The Atiyah-Singer Index Theorem: Notes and Exercises

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1 The index problem

A bounded linear map $T : V_1 \to V_2$ between Banach spaces is called *Fredholm* if the kernel is finite-dimensional and the image of T is a closed subspace of finite codimension. The index of such a map T is the integer

$$\text{ind } T = \text{dim ker } T - \text{dim coker} T.$$

In the case when V_1, V_2 are finite dimensional this is just dim $V_1 - \text{dim} V_2$; in particular it does not depend on T. In general, the space of Fredholm operators is open (in the operator norm topology) and the index is constant on each connected component.

Now let M be a compact manifold and E, F complex vector bundles over M. Let $D : \Gamma(E) \to \Gamma(F)$ be a linear differential operator of order l. We can make D into a bounded linear operator by introducing suitable function spaces, for example

$$D : L^2_l(E) \to L^2(F),$$

where L^2_l is the Sobolev space with norm defined by sum of the L^2 norms of derivatives up to order l. The *symbol* σ_D of the operator encodes the highest order terms. It is a section of the vector bundle $s^l TM \otimes \text{Hom}(E, F)$. From another point of view, for each $p \in M$ and cotangent vector $\xi \in T^* M_p$ we have a $\sigma_D(\xi) \in \text{Hom}(E_p, F_p)$ and this is a polynomial function of degree l in the variable $\xi \in T^* M_p$.

One way to define $\sigma_D(\xi)$ is to take a function f with $f(p) = 0$ and $df_p = \xi$. Then for a section s of E the value of $D(f^l s)$ at p depends only on the value of s at p and we can define

$$\sigma_D(\xi)(s(p)) = D(f^l s))(p). \tag{1}$$

For example, the Laplace operator Δ on a Riemannian manifold is elliptic of order 2 with symbol $-|\xi|^2$.

A basic fact of global analysis is that an elliptic operator D over a compact manifold defines a Fredholm operator between Sobolev spaces. Moreover

- ker D consists of smooth sections of E;
- coker D can be identified with the kernel of the formal adjoint operator $D^* : \Gamma(F) \to \Gamma(E)$.

(This means that the ensuing discussion is independent of the choice of function spaces used to set things up.)

(Recall that the formal adjoint is defined by integration-by-parts through the identity

$$\langle Df, \rho\rangle_{L^2} = \langle f, D^*\rho\rangle_{L^2},$$

where $\langle\ ,\ \rangle_{L^2}$ is the L^2 inner product.)

Thus the elliptic operator D has a index, which is independent of deformations through elliptic operators. If D is defined using a metric, connections etc. then the index does not depend on those choices. Also, two operators with the same symbol have the same index.

The index problem is to give a formula for ind D in terms of topological invariants of the data (M, E, F, σ_D).

Examples

- The Laplace operator is self-adjoint so has index zero.
- If M is a compact Riemann surface there is a $\overline{\partial}$-operator

  $$\overline{\partial} : \Omega^0 \to \Omega^{0,1}.$$

  More generally if $L \to M$ is a holomorphic line bundler we have

  $$\overline{\partial}_L : \Omega^0(L) \to \Omega^{0,1}(L).$$

  This is an elliptic operator (with $E = L$ and $F = L \otimes \overline{T}^*\Sigma$). The index is given by the Riemann-Roch formula

  $$\text{ind } \overline{\partial}_L = \dim\, H^0(L) - \dim H^1(L) = d - g + 1,$$

  where $d = \langle c_1(L), M\rangle$ and g is the genus of M, so we also have $2g - 2 = \langle c_1(T^*M), M\rangle$.
- Choose a Riemannian metric on M and let d^* be the formal adjoint of the exterior derivative d. Let E be the sum of the even degree differential forms and F the sum of the odd degree forms. Then we can write

  $$d + d^* : \Omega^{\text{even}} \to \Omega^{\text{odd}}.$$

This is elliptic and by Hodge theory the index is the Euler characteristic $\chi(M)$. If $\dim M$ is odd this is zero. When $\dim M = 2m$ and (for simplicity) M is oriented we have the "Gauss-Bonnet" formula

$$\chi(M) = \langle e(M), M \rangle,$$

where $e(M) \in H^{2m}(M)$ is the Euler class.

