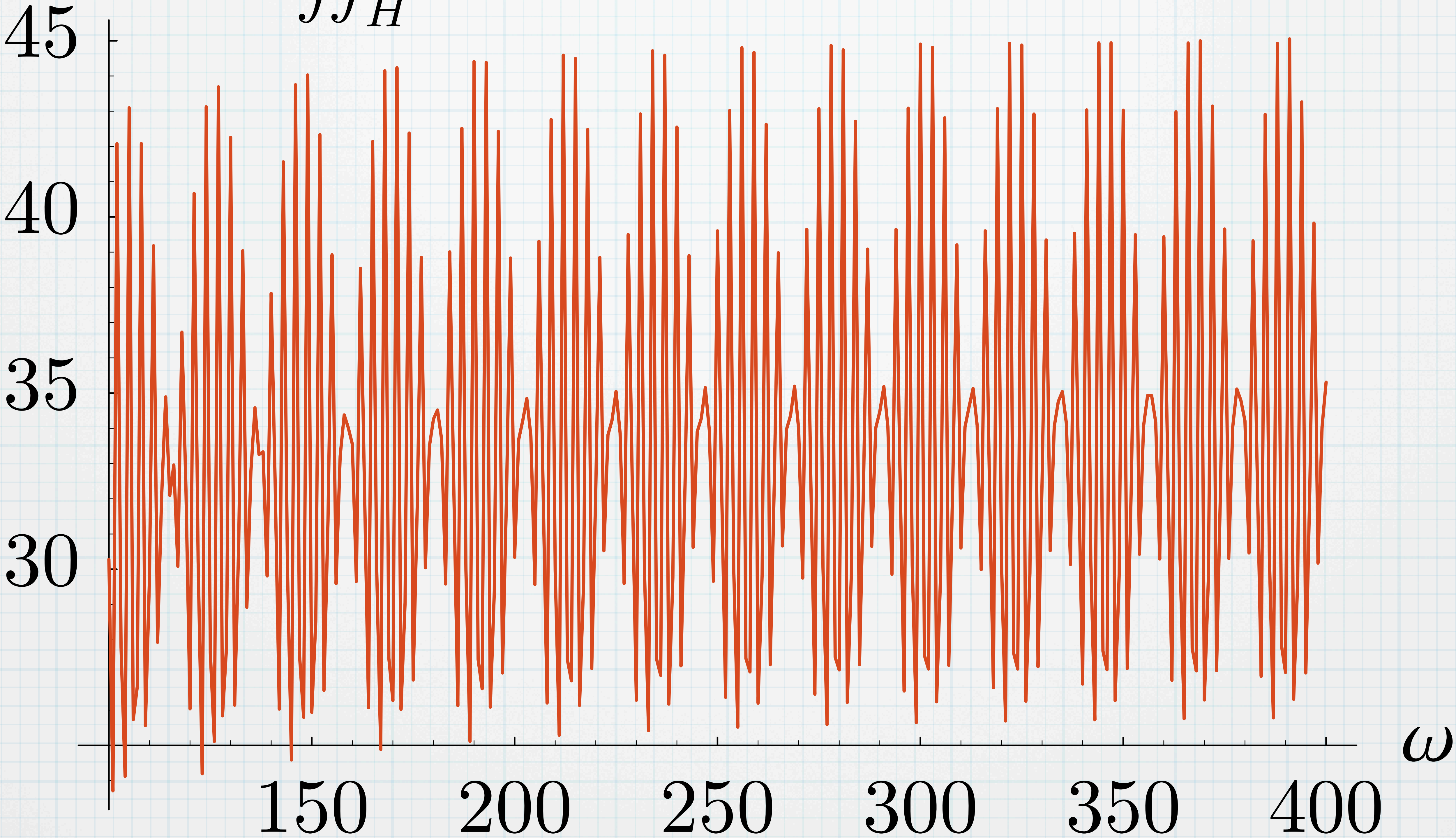


$$\omega^3\,|Error| \quad \iint_S \left(\frac{1}{x+1}+\frac{2}{y+1}\right) e^{i\omega(2x-y)}\,dV$$



Levin-type method with only vertices and multiplicities all one on a two-dimensional simplex

$$\omega^4\,|Error| \qquad \iint_H \mathrm{e}^x \cos xy \; \mathrm{e}^{\mathrm{i}\omega(x^2+x-y^2-y)}\, \mathrm{d}V$$



Levin-type method with only vertices and multiplicities all two on a quarter circle

