Gauss' Theorem Egregium, Gauss-Bonnet etc.

We know that for a simple closed curve in the plane

$$\int k\,ds = 2\pi.$$

Now we want to consider a simple closed curve C in a surface $S \subset \mathbf{R}^3$. We suppose C is the boundary of a set $Y \subset S$ homeomorphic to a disc. The local Gauss-Bonnet formula is:

$$\int_C k_g\,ds = 2\pi - \int_Y K\ dA,$$

where K is the Gauss curvature.

Set-up: we use standard polar co-ordinates (r,θ) in the plane. We suppose we have a local parametrisation $\mathbf{f}$ of S mapping the unit disc to Y and the unit circle to C. Then we may regard r,θ as local co-ordinates on S, in the obvious way. For $r \le 1$ we let C_r be the closed curve in S which is the image under $\mathbf{f}$ of the circle of radius r.. We set

$$I(r) = \int_{C_r} k_g\,ds,$$

where k_g is the geodesic curvature of C_r (with a suitable sign convention). We let $Y_r \subset S$ be the image of the disc of radius r. We compute

$$\frac{d}{dr}\left(I(r) + \int_{Y_r} K\ dA\right).$$

We will use two simple algebraic lemmas in the proof. Let $P \subset \mathbf{R}^3$ be a plane through the origin and N be a unit normal to P. For $x,y \in P$ set

$$x \wedge y = (x \times y).N$$

Then

Lemma 1 For any three vectors x,y,z in P:

$$(x \wedge y)z + (y \wedge z)x + (z \wedge x)y = 0.$$

Lemma 2 If $L : P \to P$ is a linear map then

$$L(x) \wedge L(y) = \det(L)(x \wedge y).$$

Proofs are exercises for the reader.

Let $\mathbf{t}$ be the tangent to the curves C_ρ and $\mathbf{n}$ be the normal to S. Thus $\mathbf{t}, \mathbf{n}$ are vector-valued functions of r,θ with $\mathbf{t}.\mathbf{n} = 0$ everywhere. We use suffixes $\mathbf{t}_\rho, \mathbf{t}_\theta$ etc. to denote partial derivatives with respect to ρ,θ. Then

$$I(r) = \int_{C_r} k_g\,ds = \int_0^{2\pi} (\mathbf{t}_\theta \times \mathbf{t}).\mathbf{n}\,d\theta.$$

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Thus

$$\frac{dI}{dr} = \frac{\partial}{\partial r} \left[(\mathbf{t}_\theta \times \mathbf{t}).\mathbf{n} \right] d\theta.$$

Now consider

$$S = \frac{\partial}{\partial r} \left[(\mathbf{t}_\theta \times \mathbf{t}).n \right] - \frac{\partial}{\partial \theta} \left[(\mathbf{t}_r \times \mathbf{t}).\mathbf{n} \right].$$

Then

$$S = \left[(\mathbf{t}_\theta \times \mathbf{t}_r).\mathbf{n} + (\mathbf{t}_{\theta r} \times \mathbf{t}).\mathbf{n} + (\mathbf{t}_\theta \times \mathbf{t}).\mathbf{n}_r \right] - \left[(\mathbf{t}_r \times \mathbf{t}_\theta)bn + (\mathbf{t}_{r\theta} \times \mathbf{t}).\mathbf{n} + (\mathbf{t}_r \times \mathbf{t}).\mathbf{n}_\theta \right].$$

Using the symmetry of second partial derivatives and the skew-symmetry of the cross-product, this is

$$S = 2(\mathbf{t}_\theta \times \mathbf{t}_r).\mathbf{n} + (\mathbf{t}_\theta \times \mathbf{t}).n_r - (\mathbf{t}_r \times \mathbf{t}).\mathbf{n}_\theta.$$

Now $\mathbf{t}$ is a unit vector, so the vectors $\mathbf{t}_r, \mathbf{t}_\theta$ are orthogonal to $\mathbf{t}$, as is the normal $\mathbf{n}$. Thus these three vectors lie in a plane and the triple product $(\mathbf{t}_\theta \times \mathbf{t}_r).\mathbf{n}$ is zero. So

$$S = (\mathbf{t}_\theta \times \mathbf{t}).\mathbf{n}_r - (\mathbf{t}_r \times \mathbf{t}).\mathbf{n}_\theta.$$

