Elliptic operators: lecture notes

Simon Donaldson

February 27, 2020

Abstract

Notes for a London School of Geometry and Number Theory course, January-February 2020

Contents

1 Statement of the fundamental theorem

We consider a linear differential operator of order r

$$L : \Gamma(E) \to \Gamma(F),$$

where E, F are vector bundles over a manifold M. (These can be real or complex vector bundles, we usually suppose complex.) Thus in local co-ordinates on M and local trivialisations of the bundles the operator is given by

$$Lf = \sum_{|I| \leq r} a_I \left(\frac{\partial}{\partial x} \right)^I f, \tag{1}$$

where f and Lf are vector valued functions of x, the sum runs over multi-indices I and a_I is a matrix-valued function of x. We want to consider solving the partial differential equation $Lf = \rho$ for f, where ρ is given.

We usually suppose that E and F have metrics (Euclidean or Hermitian) and that M has a volume form. Then we have L^2 inner products on sections, in the standard way. There is an adjoint operator

$$L^* : \Gamma(F) \to \Gamma(E),$$

characterised by the identity

$$\langle Lf, \sigma \rangle = \langle f, L^* \sigma \rangle \tag{2}$$

when at least one of f, σ has compact support. The metric structure is often not essential and one can work with a "transpose" operator

$$L^T : \Gamma(F^* \otimes \Lambda) \to \Gamma(E^* \otimes \Lambda),$$

where Λ is the real line bundle of volume forms. But it will usually be more convenient for us to fix metric structures.

If M is compact then it is immediate that any σ with $L^* \sigma = 0$ is orthogonal to the image of L. The "fundamental theorem" gives the converse, in the case of *elliptic* operators L (a condition we define below).

Theorem 1 *Let L be an elliptic operator over a compact manifold. Then* $\ker L, \ker L^*$ *are finite dimensional and we can solve the equation* $Lf = \rho$ *if any only if* $\rho \perp \ker L^*$.

Thus there are just a finite number of conditions on ρ for a solution to our equation to exist and the solution is unique up to a finite-dimensional ambiguity.

Example Let (M, g) be a Riemannian manifold. The derivative is a differential operator $\nabla : \Gamma(\underline{\mathbf{R}}) \to \Gamma(T^*M)$ and the Laplace operator on functions Δ_g is $\nabla^*\nabla$. This is a prototype elliptic operator. From the definition we get $\Delta^* = \Delta$. If M is compact and connected and $\Delta f = 0$ then

$$\|\nabla f\|^2 = \langle \Delta f, f \rangle = 0,$$

so f is a constant. In this case the theorem says that we can solve the equation $\Delta f = \rho$ if and only if the integral of ρ is zero and the solution is unique up to the addition of a constant.

We now recall the definition of the elliptic condition. This involves the notion of the *symbol* of a differential operator. For a point $p \in M$ and cotangent vector $\xi \in T^*M_p$ the symbol σ_ξ is a linear map

$$\sigma_\xi : E_p \to F_p.$$

In local co-ordinates, as in (1),

$$\sigma_\xi = \sum_{|I|=r} a_I \xi^I.$$

Note that the sum runs only over the highest order terms. To explain the notation: a monomial $\left(\frac{\partial}{\partial x}\right)^I$ is considered as an element of the symmetric power $s^r(TM)$ and then these are regarded as polynomial functions on the dual space T^*M. To give a co-ordinate-free definition, choose a function ϕ on M with $\phi(p) = 0$ and $d\phi(p) = \xi$. Then

$$L(\phi^r f)(p) = \sigma_\xi . f(p).$$

The significance of the symbol appears when one considers *rescaling*. Take local co-ordinates x centred at p and let $x = \epsilon\tilde{x}$, then work in $\tilde{x}$-coordinates over a ball $|\tilde{x}| < 1$ say. Thus

$$\left(\frac{\partial}{\partial x}\right)^I = \epsilon^{-|I|} \left(\frac{\partial}{\partial \tilde{x}}\right)^I.$$

For the "rescaled" operator L_ϵ

$$\epsilon^r L_\epsilon = \left(\sum_{|I|=r} a_I(\epsilon\tilde{x}) \left(\frac{\partial}{\partial \tilde{x}}\right)^I \right) + \epsilon \left(\sum_{|I|=r-1} \quad \right) + \epsilon^2 \dots \tag{3}$$

As $\epsilon \to 0$ there are two effects. One is that the terms involving derivatives of order less than r are suppressed. The other is that the co-efficients of the order r term converge to the fixed value at $\tilde{x} = 0$ i.e. the point p. In other words, the differential operators $\tilde{L}_\epsilon = \epsilon^r L_\epsilon$ converge to the constant co-efficient operator defined by the symbol of L at p.

This rescaling idea is similar to that in Riemannian geometry, where rescalings of a Riemannian metric around a point converge to the Euclidean metric on the tangent space at that point.

- The symbol of the adjoint L^* is $(-1)^r$ times the adjoint of the symbol of L. Thus L^* is elliptic if and only if L is.
- The symbol of the Laplace operator is $\sigma_\xi = -|\xi|^2$.

Definition 1 *The operator L is elliptic is $\sigma_\xi : E_p \to F_p$ is an isomorphism for all non-zero ξ.*

Note that the existence of an elliptic operator implies that the bundles E, F have the same rank.

Example

Let L_0 be a constant-coefficient operator on $\mathbf{R}^n$, of pure order r. So the a_I are constant matrices, defined for $|I| = r$. We have

$$L_0(e^{i\xi x}) = i^r \sum_I a_I \xi^I e^{i\xi x} \tag{4}$$

Now consider vector-valued functions over the torus $T^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$. These can be represented by Fourier series

$$f(x) = \sum_{\nu \in \mathbf{Z}^n} f_\nu e^{i\nu x}.$$

Recall a basic result from Fourier theory, that the smooth functions on the torus correspond to "rapidly decreasing" systems of co-efficients (f_ν) , i.e. $|f_\nu| = o(|\nu|^k)$ for all k.

Acting on functions on the torus

$$L_0 f = i^r \sum_\nu \sigma_\nu(f_\nu) e^{i\nu x}.$$

If L_0 is elliptic then for $\nu \neq 0$ we have $\sigma_\nu(f_\nu) = 0$ if and only if $f_\nu = 0$. Also for any ρ_ν

$$\sigma_\nu^{-1}(\rho_\nu)| \leq C|\nu|^{-r}|\rho_\nu|.$$

It follows that $\ker L_0$ consists of the constants and the image of L_0 consists of the smooth vector valued functions with integral 0. This confirms the fundamental theorem in this situation.

A variant of the above is to fix some $\alpha \in (\mathbf{R}^n)^*$, not in the integer lattice. This defines a flat complex line Λ_α over T^n whose sections are given by series

$$f = \sum_{\nu \in \mathbf{Z}^n} f_\nu e^{i(\nu+\alpha)x}.$$

More generally we consider sections of the tensor product of Λ_α with trivial vector bundles. Then

$$L_0 f = i^r \sum_{\nu \in \mathbf{Z}^n} \sigma_{\nu+\alpha}(f_\nu) e^{i\nu\alpha},$$

and $\sigma_{\nu+\alpha}$ is invertible for all ν. So in this setting the kernel and cokernel of L_0 are zero.

2 Proof of the fundamental theorem

There are a various possible approaches. The approach we take follows the strategy:

- Solve $Lf = \rho$ locally by perturbation from the constant co-efficient case.
- Patch a finite number of local solutions to get a global solution "modulo compact error"
- Obtain global "weak solutions" and then establish *elliptic regularity*.

2.1 Sobolev spaces

While we are ultimately interested in smooth solutions we need to go through Banach spaces, for example to apply the well-known

$$(1+T)^{-1} = 1 - T + T^2 - \dots, \tag{5}$$

for an operator with operator norm $\|T\| < 1$.

We use *Sobolev spaces*. For a bundle E over a compact manifold M and for integers $k \geq 0$, we define $L^2_k(M; E)$ to be the completion of the C^∞ sections in the norm

$$\|f\|^2_{L^2_k} = \sum_{l \leq k} \int_M |\nabla^l f|^2.$$

Here the *lth.* order derivative $\nabla^l f$ can be defined using suitable connections, and any choices give equivalent norms. Alternatively, use a partition of unity and use the ordinary derivatives of vector valued functions on $\mathbf{R}^n$: that approach also gives equivalent norms.

If D is a linear differential operator and $\sigma \in L^2$ we define a *weak solution* to the equation $Df = \sigma$ to be an L^2 section f such that for all smooth χ:

$$\langle D^* \chi, f \rangle = \langle \sigma, \chi \rangle \tag{6}$$

(If we work on a non-compact manifold we also assume that χ has compact support). In particular we can apply this to the operator ∇^l. An equivalent definition of $L^2_k(M; E)$ is to say that it consists of L^2 sections with k weak derivatives in L^2. However we do not need to use this fact.

There are two basic results about these Sobolev spaces.

Proposition 1 *(Rellich) The inclusion $L^2_k \to L^2_{k-1}$ is compact.*

Proposition 2 *(Sobolev embedding) For $k > \mathrm{dim} M/2$ there is a continuous embedding $L^2_k \to C^0$.*

To prove these we use our *standard operating technique* which is to multiply sections by a partition of unity, then transplant to the torus T^n. Using this we reduce to the case $M = T^n$. The L^2_k norm is then equivalent to a norm written in terms of Fourier co-efficients (f_ν)

$$\|f\|_k^2 = \sum_\nu |f_\nu|^2 (1 + |\nu|^2)^k. \tag{7}$$

When $k = 0$ this is the usual correspondence with the space l^2 of square-summable arrays (f_ν).