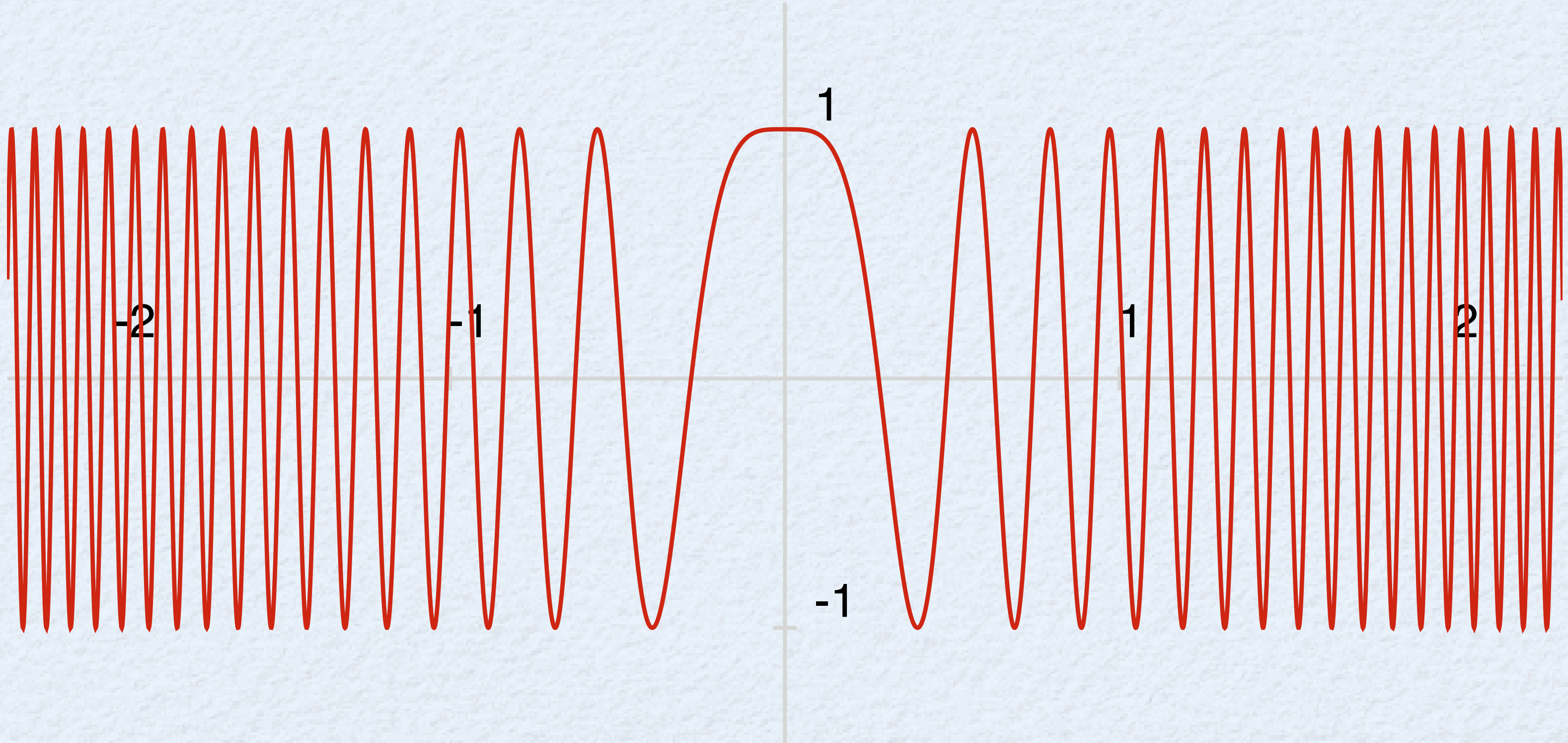
Stationary points

- Consider the integral

$$\int_{-1}^{1} f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \, \mathrm{d}x$$

where

$$0 = g(0) = g'(0) = \cdots = g^{(r-1)}(0), \qquad g^{(r)}(0) > 0$$



$\cos 20\, x^2$

Stationary points

- Filon-type methods work, but still require moments (which are harder to find since the oscillator is more complicated)
- Levin-type methods do not work
- We will combine the two methods to derive a Moment-free Filon-type method