- Let
$$\Gamma(E_0) \xrightarrow{\partial} \Gamma(E_1) \xrightarrow{\partial} \Gamma(E_2) \dots,$$
be a complex of first order differential operators (so $\partial^2 = 0$). The complex is called elliptic if the sequence of symbols is exact, for non-zero ξ. This is equivalent to the ellipticity of the operator
$$\partial + \partial^* : \bigoplus \Gamma(E_{\text{even}}) \to \bigoplus \Gamma(E_{\text{odd}}).$$

- Suppose now that M is oriented of dimension $4k$. Then we have $* : \Lambda^{2k} \to \Lambda^{2k}$ with $*^2 = 1$ and the $2k$-forms split into self-dual and anti-self-dual parts $\Lambda^{2k}_+ \oplus \Lambda^{2k}_-$. Let E be the sum of the Λ^{2k}_+, the even forms of degree less than $2k$ and the odd forms of degree greater than $2k$ and let F be the sum of Λ^{2k}_- with the odd forms of degree less than $2k$ and the even forms of degree greater than $2k$. Define $D_{\text{sign}} : \Gamma(E) \to \Gamma(F)$ using d, d^* except that for $\Omega^{2k+1} \to \Omega^{2k}_-$ we compose with the projection $\Omega^{2k} \to \Omega^{2k}_-$. Then by Hodge Theory the index of D_{sign} is the *signature* of M—the signature of the quadratic cup product form on $H^{2k}(M)$. The Hirzebruch signature theorem asserts that
$$\text{sign}(M) = \langle L_k(p_1, \dots), M \rangle,$$
where L_k is a certain polynomial in the Pontrayagin classes $p_i(M) \in H^{4i}(M)$.

2 The Atiyah-Singer index formula for Dirac operators

The double cover $\text{Spin}(n)$ of $SO(n)$ has a representation on a complex vector space S. If n is even this is a sum $S^+ \oplus S^-$. There is Clifford multiplication map of representations:

$$S \otimes \mathbf{R}^n \to S.$$

A spin structure on an oriented Riemannian n-manifold M is a principle $\text{Spin}(n)$ bundle which is a double cover of the frame bundle. Given such there is an associated vector bundle $S \to M$ with bundle map $c : S \otimes T^*M \to S$. The Levi-Civita connection defines a covariant derivative $\nabla : \Gamma(S) \to \Gamma(S \otimes T^*M)$. The Dirac operator is

$$D = c \circ \nabla : \Gamma(S) \to \Gamma(S).$$

This is a self-adjoint operator. For n even, so $S = S^+ \oplus S^-$, we have $D = D^+ + D^-$ where $D^\pm : \Gamma(S^\pm) \to \Gamma(S^\mp)$.

Example

For $n = 2$ we can regard our Riemannian manifold also as a Riemann surface. A spin structure is equivalent to a square root $K^{1/2}$ of the canonical bundle $K = T^*M$. The Dirac operator D^+ can be identified with $\overline{\partial}_L : \Omega^0(L) \to \Omega^{0,1}(L)$ for the line bundle $L = K^{1/2}$.

To get a picture of the Dirac operator in higher dimensions we can use an inductive procedure. Suppose that we have defined $D_{2m} = D^+ + D^-$ over $\mathbf{R}^{2m}$. Then we define the Dirac operator D_{2m+1} over $\mathbf{R}^{2m+1} = \mathbf{R}^{2m} \times \mathbf{R}$ to be

$$\begin{pmatrix} i\partial_t & D^- \\ D^+ & -i\partial_t \end{pmatrix},$$

where t is the coordinate in the $\mathbf{R}$ factor in $\mathbf{R}^{2m} \times \mathbf{R}$. If we have defined D_{2m+1} over $\mathbf{R}^{2m+1}$ we define a pair of operators D^+, D^- over $\mathbf{R}^{2m+2} = \mathbf{R}^{2m+1} \times \mathbf{R}$ by

$$D^+ = \partial_s + D_{2m+1} \quad , \quad D^- = -\partial_s + D_{2m+1}.$$

where s is the coordinate in the $\mathbf{R}$ factor in $\mathbf{R}^{2m+1} \times \mathbf{R}$. The process can begin in dimension 1 where the Dirac operator is $i\frac{d}{dx}$.

Now let M be a compact Riemannian manifold of even dimension $2m$ with a spin structure and let $V \to M$ a complex vector bundle of rank r. Choosing a connection on V we get a coupled Dirac operator

$$D_{+,V} : \Gamma(S^+ \otimes V) \to \Gamma(S^- \otimes V).$$

This is an elliptic operator, so has an index. The Atiyah-Singer formula in this case is

$$\text{ind } D_{+,V} = \langle \text{ch}(V)\hat{A}(M), M \rangle, \tag{2}$$

where ch $(V), \hat{A}(M)$ are polynomials in, respectively, the Chern classes of V and the Pontrayagin classes of M, defined as follows.