For Proposition 1, we just discuss the case $L^2_1 \to L^2$. In fact if $w(\nu)$ is any positive weight function with $|w(\nu)| \to \infty$ as $|\nu| \to \infty$ and we take the completion l^2_w in the norm

$$\|(f_\nu)\|_w^2 = \sum w(\nu) |f_\nu|^2,$$

then the inclusion $l^2_w \to l^2$ is compact. For suppose that $(f_\nu)^{(i)}$ is a sequence which is bounded in l^2_w norm. Then for each ν the $f_\nu^{(i)}$ are bounded. Taking a subsequence, using a diagonal argument, we can suppose that $f_\nu^{(i)} \to g_\nu$ as $i \to \infty$. The assumption on the weight function w means that for any ϵ we can find R such that for all i

$$\sum_{|\nu| > R} |f_\nu^{(i)} - g_\nu|^2 < \epsilon.$$

Using the fact that there are only a finite number of lattice points ν with $|\nu| \leq R$ one shows easily that (g_ν) is the limit of $(f_\nu)^{(i)}$ in l^2 norm.

For Proposition 2 the essential point is that the L^2_k norm on the torus should control the C^0 norm. This follows from Cauchy-Schwartz:

$$\|f\|_{C^0}^2 \leq \left(\sum |f_\nu| \right)^2 \leq \sum |f_\nu|^2 (1 + |\nu|^2)^k \sum (1 + |\nu|^2)^{-k}$$

and if $k > n/2$ the last term is finite by comparison with the integral

$$\int_{r=0}^{\infty} r^{n-1}(1+r^2)^{-k} dr.$$

By differentiation, we deduce from Proposition 2 that $L^2_k \to C^s$ if $k > n/2+s$ so

$$\bigcap_{k>0} L^2_k = C^\infty.$$

Clearly our operator L of order r over M extends to a continuous map

$$L : L^2_r \to L^2.$$

Recall from functional analysis that if $K : L^2 \to L^2$ is a compact linear operator then the image of $1 + K$ is a closed subspace of finite codimension. We say that $P : L^2 \to L^2_r$ is a (right) *parametrix* for L if $LP = 1 + K$ with $K : L^2 \to L^2$ compact. If we have such a P then:

- $$\mathrm{Im}(L : L^2_r \to L^2)$$

  contains $\mathrm{Im}(1 + K)$ so is also closed and of finite codimension.
- Since the image is closed we have $L^2 = \mathrm{Im}L \oplus (\mathrm{Im}L)^\perp$.
- From the definition, $\mathrm{ImL}^\perp$ is the set of weak solutions σ of $L^*\sigma = 0$.
- So for $\rho \in L^2$ we can solve the equation $Lf = \rho$ with $f \in L^2_k$ if and only if ρ is orthogonal to the *weak* solutions of the equation $L^*\sigma = 0$.

Thus a proof of the fundamental theorem follows if we show:

1. Every elliptic operator over a compact manifold admits a parametrix;
2. A weak solution of $L^*\sigma = 0$ is smooth and if $f \in L^2_r$ with Lf smooth then f is smooth.

(The fact that kerL is finite dimensional follows from the above discussion by interchanging L, L^*.)

2.2 Construction of a parametrix

We first develop properties of a constant co-efficient operator L_0 of pure order r. For ρ supported in the unit ball in $\mathbf{R}^n$ (say) we want to solve the equation $L_0 f = \rho$ over a slightly larger ball, with estimates on the solution. This can be done in various ways.

- *Fourier transform.* We have

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(x) e^{-i\xi x} dx,$$

and the operator L_0 goes over under the transform to multiplication by $i^r \sigma(\xi)$. So a solution f is the inverse FT of

$$(-i)^r \sigma_\xi^{-1}(\hat{\rho}(\xi)).$$

This needs some Fourier theory to carry through, for example to deal with the potential singularity at $\xi = 0$.

- *Integral operators*

The solution can be written as convolution with a Greens function G:

$$f(x) = \int_{\mathbf{R}^n} G(x-y)\rho(y) dy.$$

The function G is homogeneous of degree $r - n$ provided that $r \neq n$. For example the Laplacian on $\mathbf{R}^n$ with $n > 2$ leads to the Newton kernel

$$G(z) = C|z|^{2-n}.$$

This description has advantages when one studies other function spaces (see Section 4.1 below), but also needs some theory.

- *"Shifted" Fourier series* This is more elementary and is the approach we take in what follows in this section.

To proceed, we consider a constant coefficient elliptic operator of pure order r over the torus:

$$L_0 : \Gamma(\mathbf{C}^N \otimes \Lambda_\alpha) \to \Gamma(\mathbf{C}^N \otimes \Lambda_\alpha).$$

We have

Proposition 3 *For each k, L_0 defines an isomorphism of Banach spaces*

$$L_0 : L^2_{k+r} \to L^2_k.$$

This clear from the Fourier series description because we can find an $m > 1$ such for any f_ν we have

$$m^{-1}(1 + |\nu|^2)^r |f_\nu|^2 \leq |\sigma_{\nu+\alpha}(f_\nu)|^2 \leq m(1 + |\nu|^2)^r |f_\nu|^2.$$

For the moment we take $k = 0$ above. We have

Proposition 4 *If $\tilde{L}$ is a sufficiently small perturbation of L_0 then $\tilde{L}$ also defines an isomorphism from L^2_r to L^2.*

Here "sufficiently small perturbation" means in the obvious sense: if $\tilde{L} = L_0 + \eta$ the coefficients of the differential operator η are sufficiently small in C^rnorm. If L_0^{-1} is the inverse of L_0 then we require that the $L^2 \to L^2$ operator norm of $\eta \circ L_0^{-1}$ is less than 1 and we have

$$\tilde{L}^{-1} = L_0^{-1}(1 + \eta L_0^{-1})^{-1},$$

using (5).

To construct a parametrix we choose a partition of unity $1 = \sum_a \chi_a$ on M, where the χ_a are supported in small balls $B_a \subset M$. Choose cut-off functions $\tilde{\chi}_a$ supported in slightly larger balls $\tilde{B}_a$ and with $\tilde{\chi}_a = 1$ on the support of χ_a. We rescale $\tilde{B}_a$ to unit size and embed it in the standard torus T^n. Write $\tilde{B}_a^* \subset T^n$ for this ball in T^n. Rescaling the operator L over $\tilde{B}_a$, as in the beginning of Section 1, we get an operator $\tilde{L}_a$ over $\tilde{B}_a^*$ which is very close to a constant coefficient operator L_0. The flat line bundle Λ_α is trivial over $\tilde{B}_a^*$. Using a suitable cut-off function we can extend $\tilde{L}_a$ to an operator over T^n, acting on sections of twisted flat bundles as above, still close to the constant coefficent operator. (Here we should suppose that a slightly larger ball than $\tilde{B}_a^*$ is embedded in T^n.) Then we can suppose, by choosing our balls in M sufficiently small, that $\tilde{L}_a$ satisfies the hypotheses of Proposition 4 and is invertible from L^2_r to L^2. Let Q_a be the inverse. We define

$$P\rho = \sum \tilde{\chi}_a Q_a(\chi_a \rho) \tag{8}$$

To simplify notation in this equation we have identified sections over $\tilde{B}_a^*$ and $\tilde{B}_a$ in an obvious fashion.

For simplicity, suppose that $r = 1$ (the general case is essentially the same). Then

$$LP\rho = \sum_a \tilde{\chi}_a(LQ_a\chi_a\rho)) + \sum_a (\nabla\tilde{\chi}_a) * (Q_a\chi_a\rho)\,,$$

where $*$ denotes some bilinear algebraic expression. By construction $LQ_a\chi_a\rho = \chi_a\rho$ and $\tilde{\chi}_a\chi_a\rho = \chi_a\rho$. Since $\sum_a \chi_a = 1$ we get $LP\rho = \rho + K\rho$ where

$$K\rho = \sum_a (\nabla\tilde{\chi}_a) * (Q_a\chi_a\rho)\,. \tag{9}$$

Now Q_a is a bounded map $L^2 \to L^2_1$ and all the other ingredients in (9) are given by by multiplication by fixed smooth tensors, so K is a bounded map $K : L^2 \to L^2_1$, hence $K : L^2 \to L^2$ is compact by the Rellich Lemma.

The final step in our proof of the fundamental theorem is *elliptic regularity.*

Proposition 5 *Suppose that L is an elliptic operator of order r over a compact manifold M and that $f \in L^2$ is a weak solution of the equation $Lf = \rho$. If ρ is in L^2_k then f is in L^2_{k+r}. In particular, if $\rho \in C^\infty$ then $f \in C^\infty$.*

This is closely related to the existence of *elliptic estimates*:

Proposition 6 *With L as above, for each k there is a constant C_k such that for all $f \in \Gamma(E)$*

$$\|f\|_{L^2_{k+r}} \leq C_k \left(\|Lf\|_{L^2_k} + \|f\|_{L^2} \right). \tag{10}$$

.

In our approach to the proof of the fundamental theorem we do not use Proposition 6 explicitly, but it is an important fact so we digress to discuss it.

- One way to prove elliptic regularity is to first establish the elliptic estimate and then use a smoothing argument.
- The elliptic estimate (10) can be proved by our standard procedure, using cut-off functions and transplanting to the torus where we use Fourier series. But in particular cases there are more direct approaches which may give better results with more geometric information about the constants C_k. For example, for compactly supported functions f on $\mathbf{R}^n$, integration-by-parts shows that

  $$\int_{\mathbf{R}^n} (\Delta f)^2 = \int_{\mathbf{R}^n} |\nabla\nabla f|^2.$$

  The argument involves interchanging derivatives. On a compact Riemannian manifold (M, g) we have

  $$\int_M (\Delta_g f)^2 = \int_M |\nabla\nabla f|^2 + \mathrm{Ricci}(\nabla f, \nabla f).$$

  Here $\nabla\nabla f$ is defined using the Levi-Civita connection.

