Topology of manifolds: Notes

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November 9, 2023

The title should not be taken too seriously since we can only cover small fragments of this large subject. We will:

- Outline some main ideas;
- Give examples;
- Explain why there is a distinction between "high dimensional" and "low dimensional" manifold topology.

Other topics lectures in this series which are particularly relevent are *Chern classes and classifying spaces, Poincaré Duality, Morse Theory and the Witten complex.*

PART 1, Mostly examples

Connected sum

Let M_1, M_2 be n-dimensional manifolds. The connected sum $M_1 \sharp M_2$ is defined by removing balls from each manifold and gluing along the resulting boundaries.

Classification of closed 2-manifolds

1. Orientable: $S^2, T^2, T^2\sharp T^2, \dots, \sharp_g T^2$
2. Non-orientable: $\mathbf{RP}^2, \mathbf{RP}^2\sharp T^2, \mathbf{RP}^2\sharp T^2\sharp T^2 \ dots$.

It is unrealistic/impossible to have such complete classifications in all dimensions.

For $n \geq 4$ any finitely presented group can be realised as the fundamental group of an n-manifold (and with any given rpresentation by generators and relations).

Even if we restrict to simply connected manifolds the algebraic topology data becomes very complicated, in general.

For a (closed ,oriented, smooth) manifold we have

- The cohomology $H^*(M)$ with cup-product which is equivalent to the homology $H*(M)$ with intersection product.
- Pontrayagin classes $p_i \in H^{4i}(M)$, Stiefel-Whitney classes $w_i \in H^i(M; \mathbf{Z}/2)$.
- …Other more sophisticated gadgets

(Recall that $p_i(V) = (-1)^i c_{2i}(V \otimes \mathbf{C})$ where c_{2i} is the Chern class.)

If $\dim M = 2m$ we have an intersection form $H_m(M) \times H_m(M) \to \mathbf{Z}$ which is symmetric if m is even and skew-symmetric if M is odd.

For a closed manifold M the form is nondegenerate. In the skew-symmetric case the classification of such forms over the integers is very simple: they are sums of the block

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In the symmetric case, the classification with real co-efficients is given by the signature (number of positive eigenvalues-number of negative eigenvalues) but with integer coefficients there is a rich theory.

Projective planes

Beyond spheres the simplest cohomology of a closed manifold would occur in dimension $4k$ with a manifold M having $H^0 = H^{2k} = H^{4k} = \mathbf{Z}$ and all other cohomology zero. This occurs only when $k = 1, 2, 4$. For $k = 1$ we have the complex projective plane $\mathbf{CP}^2$. This can be obtained by attaching a 4-ball to a 2-sphere by the Hopf map $S^3 \to S^2$. For $k = 2$ we have the quaternionic projective plane constructed from a map $S^7 \to S^4$. For $k = 4$ there is a *Moufang plane* related to the Cayley numbers and exceptional Lie groups constructed from a map $S^{15} \to S^8$. The fact that no other cases occur follows from a relatively deep fact in algebraic topology (the "Hopf invariant 1 problem").

Plumbing

It is easy to construct manifolds with boundary having any given intersection form. We illustrate this with the "ADE" manifolds in 4-dimensions which are important in algebraic geometry and diferential geometry. Let Γ be the Dynkin diagram (a graph) corresponding to one of the ADE Lie algebras. For each vertex v we take a copy Σ_v of the 2-sphere and a tubular neighbourhood N_v of Σ_v in the total space of its cotangent bundle. So $\Sigma_v \subset N_v$ has self-intersection number -2. If vertices v, v' are joined by an edge in Γ we glue N_v to $N_{v'}$ in such a way that $\Sigma_v, \Sigma_{v'}$ have a single transverse intersection point, with sign $+1$. This gives a 4-manifold with boundary X_Γ.

One of the amazing facts in this area is that the ADE algebras correspond to the finite subgroups of $SU(2)$:

- $A_k \to$ cyclic of order $k + 1$;

- $D_k \to$ Dihedral of order $2k$;
- $E_6 \to$ binary tetrahedron, $E_7 \to$ binary octahedron, $E_8 \to$ binary icosahedron.