- ch (V): Take formal variables $\lambda_1, \lambda_2, \ldots, \lambda_m$ and write $\sum e^{\lambda_i}$ in terms of the elementary symmetric functions

$$\sigma_0 = 1 \ , \ \sigma_1 = \sum \lambda_i \ , \ \sigma_2 = \sum \lambda_i \lambda_j, \ldots .$$

  Now replace the σ_i by the Chern classes $c_i(V)$. The resulting formula begins

$$\text{ch } (V) = r + c_1 + (\frac{1}{2}c_1^2 - c_2) + \ldots$$

- $\hat{A}(M)$: Take formal variables μ_j and let $\hat{a}$ be the function

$$\hat{a}(z) = \left(\frac{z/2}{\sinh(z/2)} \right)^{1/2}.$$

Consider

$$\prod_j \hat{a}(\mu_j)^2.$$

Since $\hat{a}(z)$ is an even function of z this product can be written as a power series in the elementary symmetric functions of μ_j^2. Now substitute the Pontrayagin class $p_i(M)$ for the $i'th$ symmetric function of the μ_j^2. The resulting formula begins

$$\hat{A}(M) = 1 - \frac{1}{24}p_1 + \left(\frac{-4p_2 + 7p_1^2}{5760}\right) + \ dots.$$

The Chern-Weil theory represents the Chern classes and Pontrayagin classes by differential forms obtained from the Riemannian curvature R_M of M and the curvature F_V of the connection on V. The index formula (1) becomes

$$\text{index} = \int_M \operatorname{Tr}\exp(iF_V/2\pi) \ \det \ \hat{a}(R_M/2\pi). \tag{3}$$

Here the expressions are defined by considering F_V and R_M as 2-forms with values in the endomorphisms of V and TM respectively.

The index formula for Dirac operators is important because:

1. In the general theory, the calculation for any elliptic operator can be reduced to the case of Dirac operators.

2. Most of the natural operators that arise in geometry can be expressed as coupled Dirac operators.

We say something more about the first item in Section 3 below. For the second item, examples are:

- Let E be a holomorphic vector bundle over a complex Kähler manifold M. We have a $\overline{\partial}$ complex $\overline{\partial}_E : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)$. The index of the operator $P_E = \overline{\partial}_E + \overline{\partial}_E^*$ is the holomorphic Euler characteristic of E. On the other hand P_E can be identified with the Dirac operator coupled to the bundle $E \otimes K^{-1/2}$ where K is the canonical bundle. The index formula (2) yields the Riemann-Roch formula for E.

- Taking $V = S$, the coupled Dirac operator becomes the signature operator D_{sign}. The index formula (2) yields the Hirzebruch formula for the signature in terms of Pontrayagin classes.

- Taking $V = S^+ - S^-$, with an obvious interpretation, the index formula (2) yields the "Gauss-Bonnet" formula for the Euler characteristic. That is, one has an identity ch $(S^+) -$ ch $(S^-) = e(M)$.

Families

An important generalisation of the numerical index is the index of a family. Let $\pi : \mathcal{M} \to B$ be a fibration with the structure of a compact $2m$-dimensional Riemannian spin manifold on each fibre $M_b = \pi^{-1}(b)$. Let $\mathcal{W} \to \mathcal{M}$ be a complex vector bundle with a connection over each fibre M_b. Then for each $b \in B$ we have a coupled Dirac operator D_b^+ and so finite dimensional spaces $\ker\ D_b^+, \mathrm{coker}\ D_b^+$. In a case when the dimensions of these are constant, for $b \in B$, we get a pair of vector bundles Ker, Coker over B and a virtual bundle

$$\mathrm{Ind} = \mathrm{Ker} - \mathrm{Coker} \in K(B)$$

In general, when the dimensions of the spaces can jump the kernels and cokernels do not form vector bundles over B but the formal difference can be still be defined in $K(B)$. The index formula for families is

$$\mathrm{ch}\ (Ind) = \pi_* \left(\hat{A}(T_v\mathcal{M}) \mathrm{ch}\ (\mathcal{W}) \right), \tag{4}$$

where $T_v \to \mathcal{M}$ is the vector bundle given by the tangent spaces of the fibres and

$$\pi_*(H^i(\mathcal{M}) \to H^{i-2m}(B),$$

is the "integration over the fibre" map.