Asymptotic expansion

- Can still do integration by parts ($r = 2$):

$$I[f] = I[f - f(0)] + f(0)I[1]$$

$$= \frac{1}{\mathrm{i}\omega}\int_{-1}^{1}\frac{f(x)-f(0)}{g'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x + f(0)I[1]$$

$$= \left[\frac{f(1)-f(0)}{g'(1)}\mathrm{e}^{\mathrm{i}\omega g(1)} - \frac{f(-1)-f(0)}{g'(-1)}\mathrm{e}^{\mathrm{i}\omega g(-1)}\right]$$

$$- \frac{1}{\mathrm{i}\omega}I\left[\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)-f(0)}{g'(x)}\right]\right] + f(0)I[1]$$

- Unfortunately requires moments

(From Iserles & Nørsett, 2005a)

Moment-Free Methods

- Idea: find alternate to polynomials that can be integrated in closed form for general oscillators
- Can be used to find an asymptotic expansion which does not require moments (turns out to be stationary phase under a different guise)
- Can be used as an interpolation basis in a Filon-type method, to improve accuracy like before

Incomplete Gamma Functions

- Suppose $g(x) = x^r$
- Solve the differential equation

$$\mathcal{L}[v] = v' + \mathrm{i}\omega g' v = v' + r\mathrm{i}\omega x^{r-1} v = x^k$$

- Solution is known:

$$v(x) = \frac{\omega^{-\frac{1+k}{r}}}{r} \mathrm{e}^{-\mathrm{i}\omega x^r + \frac{1+k}{2r}\mathrm{i}\pi} \left[\Gamma\left(\frac{1+k}{r}, -\mathrm{i}\omega x^r\right) - \Gamma\left(\frac{1+k}{r}, 0\right)\right]$$

$$x \geq 0$$

- Now replace occurrences of x^r with $g(x)$

- We obtain

$$\phi_{r,k}(x) = D_{r,k}(\operatorname{sgn} x)\frac{\omega^{-\frac{k+1}{r}}}{r}\mathrm{e}^{-\mathrm{i}\omega g(x)+\frac{1+k}{2r}\mathrm{i}\pi}\left[\Gamma\left(\frac{1+k}{r},-\mathrm{i}\omega g(x)\right)-\Gamma\left(\frac{1+k}{r},0\right)\right]$$

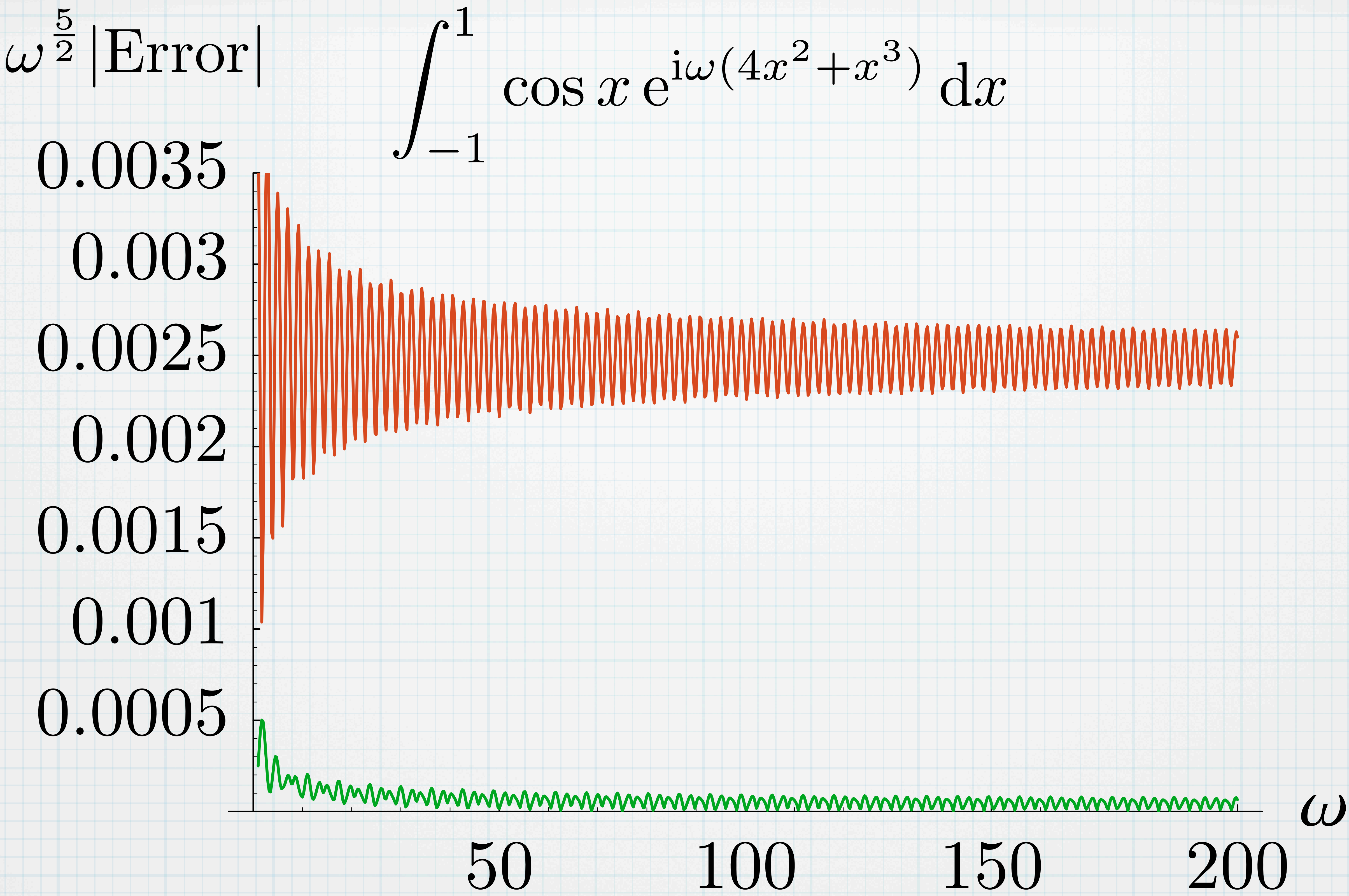
$$D_{r,k}(\operatorname{sgn} x) = \begin{cases} (-1)^k & \operatorname{sgn} x < 0 \text{ and } r \text{ even}, \\ (-1)^k \mathrm{e}^{-\frac{1+k}{r}\mathrm{i}\pi} & \operatorname{sgn} x < 0 \text{ and } r \text{ odd}, \\ -1 & \text{otherwise.} \end{cases}$$