The boundary ∂X_Γ is a closed 3-manifold which can be obtained as the quotient S^3/G where G is the group corresponding to Γ. (Recall that $SU(2)$ can be identified with S^3 and is the double cover of the rotation group $SO(3)$.)

The simplest case is A_1 where there is just one vertex. Then X_Γ is a tubular neighbourhood of S^2 in T^*S^2. Then ∂X_Γ is $S^3/\pm 1 = \mathbf{RP}^3 = SO(3)$.

In the case of E_8, the group G is perfect and ∂X_Γ is a *homology sphere* discovered by Poincaré. If we add a cone on ∂X_Γ to X_Γ we get a simply connected 4-dimensional space Z which is not a manifold but is a "homology manifold". The intersection form of Z is the E_8 quadratic form.

Rohlin's Theorem, from the 1950's, asserts that a simply-connected, smooth, closed 4-manifold with "even" intersection form has signature divisible by 16.

The E_8 form is even with signature -8, so there is no simply-connnected smooth closed 4-manifold with this intersection form.

The Pontrayagin classes and examples in dimension $7, 8$.

Let $V \to S^4$ be a rank 4 Euclidean oriented vector bundle, so the structure group of V is $SO(4)$. The unit sphere bundle of V is a simply connected closed 7-manifold Y. It is the boundary of the unit-ball bundle M which contains the zero section S^4.

Such bundles V are classified by $\pi_3(SO(4))$. Recall that the double cover of $SO(4)$ is the product $S^3 \times S^3$ so $\pi_3(SO(4) = \mathbf{Z} \oplus \mathbf{Z}$ and our bundle is specified by a pair of integers n_+, n_-. The self-intersection number d of S^4 in M is $d = n_+ - n_-$. The Pontrayagin class $p_1(M)$ evaluated on S^4 is $q = 2(n_+ + n_-)$.

The spectral sequence of the fibration $S^3 \to Y \to S^4$ shows that $H^4(Y, \mathbf{Z}) = \mathbf{Z}/d$.

If $d = 0$ we get a family of 7-manifolds Y_q, for $q > 0$ a multiple of 4, with the same cohomology ring as $S^3 \times S^4$ but of distinct diffeomorphism types since $p_1(Y_q)$ is q times the generator of H^4.

If $d = 1$ the manifold Y is a homotopy sphere. By the higher dimensional Poincare conjecture (discussed below) it is homeomorphic to S^7. Milnor showed that for certain values of n_+, n_- the manifold Y is an *exotic sphere*, not diffeomorphic to S^7. Moreover, if we add a cone on Y to M we get a topological 8-manifold which does not admit a smooth structure.

PART 2: Cobordism, surgery and the Whitney trick

Oriented cobordism is an equivalence relation on closed, oriented n-dimensional manifolds. $M_0 \sim M_1$ if there is an oriented $(n+1)$-manifold with oriented boundary $\overline{M}_0 \sqcup M_1$. The equivalence classes form a group Ω_n under the operation of disjoint union.

The Pontrayagin numbers of a closed oriented $4k$-manifold M are defined by evaluating polynomials in the Pontrayaguin classes on the fundamental class $[M] \in H_{4k}$. So when $k = 1$ we have p_1, for $k = 2$ we have p_1^2, p_2, for $k = 3$ we have $p_1^3, p_2 p_1, p_3$ etc. These numbers are cobordism invariants.

The signature $\sigma(M)$ of a $4k$-manifold M is also a cobordism invariant. The Hirzebruch Signature Theorem expresses it as a rational combination of Pontrayagin numbers. For example $\sigma(M^4) = p_1/3$, $\sigma(M^8) = (7/45)p_2 - (1/45)p_1^2$. The fact that the coefficients are rational numbers while the signature is an integer has important consequences (in a similar vein to Rohlin's Theorem).