We now fill in the proof of elliptic regularity (Proposition 5). Consider an operator $\tilde{L}$ acting on non-trivial flat bundles over the torus as above, close to a constant coefficient operator. We can suppose that $\tilde{L} : L^2_1 \to L^2$ and $\tilde{L}^* : L^2_1 \to L^2$ are isomorphisms. Thus

- If $\tilde{\rho}$ is in L^2 over T^n there *exists* an L^2_1 solution $\tilde{f}$ of the equation $\tilde{L}\tilde{f} = \tilde{\rho}$.
- Suppose that $\tilde{g} \in L^2$ is a *weak* solution of the equation $\tilde{L}\tilde{g} = 0$. Then g is orthogonal to the image of $\tilde{L}^*$; but this is the whole of L^2 and so $\tilde{g} = 0$. It follows that $\tilde{f}$ in the first item is the *unique* weak L^2 solution of the equation $\tilde{f} = \tilde{\rho}$.

Go back to our manifold M and $f \in L^2$ a weak solution of $Lf = \rho$, for some $\rho \in L^2$. We first treat the case of a first order operator L. Take a partition of unity χ_a as before, so $f = \sum f_a$ with $f_a = \chi_a f$ and we want to show that $f_a \in L^2_1$. One can check that the formula

$$L(f_a) = \chi_a \rho + (\nabla\chi_a) * f_a,$$

still holds in the weak sense. Now $(\nabla\chi_a)*f_a \in L^2$ so $L(f_a)=\rho_a$ say, with $\rho_a \in L^2$. We can now transport the discussion to the torus following our standard operating technique and, by the existence and uniqueness noted above, we see that $f_a \in L^2_1$.

Thus we have shown that for a first order operator L, if $Lf \in L^2$ then $f \in L^2_1$. The same argument shows that $Lf \in L^2_k$ implies that $f \in L^2_{k+1}$, completing the proof of Proposition for first order operators L. There is some significant further complication for higher operators, say order $r=2$. Then, following the same pattern as above, we have

$$L(\chi_a f) = \chi_a Lf + \nabla\chi_a * \nabla f + \nabla\nabla\chi_a * f. \tag{11}$$

The problem is that for $f \in L^2$ the term in (11) involving ∇f is not necessarily in L^2. To get around this we can extend the definition of the Sobolev spaces L^2_k to negative integers k. On the torus, using the Fourier series description, this can be done using the same formula (7). On a general manifold, elements of L^2_k for $k<0$ can be defined as distributions. With the definitions in place, the same argument works since we can achieve that, over the torus, $\tilde{L}: L^2_1 \to L^2_{-1}$ is an isomorphism. Our first "bootstrapping" step is: $\tilde{L}f_a \in L^2_{-1}$ implies $f_a \in L^2_1$.

3 Example:Riemann surface theory

On $\mathbf{C}$, writing $z = x_1 + ix_2$, we have a differential operator

$$\frac{\partial}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

and solutions of $\frac{\partial}{\partial\overline{z}}f = 0$ are holomorphic functions. This is an elliptic operator with symbol

$$\sigma_\xi = \xi_1 + i\xi_2.$$

If M is a Riemann surface we have an elliptic operator

$$\overline{\partial}: \Omega^0 \to \Omega^{0,1}$$

given in local holomorphic coordinates by $\overline{\partial}f = (\frac{\partial}{\partial\overline{z}}f)\, d\overline{z}$. The "transpose" can be identified with

$$\overline{\partial}: \Omega^{1,0} \to \Omega^{1,1}.$$

and the kernel of this is the space $H^{1,0}$ of holomorphic 1-forms, given in local co-ordinates by $s(z)dz$ where s is holomorphic. Our fundamental theorem tells us that, if M is compact and $\rho \in \Omega^{0,1}$ we can solve the equation $\overline{\partial}f = \rho$ if and only if

$$\int_M \rho \wedge \sigma = 0$$

for all $\sigma \in H^{1,0}$.

Now let $p \in M$ and a be a tangent vector of M at p. This is the data required to specify the residue of a meromorphic function f with a pole at p; that is, in local coordinates centered at p:

$$f(z) = \frac{\alpha}{z} + \text{holomorphic} \quad , \quad a = \alpha \frac{\partial}{\partial z}.$$

Let χ be a cut-off function supported in such a co-ordinate chart, equal to 1 near p. Then

$$\rho = \overline{\partial}(\chi \frac{\alpha}{z}) = \left(\overline{\partial}\chi\right) \frac{\alpha}{z}$$

is a smooth $(0,1)$ form, supported in an annulus. A short calculation shows that for a holomorphic 1 form $\sigma = s(z)dz$

$$\int_{\mathbf{C}} \rho \wedge \sigma = 2\pi i \alpha \; s(0). \tag{12}$$

Let $p_1, \ldots p_d$ be distinct points of M and $a_i \in TM_{p_i}$. For each point we perform the construction above to get an f_0 on M which near each p_i is meromorphic with residue a_i but with $\overline{\partial} f_0 = \rho_0$ where ρ_0 is smooth and supported in a union of annuli. If u is a smooth function on M solving the equation $\overline{\partial} u = \rho_0$ then $f = f_0 - u$ is a meromorphic function with residues a_i. Conversely such an f defines a solution $u = f_0 - f$. Equation (12) implies that for $\sigma \in H^{1,0}$

$$\int_M \rho_0 \wedge \sigma = 2\pi i \sum_i \langle \sigma(p_i), a_i \rangle,$$

where $\langle \, , \rangle$ is the dual pairing between T^*M and TM. So we deduce that

Theorem 2 *For compact M, there is a meromorphic function with at worst simple poles at p_i and residues a_i if and only if*

$$\sum_i \langle \sigma(p_i), a_i \rangle = 0$$

for all $\sigma \in H^{1,0}$.

Of course, the meromorphic function is unique up to the addition of a constant. We have an evaluation map

$$ev : H^{1,0} \to \bigoplus T^* M_{p_i}.$$

Let K be the kernel of the transposed map

$$ev^T : \bigoplus TM_{p_i} \to (H^{1,0})^*.$$

The theorem states that K is the set of residues of meromorphic functions. Linear algebra gives

$$\dim K = d - \dim H^{1,0} + \dim \ker(ev). \tag{13}$$

Write D for the divisor $p_1 + \ldots + p_d$, write $\mathcal{O}(D)$ for the space of meromorphic functions with at worst poles at the p_i and write $\mathcal{O}(K-D)$ for the space of holomorphic 1-forms vanishing at all the p_i. Also write $g = \mathrm{dim} H^{1,0}$. Then (adding 1 for the constants) the equation (13) becomes the *Riemann-Roch formula*:

$$\dim \mathcal{O}(D) = d + 1 - g + \dim \mathcal{O}(K-D). \tag{14}$$

Using our fundamental theorem, it is straightforward to show that g is the *topological genus* of M (defined as half the dimension of the first de Rham cohomology). We leave this as an exercise.

4 Some more analysis

The function spaces L^2_k and their associated norms are not adequate for many applications of elliptic theory, particularly to nonlinear problems. In this section we discuss some topics involving Hölder and L^p spaces.

4.1 The Schauder estimates

For $0 < \alpha < 1$ the $C^{,\alpha}$ seminorm on functions on a metric space is

$$[f]_{,\alpha} = \mathrm{sup}_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

On a compact manifold M we define the $C^{,\alpha}$ norm by adding $\|f\|_{L^\infty}$. In a straightforward fashion we define the $C^{k,\alpha}$ norm on sections of a vector bundle over M, taking the $C^{,\alpha}$ norm of the first k derivatives. As for the L^2_k norms there are many ways of doing this, all of which give equivalent norms. The general Schauder estimate for an elliptic operator of order r over M is similar in shape to (10):

$$\|f\|_{C^{k+r,\alpha}} \leq C_k \left(\|Lf\|_{C^{r,\alpha}} + \|f\|_{L^2} \right) \tag{15}$$

In the familiar way this can be deduced from similar results about constant co-efficient operators. For simplicity we consider the Laplace operator on $\mathbf{R}^n$ for $n > 2$; the general case is similar. For smooth ρ of compact support in $\mathbf{R}^n$ there is a unique solution $f = G\rho$ to the equation $\Delta f = \rho$ with $f \to 0$ at ∞. The operator G is given by the Newton formula:

$$(G\rho)(x) = c \int_{\mathbf{R}^n} \frac{\rho(y)}{|x-y|^{n-2}} dy \tag{16}$$

The basic estimate is

Theorem 3 *There is a constant C such that for all such ρ*

$$[\nabla^2 G\rho]_{,\alpha} \leq C[\rho]_{,\alpha}.$$

Let T_{ij} be the operator $\frac{\partial^2}{\partial x_i \partial x_j} G$. If we formally differentiate (16) we get

$$T_{ij}(\rho)(x) = \int_{\mathbf{R}^n} K_{ij}(x-y)\rho(y)dy \tag{17}$$

where K_{ij} is the relevant second derivative of $c|x-y|^{2-n}$ with respect to x. There are two problems with (17):

Problem 1

$K_{ij}(x)$ is $O(|x|^{-n})$ as $z \to 0$ so the right hand side of (17) is not defined as a Lebesgue integral. To get round this we interpret the RHS as a *singular integral*. We have

$$K_{ij}(x) = c\frac{n x_i z_j - \delta_{ij} r^2}{r^{n+2}}$$

which is homogeneous of degree $-n$. So in generalised polar co-ordinates (r, θ) with $\theta \in S^{n-1}$ we can write

$$K_{ij} = \kappa_{ij}(\theta) r^{-n}.$$

It is clear from the formula above that

$$\int_{S^{n-1}} \kappa_{ij}(\theta) d\theta = 0.$$

It is a simple exercise to see that for smooth ρ this means that the limit

$$\lim_{\delta \to 0} \int_{|x-y|>\delta} K_{ij}(x-y)\rho(y)dy.$$

exists, and this is taken as the definition of the singular integral. More generally if we have any smooth function κ on S^{n-1} of integral zero we can consider the operator T_κ defined by a singular integral of the form above with $K = r^{-n}\kappa$. The prototype is the *Hilbert transform* when $n = 1$

$$H\rho(x) = \pi^{-1} \int_{-\infty}^{\infty} \frac{\rho(y)}{x-y} dy.$$

This acts in a very simple way on Fourier tranforms

$$\widehat{H\rho}(\xi) = i\mathrm{sgn}(\xi)\hat{\rho}(\xi).$$

Problem 2

The second problem is that with this interpretation of the RHS of (17) the formula is *not true* in general. In fact it is not true when $i = j$. To see this observe that $\sum_i K_{ii} = 0$ since cr^{2-n} is the "fundamental solution" of the Laplace equation, but

$$\sum_i T_{ii}(\rho) = \Delta G \rho = \rho.$$

However if a_{ij} is a matrix of trace zero and we put $T = \sum a_{ij} T_{ij}$ and $K = \sum a_{ij} K_{ij}$ it *is true* that

$$T\rho(x) = \int_{\mathbf{R}^n} K(x-y)\rho(y)dy. \tag{18}$$

To prove Theorem 3 it suffices to consider such combinations of derivatives (since we already know that $\Delta G\rho = \rho$). The proof of (18) involves the same procedure as the usual proof of the Newton formula (16).