- Calculus can show that

$$\mathcal{L}[\phi_{r,k}]\,(x) = \operatorname{sgn}(x)^{r+k+1}\frac{|g(x)|^{\frac{k+1}{r}-1}g'(x)}{r}$$

- These functions look ugly, but have following nice properties:
    - Are smooth: $\phi_{r,k}, \mathcal{L}[\phi_{r,k}] \in C^\infty$
    - $\{\mathcal{L}[\phi_{r,k}]\}$ form a Chebyshev set (can interpolate any given nodes/multiplicities)
    - $\mathcal{L}[\phi_{r,k}]$ are independent of ω
    - Are integrable in closed form:

$$I[\mathcal{L}[\phi_{r,k}]] = \phi_{r,k}(1)\mathrm{e}^{\mathrm{i}\omega g(1)} - \phi_{r,k}(-1)\mathrm{e}^{\mathrm{i}\omega g(-1)}$$



Asymptotic expansion versus Moment-free Filon-type method with endpoints and zero and multiplicities equal to {2,3,2}

Levin-type method with asymptotic basis

- Use terms from the asymptotic expansion as the collocation basis:

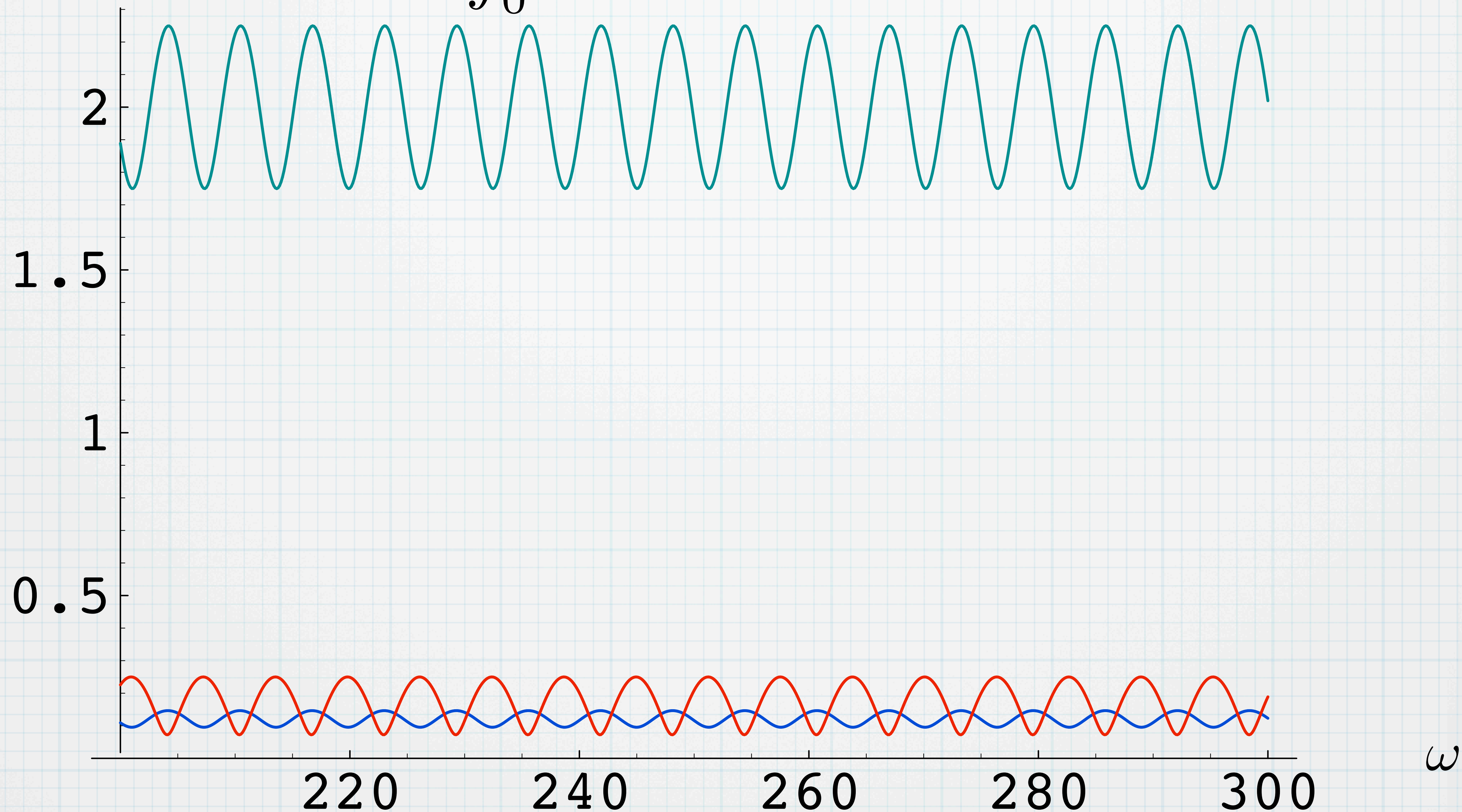
$$\nabla g \cdot \psi_1 = f, \qquad \nabla g \cdot \psi_{k+1} = \nabla \cdot \psi_k, \qquad k = 1, 2, \ldots .$$

- Captures asymptotic behaviour of the expansion while allowing for possibility of convergence

- If the regularity condition is satisfied then we obtain an order of error $\mathcal{O}\left(\omega^{-n-s-d}\right)$, where n is the size of the system and s is again the smallest endpoint multiplicity