A fundamental motivation for the notion of cobordism comes from considering families of equations. For example, suppose that $F : X \to T$ is a smooth map and for $t \in T$ let $Z_t \subset X$ be the set of solutions of the equation $F(x) = t$. For generic t the solution set Z_t is a manifold of dimension $n = \dim X - \dim T$. For different generic values t_0, t_1 the manifolds Z_{t_0}, Z_{t_1} may not be diffeomorphic but they are cobordant. To see this, join t_0, t_1 by a path $\gamma : [0, 1] \to T$ and let $W \subset X \times [0, 1]$ be the set

$$W = \{(x, s) : F(x) = \gamma(s).$$

For a generic path γ this will be a manifold of dimension $n + 1$ giving a cobordism from Z_{t_0} to Z_{t_1}. A similar discussion applies to other families of equations, such as zero sets of sections of vector bundles. The ideas can also be extended to certain infinite dimensional problems, for example moduli spaces of holomorphic curves.

Consider, for example, hypersurfaces $X \subset \mathbf{RP}^5$ defined by homogeneous polynomials of degree 4. The polynomial $x_0^4 - (x_1^4 + x_2^4 + x_3^4 + x_4^4)$ gives a 4-sphere. For different polynomials we get many other 4-manifolds but we can never get $\mathbf{CP}^2$, since it not cobordant to 0.

The equivalence relation of cobordism is generated by standard operations called *surgeries*. Let M be an n-dimensional manifold and $\Sigma \subset M$ be an embedded p-sphere with trivial normal bundle. So the boundary of a tubular neighbourhood N of P is a copy of $S^p \times S^{n-p-1}$. The manifold $N' = B^{p+1} \times S^{n-p-1}$ has the same boundary and the operation of surgery is to cut out N and replace it by N', giving a new closed manifold

$$M' = (M \setminus N) \cup_{\partial N'} N'.$$

More precisely, we have to fix a diffeomorohism $\phi : \partial N' \to \partial N$.

There is a standard cobordism from M to M' defined by adding a "handle" $H = B^{p+1} \times B^{n-p}$. We have

$$\partial(B^{p+1} \times B^{n-p}) = (S^p \times B^{n-p}) \cup (B^{p+1} \times S^{n-p-1}).$$

Take a diffeomorphism $\Phi : H \to N$ and define $W_\Sigma = M \cup_\Phi H$. After smoothing out the "corner" along ∂N this gives a cobordism from M to M'.

Take $\mathbf{R}^{n+1}$ with coordinates $x_1, \dots x_q, y_1, \dots y_{p+1}$ where $p+q = n$. Let $f : \mathbf{R}^{n+1} \to \mathbf{R}$ be the quadratic function $|x|^2 - |y|^2$. Then $f^{-1}(-1)$ contains a sphere $\Sigma = S^p$ and $f^{-1}(1)$ contains a sphere S^{q-1}. The manifold $f^{-1}(1)$ is obtained from $f^{-1}(-1)$ by surgery along Σ and $W_\Sigma = f^{-1}[-1, 1]$.

For a general cobordism W from M_0 to M_1 we choose a Morse function $F : W \to [0, 1]$ equal to 0 on M_0 and 1 on M_1. There are a funite number of critical values of F in

$[0,1]$. The level set $F^{-1}(t)$ changes by a surgery as t crosses a critical value and this gives a "factorisation" of the cobordism into a composite of surgeries. The number $p+1$ of negative directions is the *index* of the critical point.

Dimension 3

Let $K \subset S^3$ be an embedded circle: a knot. The boundary of tubular neighbourhood N is a torus T. There is a well-defined meridian μ in $H_1(T)$ which bounds a disc in N. To specify a surgery we need to define a class $\gamma \in H_1(T)$ with $\gamma.\mu = 1$ (which will bound in the solid torus N' which we glue in). There is a unique longitude class $\lambda \in H_1(T)$ with $\lambda.mu = 1$ and such that λ maps to zero in $H_1(S^3 \setminus N)$ so we can take $\gamma = \lambda + c\mu$ for any integer *surgery coefficient* c.

Now let $L \subset S^3$ be a link with components K_i. Perform surgery on each component with coefficents c_i to get a 3-manifold Y which is the boundary of a 4-manifold X. Then $H_2(X)$ has a basis σ_i in which the intersection form has diagonal entries the c_i and off-diagonal entries the linking numbers of K_i, K_j.