We have to prove that

$$[T\rho]_{,\alpha} \leq C[\rho]_{,\alpha} \tag{19}$$

for a singular integral operator T as above. A crucial property of T is *scale invariance*. For $\lambda > 0$ and a function f on $\mathbf{R}^n$ set $f_\lambda(x) = f(\lambda^{-1}x)$. Then we have

$$(T\rho)_\lambda = T(\rho_\lambda).$$

First preliminary

It suffices to prove (19) for functions ρ with $\rho(0) = 0$. To see this choose some χ with $[\chi]_{,\alpha}, [T\chi]_{,\alpha} \leq A$ say and $\chi(0) = 1$. Given any ρ set $R = \rho(0)$ and $\sigma = R\chi_{,\lambda}$. We have

$$[\chi_\lambda]_{,\alpha} = \lambda^{-\alpha}[\chi]_{,\alpha} \leq A\lambda^{-\alpha}$$

and

$$[T\chi_\lambda]_{,\alpha} = \lambda^{-\alpha}[T\chi]_{,\alpha} \leq A\lambda^{-\alpha}$$

Now $\tilde{\rho} = \rho - \sigma$ vanishes at 0. Suppose we know that (19) holds for functions vanishing at 0, then

$$[T\rho]_{,\alpha} \leq [T\tilde{\rho}]_{,\alpha} + RA\lambda^{-\alpha} \leq C[\rho]_{,\alpha} + (1+C)RA\lambda^{-\alpha},$$

and letting $\lambda \to \infty$ we deduce $[T\rho]_{,\alpha} \leq C[\rho]_{,\alpha}$.

Second preliminary

Let x_0 be some fixed unit vector in $\mathbf{R}^n$. It suffices to show the existence of a C such that for all ρ with $\rho(0) = 0$ and $[\rho]_{,\alpha} \leq 1$ we have

$$|T\rho(x_0) - T\rho(0)| \leq C. \tag{20}$$

This follows from the scale, translation and rotation invariance of the problem and the first preliminary.

To prove (20) we divide the integration region into four parts:

- $I = \{|y| > 10\}$;
- $II = \{|y| < 1/10\}$;
- $III = \{|y - x_0| < 1/10\}$;
- IV, the complement of the small balls II, III in $\{|y| < 10\}$.

We have to bound

$$|\int_{I\cup II\cup III\cup IV}\rho(y)K(-y)dy-\int_{I\cup II\cup III\cup IV}\rho(y)K(x_0-y)dy|. \tag{21}$$

For region I we take the two terms together with the bound

$$\int_I|\rho(y)||K(-y)-K(x_0-y)|dy.$$

Use the facts that for $|y|>10$ we have

$$|K(-y)-K(x_0-y)|\leq \text{const. }|y|^{-n-1},$$

and that $|\rho(y)|\leq|y|^\alpha$ since $\rho(0)=0$ and $[\rho]_{,\alpha}\leq 1$. We get our bound by comparison with

$$\int_{10}^{\infty}r^{\alpha-2}dr.$$

For the intermediate region IV all the terms are bounded and IV has finite volume so the estimate is obvious.

For the region II: the second integral in (21) is harmless. For the first integral we have $|\rho(y)|\leq|y|^\alpha$ and $K(-y)\leq|y|^{-n}$ and we get a bound by comparison with

$$\int_0^{1/10}r^{\alpha-1}dr.$$

Region III is similar to region II except that we do not assume $\rho(x_0)=0$. However if ρ is constant over III then the integral vanishes since the integral of K over spheres centred at the origin is zero. Writing $\rho=\tilde{\rho}+\rho(x_0)$ over III we can apply the same argument as for II.

4.2 Application of L^p norms

For $p>1$ the L^p norm of a function on a measure space is

$$\|f\|_{L^p}=\left(\int|f|^p\right)^{1/p}.$$

Hölder's inequality is

$$\int fg\leq\|f\|_{L^p}\|g\|_{L^q}$$

when $p^{-1}+q^{-1}=1$. If the space has finite volume then the L^p norms are of increasing strength as p increases and

$$\mathrm{lim_{p\to\infty}\|f\|_{L^p}=\|f\|_{L^\infty}}$$

There is a relatively sophisticated *Calderon-Zygmund theory* of elliptic operators on L^p spaces which we do not have time to go into in this course, but we discuss some simpler topics.

The Sobolev embedding theorem states that for compactly supported functions on $\mathbf{R}^n$:

$$\|f\|_{L^p} \leq C_{n,q}\|\nabla f\|_{L^q} \tag{22}$$

when $-n/p = 1 - n/q$. The basic case is $q = 1$ when the result is equivalent to the isoperimetric inequality

$$\mathrm{Vol}(\Omega)^{n-1/n} \leq C_{n,1}\mathrm{Area}(\partial\Omega).$$

To see the equivalence in one direction, take functions f which are smoothings of the characteristic function of Ω.

The inequality (22) for general q can be deduced from the case $q = 1$ (with a non-optimal constant). For example take $n = 4$. Applying (22) with $q = 1$ to f^3 we get

$$(\int f^4)^{3/4} = (\int (f^3)^{4/3})^{3/4} \leq C_{4,1}\int |\nabla f^3|.$$

Now $\nabla f^3 = 3f^2\nabla f$ and so by Cauchy-Schwartz:

$$(\int f^4)^{3/4} \leq 3C_{4,1}(\int f^4)^{1/2}(\int |\nabla f|^2)^{1/2},$$

which gives

$$(\int f^4)^{1/4} \leq 3C_{4,1}(\int |\nabla f|^2)^{1/2},$$

which is (22) with $q = 2$.

We get similar inequalities on a general compact Riemannian n-manifold (M, g). For simplicity take $n = 4$ where we get an inequality

$$\|f\|_{L^4} \leq C_g\|\nabla f\|_{L^2} + \|f\|_{L^2} \tag{23}$$

The best constant C_g is an important invariant of (M, g). The proof of (23) for *some* constant follows easily from the Euclidean case. To illustrate the use of these ideas we prove:

Theorem 4 *Let (M, g_0) be a compact Riemannian 4-manifold, $m > 1$ and $q > 2$. There is a constant K depending on g_0, m, q such that if g is any metric on M with $m^{-1}g_0 \leq g \leq mg_0$ and any function f on M:*

$$\|f\|_{L^\infty} \leq K\left(\|\Delta_g f\|_{L^q} + \|f\|_{L^2}\right). \tag{24}$$

This is interesting even for the fixed metric g_0. We know that the L^2 norm of Δf essentially controls the L^2_2 norm of f. We know that $L^2_2 \to L^\infty$ in dimensions $n < 4$ but this just fails when $n = 4$. The result says that if we take a slightly stronger L^q norm of Δf we do get an L^∞ bound on f. The main interest of Theorem 4 however is that the result holds for any metrics g under the weak hypothesis that it is uniformly equivalent to the reference metric g_0. There is no assumption on the smoothness or modulus of continuity of g. This means that it is impossible to prove such a result using our standard operating technique from Section 2 of approximating by constant co-efficient operators. Going further, one can consider "metrics" g that are not smooth, or even continuous, and develop some parts of elliptic theory for the Laplace operators of these. Similar remarks apply to certain other differential operators with non-smooth coefficients.

The proof of Theorem 4 is a simple example of the *Moser iteration* technique. A key point is that the L^p norms of f and ∇f computed using the metrics g and g_0 are equivalent up to a bounded factor depending on m, so in (23) we can take $C_g \leq C$ say.

In what follows we write $\| \ \|_p$ for the L^p norm. To simplify the writing, without making any essential difference, we assume that

- $f > 0$;
- $\|f\|_2 = 1$;
- $\mathrm{Vol}(M, g) = 1$;
- $\|\Delta_g f\|_q = 1$.

The second and third items imply that if $p' \geq p \geq 2$ and $\gamma' \geq \gamma > 1$ then

$$\|f\|_{p'}^{\gamma'} \geq \|f\|_p^{\gamma}. \tag{25}$$

Let $r = q/q - 1$ be the conjugate index to q so $r < 2$.

For $\alpha > 1$

$$|\int_M f^\alpha \Delta_g f| \leq \|f^\alpha\|_r \|\Delta f\|_q = \|f^\alpha\|_r = \|f\|_{\alpha r}^{\alpha}$$

By the definition of Δ_g we get

$$\int_M f^\alpha \Delta_g f = \int_M \nabla(f^\alpha).\nabla f.$$

We have

$$(\nabla f^\alpha).\nabla f = \frac{4\alpha}{(\alpha+1)^2} |\nabla f^{(\alpha+1)/2}|^2.$$

So by the Sobolev inequality (23) we get a bound on the L^4 norm of $f^{(\alpha+1)/2}$ in terms of "lower" L^p norms. After some manipulation using (25) and the inequality $\frac{4\alpha}{(\alpha+1)^2} > \alpha^{-1}$ we get

$$\|f\|_{2\alpha} \leq (2C\alpha)^{1/\alpha+1} \|f\|_{r\alpha}. \tag{26}$$

Set $k = 2/r$, so $k > 1$, and for $i \geq 0$ take $\alpha_i = k^{i+1}$. Thus $r\alpha_i = 2k^i$ and $2\alpha_i = 2k^{i+1}$. Let $\lambda_i = \log \|f\|_{2\alpha_i}$ so (26) becomes:

$$\lambda_{i+1} \leq \lambda_i + \frac{1}{k^{i+1}+1} \left(\log(2C) + i \log k\right).$$

Since

$$\sum_{i=0}^{\infty} \frac{1}{k^{i+1}+1} \left(\log(2C) + i \log k\right) < \infty,$$

this gives a bound on $\|f\|_\infty = \lim_{i\to\infty} \exp \lambda_i$.