Pseudodifferential operators

Another generalisation involves *pseudodifferential operator*. In the simplest case of an operator on vector-valued functions over $\mathbf{R}^n$ a pseudodifferential operator T is defined by a matrix valued function $P(x, \xi)$ on $\mathbf{R}^n \times \mathbf{R}^n$ and the formula

$$(Tf)(x) = \int P(x, \xi) e^{-ix\xi} \hat{f}(\xi) d\xi,$$

where $\hat{f}$ is the Fourier transform of f. The function P is required to have an asymptotic expansion as $|\xi| \to \infty$ with a leading term of some order $O(|\xi|^l)$. The degree of the operator is defined to be this number l. The symbol of the operator is the leading (order l) term and the operator is elliptic if this is invertible: *i.e.* if $P(x, \xi)$ is invertible for large ξ. If $P(x, \xi)$ is a polynomial in ξ we have a differential operator but we also get integral and "singular integral" operators and more. For example over a compact Riemannian manifold $\Delta^{1/2}$ is a pseudodifferential operator of order 1 and $(1 + \Delta)^{-1/2}$ is a pseudodifferential operator of order -1.

3 Fragmentary proof sketches

There are three approaches to the proof of the index formula, with slogans:

1. Cobordism;
2. K-theory;
3. Heat equation.

Cobordism

The basic input is that if the spin manifold M^{2m} is the boundary of a spin manifold N^{2m+1} and if the bundle V extends over N then the index of D^+_V vanishes. For simplicity consider the case when V is trivial and write D_N for the Dirac operator over N and $D^\pm$ for the operators over M. On the boundary the spin bundle of N is the sum $S^+ \oplus S^-$. Consider the subspace $L \subset \Gamma(S^+) \times \Gamma(S^-)$ of pairs (s_+, s_-) which are the boundary values of solutions of the Dirac equation $D_N\sigma = 0$ over N. One shows that L is the graph of an isomorphism $T : \Gamma(S^+) \to \Gamma(S^-)$. This is a global construction, so T is not determined by local data but it is a pseudodifferential operator of order zero and its *symbol* is determined locally. Write Δ^+ for the Laplace-type operator $D^- D^+$ on $\Gamma(S^+)$ and Δ^- for D^+D^- on $\Gamma(S^-)$. So $1+\Delta^+, 1+\Delta^-$ are invertible operators. One finds that the symbol of T is the same as that of the zero order operator

$$\tilde{T} = D^+ \circ (1 + \Delta^+)^{-1/2} : \Gamma(S^+) \to \Gamma(S^-).$$

The index of T is zero, since T is an isomorphism, and the indices of $\tilde{T}$ and D^+ are equal since $(1 + \Delta^+)^{-1/2}$ is invertible. The indices of $T, \tilde{T}$ are the same since they have the same symbol.

K-Theory

A fundamental case is that of a "compactly supported" pseudodifferential operator T over $\mathbf{R}^n$. That is, we have a function $P(x, \xi)$ on $\mathbf{R}^n \times \mathbf{R}^n$ with values in $N \times N$ matrices such that $P(x, \xi)$ is the identity for $|x|$ large. This means that for for any f we have $Tf(x) = f(x)$ for large x, so the functions in the kernel of T have compact support and similarly for the adjoint. Suppose that P is elliptic. Then $P(x, \xi)$ is invertible for (x, ξ) large in $\mathbf{R}^n \times \mathbf{R}^n$. Restriction to a large sphere gives a map $p : S^{2n-1} \to GL(N, \mathbf{C})$.

The *Bott periodicity theorem* asserts that for $N >> n$ the homotopy group $\pi_{2n-1}(GL(N, \mathbf{C}))$ is $\mathbf{Z}$. Thus we have a "degree" $\deg(p) \in \mathbf{Z}$ and the index formula in this situation is

$$\mathrm{ind}(T) = \deg(p). \tag{5}$$

To connect this to the previous discussion, consider the Dirac operator D^+_V on S^n (for n even) coupled to a rank r vector bundle $V \to S^n$. Let D^- be the Dirac operator on the trivial rank r bundle $\underline{\mathbf{C}^r}$. Then consider

$$\mathcal{D} = D^+_V \oplus D^- : \Gamma(S^+ \otimes V \oplus S^- \otimes \underline{\mathbf{C}^r}) \to \Gamma(S^- \otimes V \oplus S^+ \otimes \underline{\mathbf{C}^r}).$$

The index of $\mathcal{D}$ is the difference ind D^+_V − ind $D^+_{\underline{\mathbf{C}}^r}$. The order 0 operator $\mathcal{D}(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ can be deformed to be the identity near the point at infinity in $S^n = \mathbf{R}^n \cup \{\infty\}$, so fits into the framework above. One finds that

$$\text{ind } D^+_V - \text{ind } D^+_{\underline{\mathbf{C}}^r} = \langle \text{ch}(V), S^n \rangle,$$

which proves the Atiyah-Singer formula (2) for bundles over spheres.