By the cyclic symmetry of the triple product this is

$$S = \mathbf{t}_\theta.(\mathbf{t} \times \mathbf{n}_r) - \mathbf{t}_r.(\mathbf{t} \times \mathbf{n}_\theta).$$

Since the three vectors $\mathbf{t}, \mathbf{n}_r, \mathbf{n}_\theta$ are all orthogonal to $\mathbf{n}$ this can be written as

$$S = (\mathbf{t}_\theta.\mathbf{n})\mathbf{n}.(\mathbf{t} \times \mathbf{n}_r) - (\mathbf{t}_r.\mathbf{n})\mathbf{n}.(\mathbf{t} \times \mathbf{n}_\theta).$$

In the notation above this is

$$S = (\mathbf{t}_\theta.\mathbf{n})(\mathbf{t} \wedge \mathbf{n}_r) - (\mathbf{t}_r.\mathbf{n})(\mathbf{t} \wedge \mathbf{n}_\theta).$$

Then we can write

$$S = \mathbf{t}. \left[(\mathbf{t} \wedge \mathbf{n}_\theta)\mathbf{n}_r - (\mathbf{t} \wedge \mathbf{n}_r)\mathbf{n}_\theta \right].$$

By Lemma 1 this is

$$S = -\mathbf{t}. \left[(\mathbf{n}_\theta \wedge \mathbf{n}_r)\mathbf{t} \right],$$

and hence

$$S = -(\mathbf{n}_\theta \wedge \mathbf{n}_r).$$

Now let $\mathbf{f}_r, \mathbf{f}_\theta$ be the usual basis vectors for the tangent space to S corresponding to the local co-ordinates r, θ. Then $\mathbf{n}_r = \mathcal{L}(\mathbf{f}_r), \mathbf{n}_\theta = \mathcal{L}(\mathbf{f}_\theta)$ where $\mathcal{L} : TS \to TS$ is the linear map introduced in lectures (the derivative of the Gauss map). Lemma 2 gives

$$S = -\det(\mathcal{L})(\mathbf{f}_r \wedge \mathbf{f}_\theta).$$

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But, by definition, $\det(\mathcal{L})$ is the Gauss curvature K. So we have

$$S = -K\mathbf{f}_r \wedge \mathbf{f}_\theta.$$

Now go back to the definition of S and integrate around the circle. The integral of the $\frac{\partial}{\partial\theta}$ term vanishes, so we get

$$\frac{dI}{dr} = -\int_0^{2\pi} K(\mathbf{f}_r \wedge \mathbf{f}_\theta)d\theta.$$

Let E, F, G be the components of the first fundamental form in the r, θ coordinates, as usual. Then

$$\frac{dI}{dr} = -\int_0^{2\pi} \sqrt{EG - F^2}d\theta.$$

But

$$\int_{Y_r} KdA = \int_0^r \int_0^{2\pi} K\sqrt{EG - F^2}d\rho d\theta,$$

so

$$\frac{d}{dr}\int_{Y_r} KdA = \int_0^{2\pi} K\sqrt{EG - F^2}d\theta.$$

Hence the two derivatives cancel and we conclude that

$$I(r) + \int_{Y_r} KdA$$

is a constant, independent of r.

Finally, it is not hard to see that the limit of $I(r)$ as r tends to zero is 2π; since the geometry of the surface approximates that of the plane over very small regions. Thus the constant above must be 2π and we have proved the local version of Gauss-Bonnet.

Consequences

1. *Gauss' Theorem Egregium*: the Gauss curvature K depends only on the first fundamental form of the surface. For we can write

$$K(p) = \lim \frac{1}{\text{Area}(Y_r)}(2\pi - \int_{C_r} k_g ds),$$

where C_r is any family of circles shrinking down to p, and the RHS depends only on the first fundamental form (using our results on geodesic curvature). In fact we could there are many formulae of a similar nature that we can use to give an intrinsic characterisation of the Gauss curvature. For example let $B(p, r)$ be

the set of points of distance (measured in S) less than r from p. Then one can show that the area of $B(p,r)$ is

$$\text{Area}(B(p,r)) = \pi r^2 - \frac{K(p)}{12}\pi r^4 + O(r^6).$$

The Gauss curvature is a measure of the deviation of the geometry in S from that of the plane. One can also find an explicit formula for K in terms of the co-efficients E, F, G in a local parametrisation

$$K = \frac{1}{2J}\left[\frac{\partial}{\partial u}\left(\frac{EG_u - FE_v}{JE}\right) - \frac{\partial}{\partial v}\left(\frac{2EF_u - FE_u - EE_v}{JE}\right)\right],$$