5 Rudiments of index theory

5.1 The Fredholm index

A bounded linear map $T : H_1 \to H_2$ between Banach spaces is called *Fredholm* if $\ker T$, cokerT are finite dimensional and $\mathrm{Im} T$ is closed. The *index* of T is defined to be

$$\text{ind } T = \dim \ker T - \dim \text{cokerT}. \tag{27}$$

If H_1, H_2 are finite dimensional then ind $T = \dim H_1 - \dim H_2$. In general one can think of the index as giving a meaning to the difference of two infinite dimensions.

Proposition 7 *If T is Fredholm as above and $\tau : H_1 \to H_2$ has sufficiently small operator norm then $T + \tau$ is Fredholm and* ind $(T + \tau) =$ ind T.

To see this, consider first the case when T is surjective. We can choose a closed complementary subspaces $H_1 = H_1' \oplus \ker T$ and then the restriction of T is an isomorphism from H_1' to H_2. Without loss of generality we can suppose that $H_1 = H_2 \oplus \ker T$ and T has components $(1_{H_2}, 0)$. Let τ have components (τ_1, τ_2) so

$$(T + \tau)(h, v) = (1 + \tau_1)h + \tau_2 v.$$

If $\|\tau_1\| < 1$ then $(1 + \tau_1)$ is an isomorphism so $T + \tau$ is surjective and the kernel is given by pairs (h, v) with

$$h = -(1 + \tau_1)^{-1}\tau_2 v.$$

This gives an isomorphism from ker T to ker $(T + \tau)$ and shows that the indices are equal.

For the general case, choose a map $\lambda : \mathbf{C}^N \to H_2$ which generates coker$T$. Then the map $T \oplus \lambda : H_1 \oplus \mathbf{C}^N \to H_2$ is surjective. We have an exact sequence

$$0 \to \text{kerT} \to \text{Ker } (T \oplus \lambda) \to \mathbf{C}^N \to \text{cokerT} \to 0. \tag{28}$$

This shows

$$\text{ind } (T \oplus \lambda) = \text{ind } T + N. \tag{29}$$

We apply the previous discussion to $T \oplus \lambda$ and $(T + \tau) \oplus \lambda$ to establish the proposition.

An elliptic operator L over a compact manifold, acting on suitable Sobolev spaces is Fredholm and the index ind L is independent of the choice of Sobolev spaces. It follows from Proposition * that the index is preserved by continuous deformations through elliptic operators.

Example Rephrasing the discussion of Section 3, a divisor $D = p_1 + \dots p_d$ on a Riemann surface M defines a holomorphic line bundle $\mathcal{L} = \mathcal{L}_D$ over M. The holomorphic structure is encoded in an elliptic operator

$$\overline{\partial}_{\mathcal{L}} : \Omega^0(\mathcal{L}) \to \Omega^{0,1}(\mathcal{L}).$$

The kernel is the space $\mathcal{O}_D$ considered in Section 3 and the cokerenl $H^1(\mathcal{L})$ is the dual of $\mathcal{O}(K - D)$. The Riemann-Roch formula is

$$\text{ind } \overline{\partial}_{\mathcal{L}} = d + 1 - g$$

which is a topological invariant of $(\mathcal{L}, M)$. The dimensions of the kernel and cokernel can change, as the points p_i are varied continuously on M.

More generally, for any holomorphic line bundle $\mathcal{L} \to M$ the space $H^1(\mathcal{L})$ is dual to the holomorphic sections $H^0(K \otimes \mathcal{L}^*)$ where $K = T^*M$ is the canonical line bundle.

5.2 The index of a family

Let S be a compact, connected, space and $T_s : H_1 \to H_2$ a family of Fredholm operators parametrised by S. It might happen that the kernels and cokernels of the T_s have fixed dimensions and vary continuously with s, thus giving a pair of bundles over S, but in general the dimensions of these spaces can jump. However we can define an index of the family as a *virtual bundle* over S.

Recall that for compact X the abelian group $K(X)$ is the Grothendieck group associated to the semi-group of isomorphism classes of complex vector bundles over S, under direct sum. Thus elements of $K(X)$ can be written as formal differences $E - F$ and an exact sequence

$$0 \to E_0 \to E_1 \dots \to E_N \to 0$$

gives a relation $\sum(-1)^i E_i = 0$ in $K(X)$. Tensor product makes $K(X)$ a commutative ring.

To define the index of the family T_s we choose, using the compactness of S, a map $\lambda : \mathbf{C}^N \to H_2$ which generates the cokernel of every T_s. Then we have seen that the kernels of $T_s \oplus \lambda$ have fixed dimension and the proof of Proposition 7 shows that they vary continuously with s, forming a vector bundle over S. We define the index of the family $\text{ind}T_s \in K(S)$ by

$$\text{ind}T_s = \ker\,(T_s \oplus \lambda) - \underline{\mathbf{C}}^N.$$

In the case when $\ker T_s, \operatorname{coker} T_s$ are bundles the exact sequence (28) shows that $\text{ind } T_s = \ker\, T_s - \operatorname{coker}\, T_s$. In general is a simple exercise to show that ind T_s is independent of the choice of λ. (Consider λ, λ' and compare each with $\lambda \oplus \lambda'$.)

In the case of families of elliptic operators one usually takes S to be a manifold. We consider a fibre bundle $\mathcal{M} \to S$ with fibre M, vector bundle $\mathcal{E}, \mathcal{F} \to \mathcal{M}$ and a family of elliptic operators L_s mapping between sections of $\mathcal{E}, \mathcal{F}$ restricted to the fibres. In such a situation the operators L_s do not, at least in a natural fashion, act on fixed Banach spaces but the theory extends in a straightforward way and we get an index of the family in $K(S)$.

5.3 Bott periodicity

There is a triangle of relations between:

1. Index theory;
2. Vector bundles and K-theory;
3. The topology of the unitary groups $U(N)$.

First recall another point of view on K-theory. We have $K(\text{pt}) = \mathbf{Z}$, induced by the rank of bundles, and there is a natural splitting $K(X) = \mathbf{Z} \oplus \tilde{K}(X)$. The "reduced" group $\tilde{K}$ can also be defined as stable equivalence classes of bundles, under the relation generated by $E \sim E \oplus \underline{\mathbf{C}}$. Now consider the sphere S^n. Rank r complex vector bundles correspond to homotopy classes of maps $S^{n-1} \to U(r)$ and so $\tilde{K}(S^n)$ can be identified with $\pi_{n-1}(U)$ where U is the infinite unitary group (or $\pi_{n-1}(U(N)$ for $N >> n$). The fundamental result in the area is *Bott periodicity* which asserts that

$$\pi_{2m-1}(U) = \mathbf{Z} \quad , \quad \pi_{2m}(U) = 0. \tag{30}$$

More precisely, Bott defined a map $B : \pi_i(U) \to \pi_{i+2}(U)$ and showed that it is an isomorphism. Then (30) follows from the simple facts that $\pi_0(U) = 0, \pi_1(U) = \mathbf{Z}$.

An alternative formulation starts with the fact that $K(S^2) = \mathbf{Z} \oplus \mathbf{Z}$. Then the statement is that

$$K(X \times S^2) = K(X) \otimes K(S^2) = K(X) \oplus K(X). \tag{31}$$

Recall that the smash product $X \wedge Y$ of based spaces is defined by collapsing $X \times \text{pt} \cup \text{pt} \times Y$ to a point. Equation (31) implies that $\tilde{K}(X \wedge S^2) = \tilde{K}(X)$. WE have $S^n \wedge S^2 = S^{n+2}$ so

$$\tilde{K}(S^{n+2}) = \tilde{K}(S^n), \tag{32}$$

which implies (30).

In this framework, the Bott map takes the form of a map

$$\beta : K(X) \to K(X \times S^2),$$

different from the obvious map p^* induced by projection $p : X \times S^2 \to X$. To define β, let $H^{-1} \to S^2$ be the tautological line bundle (dual of the hyperplane bundle H). Let $b = q^*(1 - H^{-1}) \in K(X \times S^2)$ where $q : X \times S^2 \to S^2$ is the projection. Then the Bott map is $\beta(E) = bp^*(E)$. The family index construction gives a way to define a map $\alpha : K(X \times S^2) \to K(X)$ which will turn out to be inverse to β. Then

$$\alpha \oplus \iota^* : K(X \times S^2) \to K(X) \oplus K(X)$$

is inverse to

$$\beta \oplus p^* : K(X) \oplus K(X) \to K(X \times S^2),$$

(where ι^* is induced by the inclusion $\iota : X \to X \times S^2$) which establishes (31).

To define α, let V be a vector bundle over $X \times S^2$. Using a connection or otherwise we can define a family of $\overline{\partial}$-operators, so for each $x \in X$

$$\overline{\partial}_x : \Gamma(V_x) \to \Omega^{0,1}(V_x),$$

where V_x is the restriction of V to $\{x\} \times S^2$. Then we set

$$\alpha(V) = \text{ind}\, \overline{\partial}_x.$$

To see that $\alpha\beta = 1$ we can reduce to the case when X is a point. Then the result follows from the fact that for the trivial bundle line over S^2

$$\ker\overline{\partial} = \mathbf{C} \quad , \text{coker}\overline{\partial} = 0,$$

while for the bundle H^{-1}

$$\ker\overline{\partial} = 0 \quad , \text{coker}\overline{\partial} = 0.$$

Once we know that $\alpha\beta = 1$ there is a simple trick, using the formal properties of the constructions, to show that $\beta\alpha = 1$ (see [1]).