Now go back to the case of an elliptic operator $D : \Gamma(E) \to \Gamma(F)$ over a manifold M. Writing $\pi : T^*M \to M$, the symbol gives a bundle isomorphism $\sigma : \pi^*(E) \to \pi^*(F)$ over the complement of the zero section on T^*M. Let $\Sigma M \to M$ be the compactification of T^*M obtained by adding a point at infinity in each fibre. We use σ to define a bundle $E \sharp_\sigma F$ over ΣM, equal to $\pi^* E$ on the "zero" disc bundle and $\pi^*(F)$ on the "infinity" disc bundle, glued together using σ. The K group with compact supports $K_c(T^*M)$ can be defined as the kernel of the restriction map

$$K(\Sigma M) \to K(M_\infty)$$

where $M_\infty \subset \Sigma M$ is the section at infinity. The virtual bundle $E \sharp_\sigma F - \pi^* F$ lies in this kernel so we have an element $[E, F, \sigma] \in K_c(T^*M)$. The deformation invariance of the index implies that ind$D$ depends only on $[E, F, \sigma]$. Moreover the index defines a homomorphism

$$\text{ind} : K_c(T^*M) \to \mathbf{Z}.$$

From this point of view, the index problem is to define, using algebraic topology, a homomorphism $K_c(T^*M) \to \mathbf{Z}$ and to show that this coincides with analytically defined index.

The importance of the Dirac operator, in this setting is that it defines the K-theory analogue of the Thom class in compactly supported cohomology $H^*_c(T^*M)$. This becomes the statement that any elliptic operator is equivalent, in a suitable sense, to a coupled Dirac operator.

Heat equation

Go back to a compact, even-dimensional, Riemannian manifold M and bundle V. The Laplace-type operator Δ_+ on $\Gamma(S^+ \otimes V)$ decomposes this space into a sum of eigenspaces

$$E^+_\lambda = \{ s \in \Gamma(S^+) : \Delta^+ s = \lambda s \}.$$

Similarly for $\Gamma(S^- \otimes V)$. For non-zero λ, the operator D^+ defines an isomorphism from E^+_λ to E^-_λ. For $t > 0$ we can define an operator $\exp(-t\Delta^+)$. For any section s the 1-parameter family $s_t = \exp(-t\Delta^+)s$ is the solution of the "heat equation"

$$\frac{\partial s_t}{\partial t} = -\Delta^+ s_t,$$

with initial condition $s_0 = s$. The operators are represented by a heat kernel

$$s_t(x) = \int_M H_t^+(x, y) s(y)\ dy.$$

The trace of $\exp(-t\Delta^+)$ is given on the one hand by the sum $\sum e^{-\lambda t}$ over eigenvalues and on the other hand by the integral

$$\int_M \text{tr}\ (H_t^+(y, y))\ dy.$$

Thus we have that for all t

$$\text{index } D^+ = \int_M \text{tr } H_t^+(y, y) - \text{tr } H_t^-(y, y)\ dy. \tag{6}$$

The ordinary heat kernel on $\mathbf{R}^n$ is

$$H_t(x, y) = (4\pi t)^{-n/2} \exp(-(x - y)^2/4t).$$

In general there is an asymptotic expansion as $t \to 0$:

$$\text{tr } H_t^+(y, y) \sim a_0^+ t^{-n/2} + a_1^+ t^{-n/2+1} + a_2^+ t^{-n/2+2} + \ldots,$$

where a_i are functions of y. The first term a_0 is $(4\pi)^{-n/2}$ times the rank of the bundle V, just like the model case. By general theory all the a_i can be computed locally, in terms of the metric and its derivatives in local coordinates. Similarly for H_t^-. Then (6) implies that there must be a cancellation $a_i^+ = a_i^-$ for $i < n/2$ and

$$\text{index } D^+ = \int_M a_{n/2}^+(y) - a_{n/2}^-(y)\ \ dy \tag{7}$$

where the integrand can be computed by local differential geometry. The problem is to show that this integral gives the the combination of Pontrayagin and Chern classes appearing in $\text{ch}(V)\hat{A}$. This is *a priori* difficult because the asymptotic expansion theory is inductive and calculations get very long in higher dimensions, but there are ingenious ways to simplify the calculations. In the end this gives a stronger statement (the local index theorem): the integrand is exactly the Chern-Weil integrand in (3).