The fact that the cobordism group Ω_3 is zero implies that any compact 3-manifold is obtained in this way.

We get another description of the ALE manifolds X_Γ by taking a link with unknotted components, linking matrix corresponding to the graph Γ and all coefficents -2.

The Whitney trick

Suppose that P, Q are connected submanifolds of dimensions p, q in a simply connected $(p+q)$-dimensional manifold M, intersecting transversally in a finite set of points. Suppose that $p, q \geq 3$. If two intersection points a, b have opposite signs then there is an isotopy of $P \subset M$ to P' which reduces the number of intersection points by 2.

Choose paths in P and Q between a and b. Since M is simply connected the union of these paths bounds a disc D in M. Since $\dim M > 4$ a generic disc is embedded and since $p, q \geq 3$ does not meet P or Q in its interior. By choosing a suitable framing of the normal bundle of $D \subset M$ compatible with the given data on the boundary one finds that a neighbourhood of D in M is standard and one can write down the isotopy which cancels the intersection points.

This the basic fact that distiguishes high dimensional and low-dimensional manifold topology. In dimension 4 with $p = q = 2$ the argument fails (and the conclusion is false). The disc could meet P, Q and have self-intersection points. There is also a framing problem.

(In dimension 3 there are famous results (Dehn's Lemma, The loop theorem) in a similar spirit, but proved in an entirely different way.)

The h-cobordism theorem

Suppose that M_0, M_1 are simply connected manifolds of dimension $n \geq 5$ and W is a cobordism from M_0 to M_1. Suppose that W is simply connected and that the inclusion maps $M_0, M_1 \subset W$ induce isomorphisms on homology. Then M_0, M_1 are diffeomorphic and $W = M_0 \times [0,1]$.

Choose a Morse function f_0 to represent W as a composite of surgeries. If there are no critical points the conclusion follows so the problem is to adjust f_0 to remove critical points. To give the main idea, suppose $n = 6$ and there are just two critical points $u, v \in W$ of f_0. The hypothesis on homology implies that u, v have index-difference 1. Let's suppose that u has index 3, v has index 4 and $f_0(u) = 1/4, f_0(v) = 3/4$. Write $M = f_0^{-1}(1/2)$ .There are a pair of 3-spheres $P_u, P_v \subset M$ so that M_0 is obtained from M by surgery along P_u and M_1 is obtained from M by surgery along P_v. The hypothesis on homology implies that the intersection number of P_u, P_v is ± 1. Using the Whitney trick we can suppose that P_u, P_v meet transversally in just one point. Then a neighbourhood of $P_u \cup P_v \subset M$ is standard and one can write down a deformation of f_0 which cancels the critical points.

One application is the higher dimensional topological Poincare conjecture: for $n \geq 5$ a manifold M^n homotopy equivalent to S^n is homeomorphic to S^n.

The theorem also leads to classification results for manifolds with simple topolology. For example any 2-connected 6 manifold is diffeomorphic to a connected sume of copies of $S^3 \times S^3$.

Exercises

1. Let F_r be the free group on r generators and $\gamma \in F_r$. For $n \geq 4$ use surgery on a connected sum of r copies of $S^1 \times S^{n-1}$ to construct an n-dimensional manifold with fundamental group $F_r / < \gamma >$. Why does this not work if $n = 3$?

2. Show that $\mathbf{RP}^2 \sharp \mathbf{RP}^2 \sharp \mathbf{RP}^2$ and $T^2 \ \sharp \mathbf{RP}^2$ are homeomorphic.

Let $\overline{\mathbf{CP}}^2$ denote $\mathbf{CP}^2$ with reversed orientation. Show that no two of

$$S^2 \times S^2, \mathbf{CP}^2 \sharp \mathbf{CP}^2, \mathbf{CP}^2 \sharp \overline{\mathbf{CP}}^2$$

are homeomorphic but $S^2 \times S^2 \ \sharp \overline{\mathbf{CP}}^2$ is homeomorphic to $\mathbf{CP}^2 \sharp \overline{\mathbf{CP}}^2 \sharp \overline{\mathbf{CP}}^2$.