5.4 The index problem

Consider a general elliptic operator $L : \Gamma(E) \to \Gamma(F)$ over a compact n-manifold M. Let $S(T^*M)$ be the unit sphere bundle in T^*M with projection $\pi : S(T^*M) \to M$. The symbol of L defines an isomorphism

$$\sigma : \pi^*E \to \pi^*F. \tag{33}$$

Let $p : \Sigma M \to M$ be the S^n bundle over M defined by adding a point at infinity to each fibre of T^*M. Using σ as a clutching map, we define a bundle $W_L \to \Sigma M$, isomorphic to E over the zero section and to F over the ∞ section.

We mention briefly the notion of a *pseudo-differential operator* over M. Roughly speaking, these are defined as follows. Let $B \subset M$ be a co-ordinate neighbourhood. A pseudo differential operator P over M has the property that if f is supported in B then:

- Away from B, Pf is given by an integral operator

$$Pf(x) = \int G(x,y)f(y)dy.$$

- Over B there is a (matrix valued) function $a(x,\xi)$ and Pf is given by

$$Pf(x) = \int a(x,\xi)\hat{f}(\xi)e^{i\xi x}d\xi.$$

So when a is a polynomial in ξ this is just a differential operator. Using pseudo-differential operators we can extend the whole discussion above to the case when the symbol yields a general bundle isomorphism (33) and it follows from the deformation invariance and other formal properties that the index of L depends only on the class of W_σ in $K(\Sigma M)$. In fact it is better to use "K-theory with compact supports". For a locally compact space Z one defines $K_c(Z) = \tilde{K}(Z^+)$ where Z^+ is the one point compactification of Z. Then $K_c(T^*M)$ can be identified with the kernel of

$$K(\Sigma M) \to K(M),$$

defined by restriction to the ∞-section. The index of L depends only on the class of $S(L) = W_L - p^*F$ in $K_c(T^*M)$.

Remark A minor variant of the previous K-theory formulation of Bott periodicity is the statement that, for compact X,

$$K_c(X \times \mathbf{R}^2) = K(X). \tag{34}$$

Conclusion

The indices of elliptic operators over M define a homomorphism

$$\mathrm{ind} : K_c(T^*M) \to \mathbf{Z}$$

and the general "index problem" is to identify this (in a form amenable to explicit calculation).

6 Dirac operators and the index formula

6.1 Spinors

For each $m \geq 1$ we will construct a complex vector space S_m and a map $\gamma : \mathbf{R}^{2m} \otimes S_m \to S_m$ defined by complex linear maps $\gamma_i : S_m \to S_m$. These maps will have the property that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}. \tag{35}$$

To do this we start with $S_1 = S_1^+ \oplus S_1^-$ with $S_1^{\pm} = \mathbf{C}$ and

$$\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{36}$$

Inductively if we have constructed S_m we define $S_{m+1} = S_m \otimes S_1$ with, schematically

$$\tilde{\gamma} = \gamma \otimes 1 \pm 1 \otimes \gamma.$$

These spaces and maps have the properties

- dim $S_m = 2^m$
- with respect to the natural Hermitian metrics $\gamma_i^* = -\gamma_i$;
- $S_m = S_m^+ \oplus S_m^-$
- γ_i interchange $S^{\pm}$.

Recall that the Lie algebra of $SO(2m)$ can be identified with $\Lambda^2 \mathbf{R}^{2m}$. One checks from (35) that the map

$$e_i \wedge e_j \mapsto \frac{1}{2}\gamma_i \gamma_j$$

is a Lie algebra homomorphism from Λ^2 to $\mathrm{End}(S)$. This makes S a representation of the double cover $Spin(2m)$ of $SO(2m)$ and γ is a $Spin(2m)$ equivariant map. This representation is a sum $S^+ \oplus S^-$.

The *Dirac operator* on $\mathbf{R}^{2m}$ acts on S_{2m} valued functions as

$$D = \sum_i \gamma_i \frac{\partial}{\partial x_i}.$$

From the algebraic properties above

- D is self-adjoint;
- D is the sum of $D^+ : \Gamma(S^+) \to \Gamma(S^-)$and $D^-(\Gamma(S^-) \to \Gamma(S^+)$;
- $D^2 = \Delta$.

It is clear that D^+ is an elliptic operator. In fact the map $S^{2m-1} \to U(2^{m-1})$ defined by its symbol gives the generator of $\pi_{2m-1}(U)$.

Now let (M^{2m}, g) be a compact oriented Riemannian manifold. There is an $SO(2m)$- bundle of oriented orthnormal frames $P \to M$. A *spin structure* on M is a $Spin(2m)$ bundle giving a double cover $\tilde{P} \to M$. Given a spin structure we can form associated complex Hermitian vector bundles $S = S^+ \oplus S^-$ over M. These come with a bundle map

$$\gamma : T^*M \otimes S \to S. \tag{37}$$

The Levi-Civita connection induces a connection on $\tilde{P}$ and we have a covariant derivative
$$\nabla : \Gamma(S) \rightarrow \Gamma(T^*M \otimes S).$$
The Dirac operator is the composite of this with (37). More generally, if E is a complex vector bundle with a unitary connection over M there is a Dirac operator coupled to E and in particular
$$D_E^+ : \Gamma(S^+ \otimes E) \rightarrow \Gamma(S^- \otimes E).$$

For the rest of the course we consider the index of this operator which, by general principles discussed above, is a topological invariant of $(M, E, \text{spin structure})$.

6.2 Characteristic classes and the index formula

Recall that a complex vector bundle $E \rightarrow M$ has Chern classes $c_i(E) \in H^{2i}(M; \mathbf{Z})$ and a real vector bundle V has Pontrayagin classes $p_j(V) = (1)^j c_{2j}(V \otimes \mathbf{C}) \in H^{4j}(M; \mathbf{Z})$.

The Chern character of E is a class in $H^{\text{even}}(M; \mathbf{Q})$ defined as follows. Take formal variables λ_a. The expression $\sum_a \exp(\lambda_a)$ is symmetric in the λ_a and so can be written as a power series in the elementary symmetric polynomials. Then $ch(E)$ is given by the same power series, substituting the Chern classes for the elementary symmetric polynomials
$$\sum \lambda_a = c_1 \quad \sum \lambda_a \lambda_b = c_2 \ldots.$$
Thus
$$ch(E) = 1 + c_1 + (\frac{c_1^2}{2} - c_2) + \frac{1}{6}(3c_3 - 3c_1c_2 + c_1^3) + \ldots$$

A general result from algebraic topology is that the Chern character defines an isomorphism $K(M) \otimes \mathbf{Q} \rightarrow H^{\text{even}}(M; \mathbf{Q})$. The map $E \mapsto \text{ind} D_E^+$ induces a homomorphism $K(M) \rightarrow \mathbf{Z}$ and it follows from Poincare duality that there is a class $\hat{A}(M) \in H^{\text{even}}(M; \mathbf{Q})$ such that
$$\text{ind} D_E^+ = \langle ch(E)\hat{A}(M), [M] \rangle \tag{38}$$

The class $\hat{A}(M)$ is determined by the differentiable manifold M and *a priori* the spin structure. The problem is to determine it. It is not surprising that it is given by the Pontrayagin classes $p_j = p_j(TM) = p_j(M)$. In fact the formula begins
$$\hat{A} = 1 - \frac{1}{24} p_1 + \left(\frac{-4p_2 + 7p_1^2}{5760} \right) + \ldots.$$

Remark We saw above that an elliptic operator L defines an element $S(L) \in K_c(T^*M)$. When M is a spin manifold there is a “Thom isomorphism” $K(M) \rightarrow K_c(T^*M)$ induced by $E \mapsto S(D_E^+)$. In this sense, any elliptic operator is equivalent to a Dirac operator and the index formula for the D_E^+ gives a formula for the index of any elliptic operator.

The recipe for producing the class $\hat{A}(M)$ in (38) goes as follows. In general, let $f(z)$ be any even power series with $f(0) = 1$. Take formal variables λ_a and consider $\prod f(\lambda_a)^2$. This can be written as a power series in the elementary symmetric functions of λ_a^2. Then we get a power series in the Pontrayagin classes by substituting these for the elementary symmetric functions of λ_a^2. In our case we want to take the power series

$$\hat{A}(z) = \left(\frac{z/2}{\sinh z/2}\right)^{1/2} \tag{39}$$

which defines $\hat{A}(M) = 1 - p_1/24 + \ldots$.

The Chern-Weil construction represents $ch(E)$ and $\hat{A}(M)$ by expressions in the curvature F_A of the connection on E and the Riemann curvature R of (M, g). We have

$$ch(E, A) = \text{tr} \exp(iF/2\pi) \quad \hat{A}(M, g) = \det\hat{A}\ (R/2\pi).$$

So the Index formula (38)—due to Atiyah and Singer—is

$$\text{ind}D_E^+ = \int_M \text{tr} \exp(iF/2\pi)\det\ \hat{A}(R/2\pi). \tag{40}$$

6.3 The heat equation approach

We digress with some generalities on parabolic differential equations. Let $V \to M$ be some bundle with connection. Let $\Lambda : \Gamma(V) \to \Gamma(V)$ be a positive self-adjoint operator equal to $\nabla^*\nabla$ plus lower order terms. Then $Q = (1 + \Lambda)$ is invertible and its inverse is a compact self-adjoint operator from L^2 to L^2. From functional analysis, we know that there is an orthonormal basis of eigenfunctions of Q with eigenvalues tending to zero. These are eigenfunctions of Λ with eigenvalues $\lambda \geq 0$ tending to infinity:

$$\Lambda\phi_\lambda = \lambda\phi_\lambda.$$

For $t > 0$ we define an operator $H_t = e^{-t\Lambda}$ acting as $e^{-\lambda t}$ on ϕ_λ. Our elliptic estimates show that

- For fixed t, the operator H_t is bounded from L^2 to L^2_k, for all k.
- For all k, the H_t are uniformly bounded (independent of t) from L^2_k to L^2_k.