$\omega^4 \, |Error|$

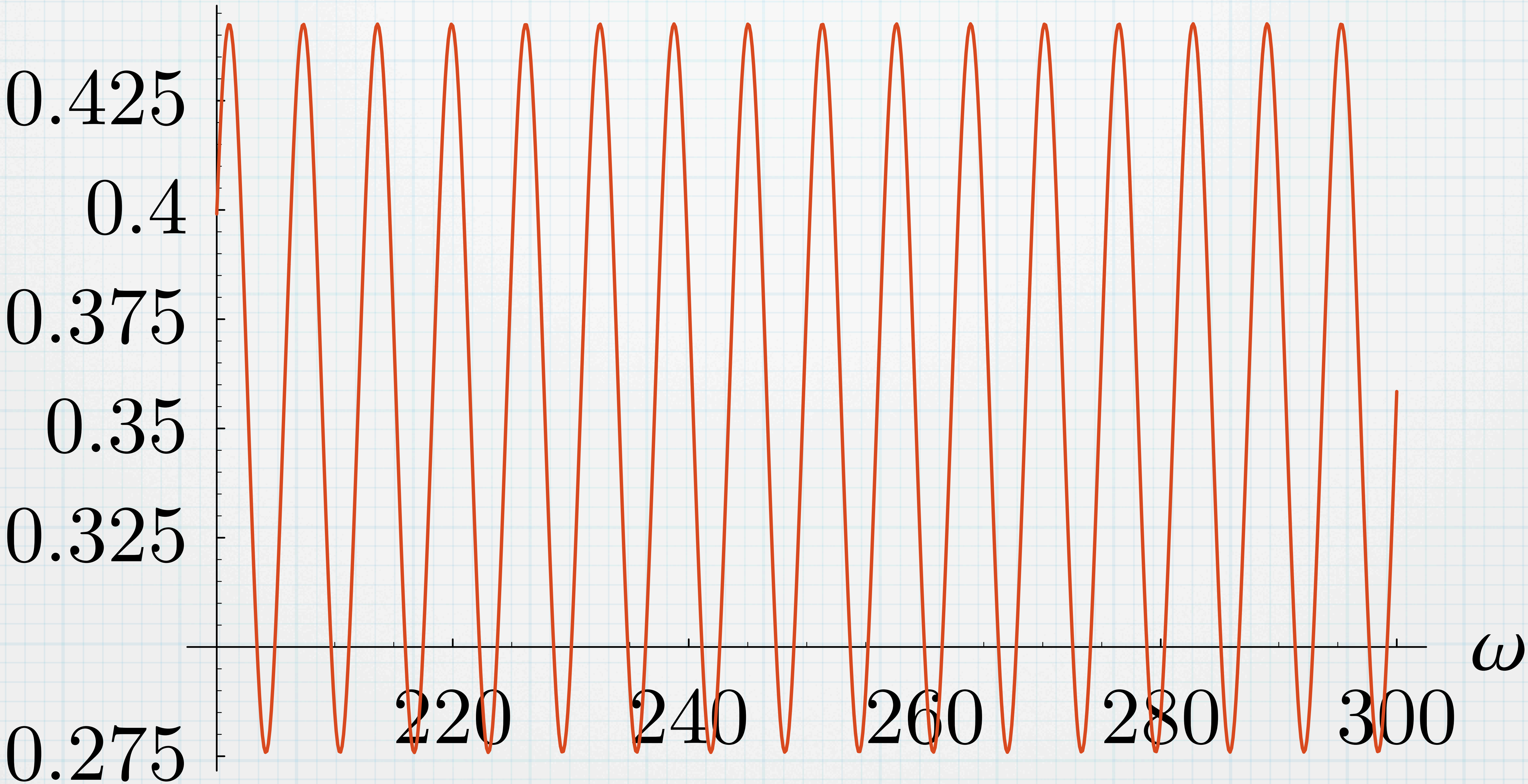
$$\int_0^1 \log(x+1)\, e^{i\omega x}\, dx$$



Asymptotic expansion, **Filon-type method** with only endpoints and multiplicities equal to 3, and **Levin-type method with asymptotic basis** with nodes $\{0, \frac{1}{2}, 1\}$ and multiplicities all one

$$\int_0^1 \log(x+1)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}x$$

$$\omega^4|\text{Error}| \qquad \iint_S \left(\frac{1}{x+1} + \frac{2}{y+1}\right) e^{i\omega(2x-y)}\, dV$$