(Complex geometry may be helpful here. Blowing up a point on complex surface correponds to connected sum with $\overline{\mathbf{CP}}^2$.)

3. Let X be the 4-manifold corresponding to the A_2 graph (two vertices joined by one edge). Show that $\pi_1(X) = \mathbf{Z}/3$.

4. Let W be an oriented $(2n+1)$ manifold with boundary M. There is a boundary map

$$\partial : H_{n+1}(W, M) \to H_n(M).$$

Use duality and the long exact sequence in cohomology to show that the image of ∂ is an isotropic subspace in $H_n(M)$ with respect to the intersection form (i.e. $\partial a . \partial b = 0$ for all a, b) of dimension $\frac{1}{2}\mathrm{dim} H_n(M)$. In the case when n is even, deduce that the signature of M is zero.

For an oriented surface Σ of genus g denote $\lambda^i(\Sigma) = \Lambda^{g+i}(H^1(\Sigma))$. Let Σ_0, Σ_1 be oriented surfaces of genus g_0, g_1 respectively and let W be a cobordism from Σ_0 to Σ_1. Show that W defines, up to an overall factor, linear maps

$$\lambda^i_W : \lambda^i(\Sigma_0) \to \lambda^i(\Sigma_1).$$

Investigate how your construction behaves with respect to composition of cobordisms.

(Hint: A a p-dimensional subspace of a vector space V defines an element in $\Lambda^p(V)$, up to a factor.)

5. Show that $\mathbf{CP}^n$ is not a boundary when n is even. Construct manifolds W_m with $\partial W_n = \mathbf{CP}^{2m+1}$.

(Hint: consider a map from $\mathbf{CP}^{2m+1}$ to quaternionic projective space $\mathbf{HP}^m$.)

6. The Hopf invariant of a smooth map $f : S^{4k-1} \to S^{2k}$ can be defined as follows. Choose a k-form ω on S^{2k} of integral 1. Then choose a $(k-1)$-form α on S^{4k-1} such that $f^*(\omega) = d\alpha$. The Hopf invariant is

$$H(f) = \int_{S^{4k-1}} \alpha \wedge f^*(\omega).$$

Show that this is well-defined, independent of choices of ω, α.

Let X be a closed manifold of dimension $4k$ which has a decomposition $X = B^{4k} \cup N$ where N is a tubular neighbourhood of a $2k$-sphere $\Sigma \subset X$ with $\Sigma.\Sigma = 1$. So ∂N is a $(4k-1)$-sphere. If Ω is a closed $2k$-form on X with integral 1 overΣ show that

$$\int_X \Omega^2 = 1.$$

By constructing a suitable form Ω, show that the Hopf invariant of the map $S^{4k-1} = \partial N \to S^{2k}$ is 1.

7. Let M be the 8-manifold constructed in Part 1, from a vector bundle $V \to S^4$, with $d = n_+ - n_- = 1$ and $Y = \partial M$. Let Z be the space obtained by adding a cone over Y to M. Suppose that Y is diffeomorphic to S^7 so that Z is a smooth 8-manifold. use the signature theorem to show that $p_2 = (1/7)(45 + q^2)$ where $q = 2(n_+ + n_-)$. Hence derive a contradiction to the assumption that Y is diffeomorphic to S^7, for certain values of n_+, n_-.

8. Use Alexander Duality to show that a knot $K \subset S^3$ bounds an oriented surface Σ, embedded in $S^3 \setminus K$ (a *Seifert surface*). The surface Σ is homeomorphic to closed surface of some genus g, minus a disc. Now perform surgery on K with coefficient $c > 0$ to construct a 3-manifold Y which is the boundary of a 4-manifold X. Show that

- $H^2(X) = \mathbf{Z}$ and a generator is represented by an embedded surface in X of genus g and self-interection c.
- $H^1(Y) = \mathbf{Z}/c$.

(For the first part, recall that $H^1(S^3 \setminus K)$ can be identified with homotopyclasses of maps from $S^3 \setminus K$ to S^1.)