The first item implies that the operator is represented by a smooth kernel $K_t(x, y) \in V_x \otimes V_y^*$ as

$$H_t(f)(x) = \int_M K_t(x, y)f(y)dy.$$

The trace of H_t has two expressions giving the identity

$$\sum_\lambda e^{-\lambda t} = \int_M \text{tr}K_t(y, y)dy. \tag{41}$$

Recall that the heat kernel on $\mathbf{R}^{2m}$ is

$$(4\pi t)^{-m}\exp(-r^2/4t) \tag{42}$$

One shows that there is an asymptotic expansion, for small t,

$$K_t(y,y) \sim (4\pi t)^{-m}(1+\theta_1 t+\ldots) \tag{43}$$

where $\theta_i = \theta_i(y) \in V_y \otimes V_y^*$. The series is produced by entirely algebraic infinitesimal calculations and the proof that it does represent an asymptotic series for $K_t(x,x)$ uses the estimates above for the operators H_t.

For our application of this theory we consider the operator D^2 on $\Gamma(S \otimes E)$ (dropping the bundle E from our notation). This is a sum

$$D^2 = D_+^2 + D_-^2.$$

We define the "supertrace"

$$\mathrm{tr}_s\exp(-tD^2) = \mathrm{tr}\exp(-tD_+^2) - \mathrm{tr}\exp(-tD_-^2).$$

The operator D_E^+ gives an isomorphism between the non-zero eigenspaces of D_+^2, D_-^2 so we have that for all t:

$$\mathrm{tr}_s\exp(-tD^2) = \mathrm{ind}D_E^+. \tag{44}$$

Taking $t \to 0$ we get

$$\mathrm{ind}D_E^+ = (4\pi)^{-m}\int_M \mathrm{tr}_s(\theta_m(y))dy.$$

The **local index theorem** states:

Theorem 5

$$(4\pi)^{-m}\mathrm{tr}_s(\theta_m)d\mathrm{vol}_g = \Big(\mathrm{tr}\ \exp(iF/2\pi)\ \det\hat{A}(R/2\pi)\Big)_{2m}$$

where $(\ )_{2m}$ denotes the top-dimensional component.

Integrating, this implies (40).

7 Proof of the local index formula

7.1 Clifford algebras and the Lichnerowicz formula

If V is an oriented Euclidean vector space we define the Clifford algebra $Cl(V)$ to be the algebra generated by $V_{\mathbf{C}} = V \otimes \mathbf{C}$ subject to the relation $v^2 = -(v.v)1$. If e_a is an orthonormal basis of V it is clear that $Cl(V)$ has a basis

$$1, e_a, e_a e_b \ldots \tag{45}$$

For our purposes we take V to have even dimension $2m$ and we have a spin space $S = S^+ \oplus S^-$. The following facts are true

- As vector spaces, there is a natural isomorphism $Cl(V) = \Lambda^* V_{\mathbf{C}}$ (as representations of $SO(V)$).
- The map $V \otimes S \to S$ induces an isomorphism $\gamma : Cl(V) \to \mathrm{End}(S)$
- For $E \in Cl(V)$
$$\mathrm{tr}_s(\gamma(E)) = (-2i)^m E_{top}$$
where E_{top} is the component of E in $\Lambda^{2m} V_{\mathbf{C}} = \mathbf{C}$.

We will think of $Cl(V)$ as $\Lambda^* V$ with a different product structure. The Clifford product maps $\Lambda^p \otimes \Lambda^q$ to
$$\Lambda^{p+q} \oplus \dots \Lambda^{|p-q|}.$$
The top-dimensional component is the exterior product and the other components are defined by combinations of contraction and exterior product.

One application of these ideas is to the *Lichnerowicz formula*. First consider the case of the Dirac operator defined by a non-trivial connection A on a bundle E over $\mathbf{R}^{2m}$. Then we have:
$$D_E^2 = \sum \nabla_i \nabla_j \gamma_i \gamma_j$$
and $[\nabla_i, \nabla_j] = F_{ij}$. The "rough Laplacian" $\nabla_A^* \nabla_A$ is $-\sum \nabla_i^2$ and
$$D_E^2 = \nabla_A^* \nabla_A + \sum F_{ij} \gamma_i \gamma_j. \tag{46}$$
For the case of a curved Riemannian base manifold (but trivial bundle E) the differential geometry is a little more complicated but one gets
$$D^2 = \nabla^* \nabla + \frac{1}{4} \sum R_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l \tag{47}$$
To understand the curvature term we have to understand the element
$$\mathcal{R} = \sum R_{ijkl} e_i e_j e_k e_l$$
in the Clifford algebra. This brings in the Bianchi identities for the curvature tensor R. One of way of expressing these is that R is in the kernel of the wedge product map $s^2 \Lambda^2 \to \Lambda^4$. Now $\mathcal{R}$ clearly lies in the image of $\Lambda^2 \otimes \Lambda^2$ under the Clifford product. This Clifford product has components

1. $\Lambda^2 \otimes \Lambda^2 \to \Lambda^4$, the wedge product, which is symmetric.
2. $\Lambda^2 \otimes \Lambda^2 \to \Lambda^2$, up to a factor this is the bracket on the Lie algebra of $SO(2m)$ and is antisymmetric.
3. $\Lambda^2 \otimes \Lambda^2 \to \Lambda^0$, the inner product, which is symmetric.

We see then that the Bianhci identities imply that the first two components vanish and $\mathcal{R}$ lies in Λ^0. Up to a factor it is the scalar curavture S of the metric and we get the Lichnerowicz formula
$$D^2 = \nabla^* \nabla + \frac{S}{4}. \tag{48}$$

7.2 Local index formula 1, bundle curvature

We describe the proof by E. Getzler. Book references are [2], [3]. The main ideas are to work in the Clifford algebra and exploit the grading.

As a first step in this direction, recall that at each point $y \in M$ we have

$$\theta_p \in \mathrm{End}(S_y \otimes E_y) = \mathrm{End}(S_y) \otimes \mathrm{End}(E_y).$$

But we saw above that $\mathrm{End}(S) = \Lambda^*$ and we can write $\theta_p = \sum \theta_{p,q}$ where $\theta_{p,q} \in \Lambda^{2q} \otimes \mathrm{End}E$. In particular the quantity we want to calculate is

$$\mathrm{tr}_s \theta_m = (-2i)^m \mathrm{Tr}_E(\theta_{m,m}). \tag{49}$$

It will transpire that

- $\theta_{p,q} = 0$ for $q > p$;
- The terms $\theta_{p,p}$ can be found explicitly.

We consider ijn this subsection the case when the manifold M is flat but the bundle E has a connection. The spin spaces for nearby points x, y can be identified. Write $K_t(x) = K_t(x, y)$ (with y fixed). Then $K_t(x) \in \Lambda^* \otimes E_y^* \otimes E_x$. Let $\mathcal{D}$ be the operator on sections of $\Lambda^* \otimes E_y^* \otimes E$ given by

$$\mathcal{D} = \sum e_a \nabla_a,$$

where e_a acts by Clifford multiplication in Λ^* and ∇_a is the covariant derivative on E. Then K_t is the solution of

$$(\frac{\partial}{\partial t} + \mathcal{D}^2)K = 0,$$

which tends to the δ-function at y as $t \to 0$. That is, for all σ supported near y

$$\int K_t \sigma \to \mathrm{Tr}(\sigma(y))$$

as $t \to 0$.

We now go back to the discussion at the beginning of the course on *rescaling*. Let U be a vector space and consider U-valued functions on $\mathbf{R}^{2m}$ (in fact on small neighbourhoods of the origin). For $\epsilon > 0$ we define an operator on such functions by

$$m_\epsilon(f)(x) = f(\epsilon^{-1}x).$$

Then our rescaling observation from Section 1 is that for a differential operator L of order r:

$$\epsilon^r m_\epsilon^{-1} L m_\epsilon \to L_0$$

as $\epsilon \to 0$, where L_0 is the constant co-efficient operator given by the symbol of L. If we change the definition of m_ϵ to

$$m_\epsilon f(x) = \epsilon^p f(\epsilon^{-1}x),$$

for some p, we would not change the conjugation $m_\epsilon^{-1} L m_\epsilon$. Suppose now that $U = \bigoplus_{k\geq 0} U_k$ and define

$$M_\epsilon(f)(x) = \epsilon^k f(\epsilon^{-1}x)$$

on sections of U_k. Conjugation by M_ϵ acts in the same way as before on components of L mapping sections of U_k sections of U_k but the components mapping sections of U_k to sections of U_{k+1} (for example) are scaled by an extra factor of ϵ^{-1}.

In our situation, choose a standard local trivialisation of the bundle E so that the operator $\mathcal{D}$ can be thought of as acting on U-valued functions with

$$U = \Lambda^* \otimes \mathrm{End}(E_y),$$

with grading from Λ^*. Define

$$\mathcal{D}_\epsilon = \epsilon M_\epsilon^{-1} \mathcal{D} M_\epsilon.$$

Proposition 8 *$\mathcal{D}_\epsilon^2 \to \Delta_0 + \tilde{F}$ as $\epsilon \to 0$ where Δ_0 is the Euclidean Laplace operator and*

$$\tilde{F} = \sum F_{ij}(0) e_i e_j$$

where $e_i e_j$ acts by exterior *multiplication.*

To see this we write $\sum -\nabla_i^2 = -\sum \partial_i^2 + \eta$ where η is a first order operator

$$\eta = -\sum A_i \partial_i + \partial_i A_i + A_i^2.$$

All these operators act on sections of U preserving the grading. We have

$$\mathcal{D}^2 = \Delta_0 + \eta + \sum F_{ij} e_i e_j$$

where $e_i e_j$ acts by *Clifford* multiplication. When we do the rescaling $\mathcal{D}_\epsilon^2$:

- Δ_0 is preserved,
- η is scaled down by a factor of ϵ (or smaller),
- the components of $F_{ij} e_i e_j$ mapping λ^k to Λ^k and Λ^{k-2} are scaled by factors of ϵ^2, ϵ^4 respectively
- the component of $F_{ij} e_i e_j$ mapping Λ^k to Λ^{k+2} is preserved (except that the curvature F is evaluated at ϵx).