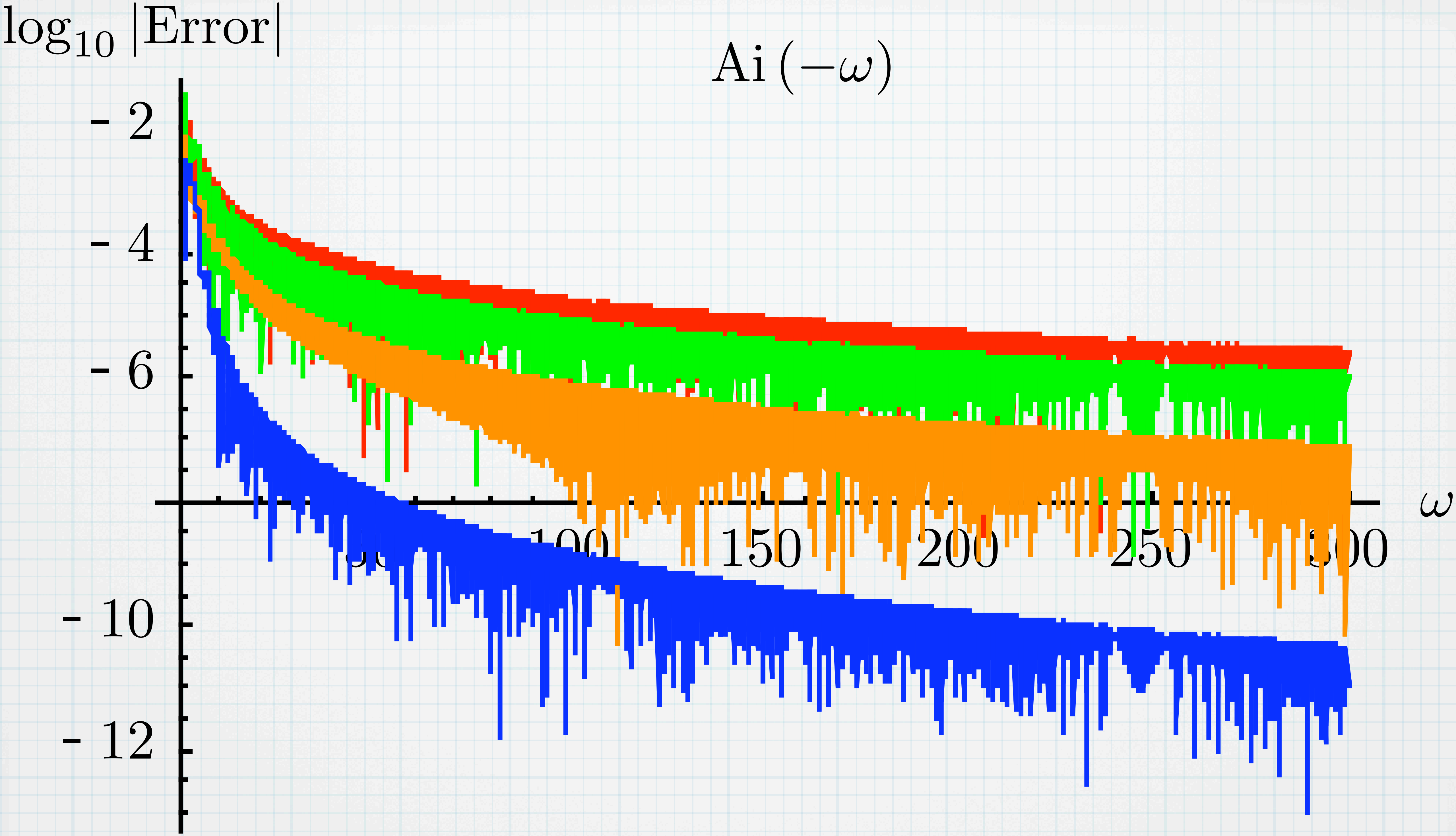
Levin-type method with asymptotic basis with only vertices and multiplicities all one on a two-dimensional simplex