where $J = \sqrt{EG - F^2}$. The formula becomes much simpler in a co-ordinate system chosen so that the first fundamental form has a convenient shape. For example in "isothermal co-ordinates" where $F = 0$ and $E = G = e^f$ say,

$$K = -\frac{e^{-f}}{2}(f_{uu} + f_{vv}).$$

2.*Polygons.* Now let $Y \subset S$ be a polygonal region, homeomorphic to the disc, whose boundary is made up of a number of smooth curve segments Γ_i meeting at corners. We can approximate the boundary of Y by smooth curves and arrive at the formula

$$\int_Y K dA = 2\pi - \sum \int_{\Gamma_i} k_g ds - \sum(\pi - \phi_j),$$

where ϕ_j is the internal angle at corner j. In particular we find that the sum of the angles of a geodesic triangle T in S is

$$\pi + \int_T K dA.$$

Especially interesting cases are the sphere, with $K = 1$ and the pseudosphere/upper half plane with $K = -1$.

3. *The Global Gauss-Bonnet formula.* Let S be a compact surface and suppose that S has a *triangulation*. Applying the formula above to each triangle and summing, using the fact that the sum of the angles at each vertex is 2π one finds that

$$\int_S K dA = \chi(S),$$

where $\chi(S)$ is the Euler characteristic of the triangulation

$$\chi(S) = V - E + F$$

where V is the number of vertices, E the number of edges and F the number of faces. This shows that the LHS does not depend on the particular way the surface is embedded in $\mathbf{R}^3$ and the RHS does not depend on the triangulation: it is a *topological* invariant of the surface.

4. *Classification of flat surfaces* Let S be a surface which is locally isometric to the plane. Gauss' theorem shows that $K = 0$. This means that the linear map $\mathcal{L}$ has determinant zero at each point of S. Thus for each p in S there is a non-trivial kernel of $\mathcal{L}$. If this kernel is the whole tangent space, at each point of S, the second fundamental form is zero and it is easy to show that S is contained in a *plane*. Suppose on the other hand that this kernel has dimension 1 (that is, the second fundamental from is not zero). We can choose local co-ordinates (u, v) so that this kernel is spanned by $\mathbf{f}_u$. Then if $\mathbf{n}$ is the unit normal we have $\mathbf{n}_u = 0$. By the symmetry of the second fundamental form $\mathbf{n}_v.\mathbf{f}_u = \mathbf{n}_u.\mathbf{f}_v = 0$. So $\mathbf{n}, \mathbf{n}_v$ are vectors orthogonal to $\mathbf{f}_u$ and

$$\mathbf{f}_u = \lambda \mathbf{n} \times \mathbf{n}_v,$$

for some function λ. But now $\mathbf{n}_{vu} = \mathbf{n}_{uv} = 0$ so $\mathbf{n}$ and $\mathbf{n}_v$ are both independent of u. This means that the *direction* of $\mathbf{f}_u$ does not vary with u. Thus the curves $v = $ constant are a family of *straight line* segments in S. Now choose some curve $\sigma(t)$ in S which meets each of these segments once (for example a curve $u = $ constant). Let $X(t)$ be a vector pointing along the line segment through $\sigma(t)$. Then we have a new parametrisation

$$\mathbf{g}(r, t) = \sigma(t) + rX(t)$$

of S. The condition that the normal be independent of r tells us that X' must lie in the plane spanned by σ' and X, so

$$X' = \alpha X + \beta \sigma',$$

for functions $\alpha(t), \beta(t)$. If β vanishes everywhere then the direction of X is constant and S is a *cylinder*. Suppose on the other hand that β does not vanish anywhere. Then put $\gamma(t) = \sigma(t) - f(t)X$ where $f = \beta^{-1}$. We have

$$\gamma' = \sigma' - f'X - f(\alpha X + \beta \sigma') = -(f' + f\alpha)X.$$

Suppose $f' + f\alpha$ vanishes everywhere. Then γ is constant and all the line segments pass through a fixed point: the surface S is a *cone*. Suppose on the other hand that $f' + f\alpha$ does not vanish anywhere. Then X is a multiple of γ' and the surface is the *developable surface associated to the curve* γ. Thus we see that the three types of flat surfaces we studied essentially include all the possibilities. (This is slightly imprecise because one ought to discuss what happens when the various functions vanish at some points but not everywhere.)