This proves Proposition 8.

Let K_t^ϵ be the fundamental solution corresponding to $\mathcal{D}_\epsilon^2$. One finds from the definition that

$$K_t^\epsilon = \epsilon^{2m} M_\epsilon^{-1} K_{\epsilon^2 t}.$$

Thus if

$$K_t^\epsilon(0) \sim (4\pi t)^{-m}(\theta_0^\epsilon + t\theta_1^\epsilon + \ldots)$$

and we write $\theta_p^\epsilon = \sum \theta_{pq}^\epsilon$ as before then

$$\theta_{pq}^\epsilon = \epsilon^{2p-2q}\theta_{pq}.$$

So we can compute the θ_p using $\mathcal{D}_\epsilon^2$ for any ϵ and taking the limit as $\epsilon \to 0$ from $\Delta_0 + \tilde{F}$. Thus we have to compute the heat operator $\exp(-t(\Delta_0 + \tilde{F}))$. But $\tilde{F}$ commutes with Δ_0 so this is

$$\exp(-t\Delta_0)\exp(-t\tilde{F}).$$

The expansion of the kernel at the origin is just

$$(4\pi t)^{-m}(1 - t\tilde{F} + \frac{1}{2!}t^2\tilde{F}^2 + \ldots).$$

Finally taking the trace on the $\mathrm{End}E$ component we get

$$\mathrm{Tr}_E\theta_{mm} = \frac{1}{m!}\mathrm{Tr}F^m,$$

which gives the local index formula in this situation, using (49).

7.3 Local index formula 2, manifold curvature

Now we suppose that the bundle $E = \mathbf{C}$ is trivial but the manifold (M, g) is not flat. The same strategy works but the differential geometry is more complicated. We define $\mathcal{D}$ as before and we have to choose a local trivialisation of the spin bundle to achieve our grading. We have

$$\mathcal{D}^2 = \nabla^*\nabla + S/4,$$

where S is the scalar curvature. The curvature term preserves the grading so is scaled away and this time the interest comes from the term $\nabla^*\nabla$. In a suitable local frame one finds that

$$\nabla^*\nabla = -\sum\left(\partial_i + \frac{1}{4}R_{ijkl}e_ke_lx_j\right)^2 + LOT,$$

where LOT are lower order terms. We get $\mathcal{D}^2 \to P$ as $\epsilon \to 0$ where the model operator P is

$$P = -\sum_i\left(\partial_i + \frac{1}{4}\sum_j \Omega_{ij}x_j\right)^2 \tag{50}$$

This is an operator acting on Λ^* valued functions on $\mathbf{R}^{2m}$. The Ω_{ij} are 2-forms $\Omega_{ij} = R_{ijkl}e_ke_l$ acting by exterior multiplication. Write Ω for the skew-symmetric $2m \times 2m$ matrix of 2-forms (Ω_{ij}). Also write $B(z)$ for the function

$$B(z) = \frac{z/2}{\tanh z/2}.$$

Proposition 9 *The heat kernel associated to P is*

$$(4\pi t)^{-m} \det \hat{A}(t\Omega) \exp\left(-\frac{1}{4t}\langle x, B(t\Omega)x\rangle\right). \tag{51}$$

Putting $x = 0$, this implies the local index theorem.

The proof of the Proposition 9 is in principle a calculation. Simple observations reduce it to the case of the heat kernel associated to the harmonic oscillator Schrodinger operator on $\mathbf{R}$

$$-\left(\frac{d}{dx}\right)^2 + \frac{a^2x^2}{16}.$$

Mehler's formula for this heat kernel is

$$(4\pi t)^{-1/2}\hat{A}(at)\exp(-\frac{1}{4t}B(at)x^2).$$

8 Applications

8.1 The signature and Riemann-Roch formulae

If M is a Kähler manifold the Dirac operator can be interpreted in complex geometry. A spin structure is a choice of square root $K^{1/2}$ of the canonical line bundle. One can identify S with the differential forms $\Omega^{0,*}(K^{1/2})$ and (up to a factor) the Dirac operator with $\overline{\partial} + \overline{\partial}^*$. More generally, if E is a holomorphic vector bundle over M then

$$\mathrm{ind} D_E^+ = \chi(E \otimes K^{1/2}),$$

where the right hand side is the alternating sum of the dimensions of the sheaf cohomology groups. The Atiyah-Singer index formula becomes the Riemann-Roch formula:

$$\chi(E) = \langle ch(E)Td(M), [M]\rangle \tag{52}$$

where $Td(M)$ is the Todd class, which is a power series in the Chern classes $c_i(M)$. This is defined by the function $f(z) = z/1 - e^{-z}$. One takes the product of $f(\lambda_i)$ and substitutes the Chern classes for the elementary symmetric functions:

$$Td(M) = 1 - \frac{1}{2}c_1 + \frac{1}{12}(c_2 + c_1^2) + \dots$$

(The formula from Section 3 for Riemann surfaces is an example.)

If M has dimension $4k$ the signature $\tau(M)$ of M is the signature of the cup-product form on H^{2k}. We have a sequence of operators:

$$\Omega^0 \to \Omega^1 \dots \to \Omega^{2k-1} \to \Omega^+,$$

where Ω^+ denotes the self-dual forms and the last operator is the projection of d. Taking the corresponding "$d + d^*$" operator gives the Dirac operator on $S \otimes S^+$.

By simple Hodge theory, the index is $\frac{1}{2}(\chi+\tau)$, where χ is the topological Euler characteristic. Combining with the similar operator defined by the anti-delf dual forms we get a signature operator D_{sign} of index τ. The general formula gives the Hirzebruch signature formula

$$\tau(M) = \langle L(M), [M] \rangle$$

where L is the expression in the Pontrayagin classes defined like $\hat{A}$ but using the function $(z/\tanh z)^{1/2}$. Thus

$$L(M) = 1 + \frac{1}{3}p_1 + \frac{7p_2 - p_1^2}{45} + \dots.$$

8.2 Scalar curvature in Riemannian geometry

If M is a spin manifold with scalar curvature $S > 0$ the Lichnerowicz formula shows that the kernel of D is zero so ind $D^+ = 0$.

- The quaternionic projective spaces are spin and have $S > 0$ so we see that the top $\hat{A}$ class must vanish (a relation between Pontrayagin numbers).
- The complex projective plane has $S > 0$ but $p_1 \neq 0$. This shows that the spin condition is necessary.
- The argument can be extended to manifold such as the torus T^{2m}. Suppose that there is a metric with $S \geq \epsilon > 0$. It is easy to see that for any $\delta > 0$ we can take a covering $\tilde{T}^{2m} \to T^{2m}$ such that there is a degree 1 map $\Phi : \tilde{T} \to S^{2m}$ with $|d\Phi| \leq \delta$. Take a bundle $E_0 \to S^{2m}$ whose Chern character in the top dimension is non-zero. Then we get a bundle $E = \Phi^*(E_0)$ over $\tilde{T}$ with connection having curvature $|F| < C\delta$. Taking δ small enough we get $|F| \leq \epsilon$. Then the Lichnerowicz formula shows that $\text{ind} D^+_E = 0$ which contradicts the index formula, so no such metric exists. (The argument extends to show that any metric with $S \geq 0$ is flat.)

8.3 Manifold topology

The right hand side of the index formula is *a priori* a rational number, so the fact that it is an integer gives constraints on the Pontrayagin classes.

- Let M be a 4-dimensional spin manifold. The index of the Dirac operator is $p_1/24$ while the signature is $p_1/3$ so we see that the signature is divisible by 8. This follows in a more elementary way from the fact that the quadratic form is *even*. But we can go further. The spin bundles $S^{\pm}$ can be viewed as quaternionic bundles, so the kernel and cokernel are quaternionic and hence have even complex dimension. We get *Rohlin's Theorem*, that the signature of M is divisible by 16.
- An Enriques surface is the quotient of a K3 surface by a fixed point free involution. It has even quadratic form of signature 8 but is not spin.

- For any j equal to 2 modulo 4 we can construct a rank 4 real vector bundle $N \to S^4$ with Euler class 1 and Pontrayagin class j. Let N_0 be a tubular neighbourhood of the zero section in N. The boundary of N_0 is a homotopy 7-sphere Σ. Suppose it is diffeomorphic to S^7. Then we can attach an 8-ball to get a closed manifold M with $\tau(M) = 1$. The signature formula gives
$$45 = 7p_2 - j^2,$$
which is clearly impossible if $j^2 \neq 4 \bmod 7$. (For example if $j = 6$.) So we deduce that Σ is not diffeomorphic to S^7. In fact it is homeomorphic to S^7 (Milnor's exotic spheres.)

- We can couple the signature operator to a bundle E and we have
$$\mathrm{ind} D^E_{\mathrm{sign}} = \langle ch(E)L(M), [M]\rangle.$$
It is a theorem of Sullivan that in dimensions > 4 any topological manifold has a *Lipschitz structure*. Thus the transition functions between charts have derivatives almost everywhere which are bounded, but not necessarily continuous. So a tangent bundle does not exist but one can define differential forms and Riemannian metrics etc. In the spirit of the second part of Section 4 one can extend elliptic theory to work with these non-smooth structures and in particular define the indices of coupled signature operators D^E_{sign}. We can use these to *define* the L-class $L(M)$ and thence to define the Pontrayagin classes as *rational classes.* This gives a proof of a famous theorem of Novikov on the topological invariance of the rational Pontryagin classes.

References

[1] M. F. Atiyah *Bott periodicity and the index of elliptic operators* Quart. Jour. Math 19 (1968) 113-140

[2] N. Berline, E. Getzler and M. Vergne *Heat kernels and Dirac operators* Springer 1992

[3] J. Roe *Elliptic operators, topology and asymptotic methods* Pitman Research Notes in Mathematics, Longman Scientific 1988