AIRY FUNCTION

- Write the Airy function as

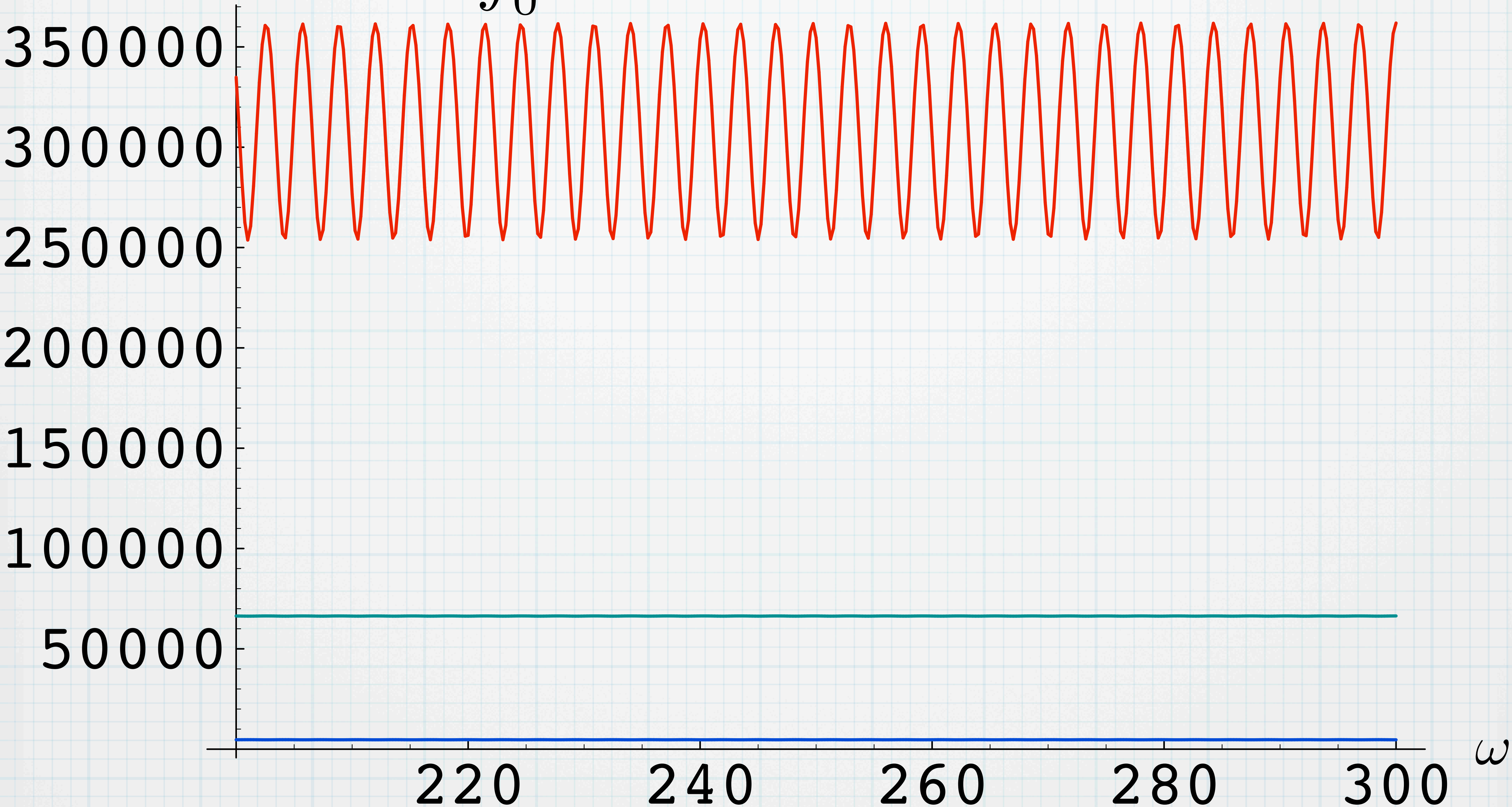
$$\mathrm{Ai}\,(x) = \Re\left[\frac{1}{\pi}\sqrt{x}\int_0^\infty \mathrm{e}^{\mathrm{i}x^{3/2}\left(t^3-t\right)}\,\mathrm{d}t\right]$$
$$= \Re\left[\frac{1}{\pi}\sqrt{x}\int_0^2 \mathrm{e}^{\mathrm{i}x^{3/2}\left(t^3-t\right)}\,\mathrm{d}t + \frac{1}{\pi}\sqrt{x}\int_2^\infty \mathrm{e}^{\mathrm{i}x^{3/2}\left(t^3-t\right)}\,\mathrm{d}t\right]$$

- Approximate first integral with Moment-free Filon-type method
- Approximate second integral with Levin-type method with asymptotic basis



One-term asymptotic expansion, Moment-free Filon-type method & Levin-type method with asymptotic basis with nodes $\{0,1,2\}$, $\{0,0.5,1,1.5,2,3\}$ and $\{0,1,2\}$ **with multiplicities** $\{2,3,2\}$ compared to $\text{Ai}(-\omega)$

$\omega^4 \, |Error|$ $\qquad \int_0^1 e^{10x} \, e^{i\omega(x^2+x)} \, dx$



Asymptotic expansion , Filon-type method with only endpoints and multiplicities equal to 3, and Levin-type method with asymptotic basis with only endpoints and multiplicities all one

$\omega = 200$

$$\int_0^1 \mathrm{e}^{10x}\, \mathrm{e}^{\mathrm{i}\omega(x^2+x)}\, \mathrm{d}x$$

Order	4	5	7
Filon-type	0.042	0.0016	$1.3 \cdot 10^{-6}$
Levin-type	0.015	0.00043	$3 \cdot 10^{-7}$
Asymptotic expansion	0.0083	0.00011	$1.7 \cdot 10^{-8}$
Levin asymptotic basis	0.00059	$2.8 \cdot 10^{-6}$	$9.9 \cdot 10^{-12}$

Miscellaneous

- Filon-type and Levin-type methods do not need derivatives to obtain high asymptotic orders

- Can use incomplete Gamma functions for multivariate integrals with stationary points
- Can replace collocation with least squares
- Collocation with WKB expansion can be used to approximate oscillatory differential equations

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