Exercises

1. Let $T_0 : V_1 \to V_2$ be a Fredholm operator. Show that any small deformation of T_0 is also Fredholm and has the same index.

(*Hint*: First do the case when T_0 is surjective. In general, choose a map $\tau : \mathbf{C}^N \to V_2$ such that $T_0 + \tau : V_1 \oplus \mathbf{C}^N \to V_2$ is surjective and thence reduce to the first case.)

2. Check that the definition (2) of the symbol is independent of choices.

3. Compute the symbols of the exterior derivative d and its adjoint d^*. Show that $d+d^* : \Omega^{\text{even}} \to \Omega^{\text{odd}}$ is elliptic. Use Hodge theory to show that the index, on a compact Riemannian manifold, is the Euler characteristic.

4. (i) Given that there is a number a such that sign $M^4 = ap_1$ for any compact oriented 4-manifold M^4, find a.

(ii) Given that there are numbers b, c such that sign $M^8 = bp_1^2 + cp_2$ for any compact oriented 8-manifold M^8 find b and c by calculating with the examples $\mathbf{CP}^4$ and $\mathbf{CP}^2 \times \mathbf{CP}^2$.

5. Let M be a compact oriented manifold of dimension $4k$ which is the oriented boundary of a $(4k+1)$-manifold N. Use Poincaré duality and the exact sequence of the pair (N, M) to show that the signature of M is zero.

6. Show that for a spin manifold of dimension 4 the bundles S^+, S^- are naturally quaternionic vector bundles and $D^\pm$ are quaternion-linear. Use the index formula to deduce *Rohlin's Theorem*: the signature of a spin 4-manifold is divisible by 16.

7. Show that on a spin 4-manifold the difference $\text{ch}(S^+) - \text{ch}S^-$ is the Euler class.

8. The *Hilbert transform* H is the linear map from functions on $\mathbf{R}$ to functions on $\mathbf{R}$ defined by

$$(Hu)(x) = \pi^{-1} \lim_{\delta \to 0} \int_{|y-x|>\delta} \frac{u(y)}{x-y} dy.$$

(You might want to think of this as defined initially on smooth functions u of compact support.)

Show that there is a holomorphic function on the upper half-plane with boundary value $u + iH(u)$.

Show that H is a pseudodifferential operator of order 0 with symbol $\sigma(\xi) = i\text{sgn}(\xi)$.

(*Hint.* Show that

$$(Hu)(x) = \pi^{-1} \lim_{\epsilon \to 0} \int \frac{(x-y)u(y)}{(x-y)^2 + \epsilon^2}\, dy,$$

and that

$$F(x+it) = \pi^{-1} \int \frac{tf(y)}{(x-y)^2 + t^2}\, dy,$$

is a harmonic extension of f over the upper half-plane.)

9. Let σ be the symbol of the Dirac operator D^+ over $\mathbf{R}^4$, so for each $\xi \in \mathbf{R}^4$ we have $\sigma_\xi : \mathbf{C}^2 \to \mathbf{C}^2$. Show that the map from S^3 to $GL(2, \mathbf{C})$ obtained by restricting to ξ in the unit sphere defines the generator of $\pi_3(GL(2, \mathbf{C}))$.

10. Let L be a complex line bundle over the flat torus $\mathbf{C}/\mathbf{Z} + 2\pi i\mathbf{Z}$ with curvature $-idxdy$. Consider the $\overline{\partial}$-operator $\overline{\partial}_L : \Omega^0(L) \to \Omega^{0,1}(L)$ and the second order operators $\Delta_+ = \overline{\partial}_L^* \overline{\partial}_L, \Delta_- = \overline{\partial}_L \overline{\partial}_L^*$. Use the constant form $d\overline{z} = dx - idy$ to identify $\Omega^0(L)$ with $\Omega^{0,1}(L)$ so that Δ_+, Δ_- can be regarded as acting on the same space. With that interpretation, show that $\Delta_- = \Delta_+ + 2$ and hence relate the the heat kernels of $\Delta_\pm$. Use this to show that the index of $\overline{\partial}_L$ is 1.