Differential Geometry,Lecture Notes

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1 Basics

A Riemannian metric g on an n-dimensional manifold M is a smooth section of S^2T^*M which gives a positive definite quadratic form on each tangent space. In local coordinates it can be written as

$$g = \sum g_{ij} dx_i dx_j .$$

We discuss two approaches to the foundations of Riemannian geometry.

1. Jets.

Given $p \in M$ and integer $k \geq 0$ we define an equivalence relation on smooth maps from M to N: $f \sim_k g$ if f and g agree to order k at p: that is, in local coordinates if the Taylor series up to order k are the same. One checks that this is independent of local coordinates. The equivalence classes are the k-jets of maps from M to N at p.

We want to consider Riemannian metrics up to the action of diffeomorphisms. To approach this we look at the jet version

$$\mathcal{M}_{k,p} = \frac{\mathrm{k-jets of metrics at} p}{\mathrm{(k+1)-jets of diffeomorphisms fixing} p}.$$

By the classification of forms $\mathcal{M}_{0,p}$ is a point: we can choose coordinates centred at p so that $g_{ij}(0) = \delta_{ij}$.

The fundamentals of Riemannian geometry are contained in the discussions of $k = 1, 2$.

$k = 1$.

Suppose in coordinates x_i we have

$$g_{ij} = \delta_{ij} + \sum_k P_{ijk} x_k + O(x^2).$$

Here P_{ijk} is symmetric in ij. Change coordinates to $\tilde{x}_i$ with

$$x_i = \tilde{x}_i + \sum a_{ijk} \tilde{x}_j \tilde{x}_k .$$

Here a_{ijk} is symmetric in jk. In the new coordinates the metric has first order term

$$\tilde{P}_{ijk} = P_{ijk} + 2(a_{ijk} + a_{jik}).$$

So, writing $v = TM_p$, we have to consider the map $V \otimes s^2(V) \to s^2(V) \otimes V$ defined by the symmetrisation above. The basic fact is that this is an isomorphism, so $\mathcal{M}_{1,p}$ is a point and there is a unique way to fix the 2-jet of local coordinates to make $\tilde{P}_{ijk} = 0$.

$k = 2$

Now we consider

$$g_{ij} = \delta_{ij} + \sum_{kl} Q_{ijkl} x_k x_l,$$

and the change of coordinates

$$x_i = \tilde{x}_i + b_{ijkl} \tilde{x}_j \tilde{x}_k \tilde{x}_l.$$

So Q_{ijkl} is symmetric in (ij) and (kl) while b_{ijkl} is symmetric (jkl). In the same way we get

$$\tilde{Q}_{ijkl} = Q_{ijkl} + 2(b_{ijkl} + b_{jikl}),$$

and we have to consider the map

$$V \otimes s^3(V) \to s^2(V) \otimes s^2(V)$$

defined by this symmetrisation. The basic fact is that this map is an injection. We define $\mathcal{R}$ to be the cokernel

$$\mathcal{R} = \frac{s^2(V)] \otimes s^2(V)}{V \otimes s^3(V)},$$

and computing dimensions we find

$$\dim \mathcal{R} = \frac{n^2(n^2 - 1)}{12}.$$

So for example $\mathcal{R}$ is one dimensional if $n = 2$. We have

$$\mathcal{M}_{2,p} = \frac{\mathcal{R}}{O(V)}.$$

Understanding Riemannian metrics to second order about a point is the same as understanding the representation $\mathcal{R}$ of $O(V)$. We can also think of $\mathcal{R}$ as the kernel of the adjoint map

$$\mathcal{R} = \{Q_{ijkl} \in S^2 \otimes s^2 : Q_{ijkl} + Q_{jkil} + Q_{kijl} = 0\}.$$

For the second discussion we begin with another fundamental notion in differential geometry: the failure of integrability. Let N be a manifold and H a p-dimensional sub-bundle of TM so for each $p \in N$ we have $H_p \subset TN_p$. In this situation there is an invariant Φ which is a section of

$$\Lambda^2 H^* \otimes (TN/H).$$

This is defined using the observation that if v_1, v_2 are sections of H the Lie bracket $[v_1, v_2]$ reduced modulo H_p at a point p depends only on the values of v_i at p.

The Frobenius Theorem asserts that $\Phi = 0$ if and only if H is integrable which is to say that through there is a p-dimensional submanifold everywhere tangent to H.

Now let G be a Lie group and recall the notion of a principle G-bundle $P \to M$ (G acts freely on the right on P and $M = P/G$). A connection on P is a subbundle $H \subset TP$ which is G invariant and complementary to the fibres. The tensor Φ above can be regarded as a section of a bundle over M called the curvature F of the connection:

$$F \in \Gamma(M; \Lambda^2 T^* M \otimes \mathrm{adP}).$$

Here ad$P$ is vector bundle over M associated to P via the adjoint representation, so the fibres of ad$P$ are copies of the Lie algebra Lie(G).

It is usually more convenient for us to work with the essentially equivalent notion of a covariant derivative. Let P have a connection as above and let E be a vector bundle over M associated to some representation $\rho : G \to GL(W)$. Then ρ induces $\mathrm{Lie}(G) \to \mathrm{End}(W)$ and $\mathrm{adP} \to \mathrm{End}E$. For any $\xi \in TM_p$ the connection gives a way to define the covariant derivative $\nabla_\xi s$ of a section s of E. One way to define this is to choose a local trivialisation of P compatible with H at the gives point p. This defines a local trivialisation of E so sections of E are identified with W-valued functions and we define $\nabla_\xi s$ to be the ordinary derivative, in this trivialisation. Obvious variants of this notation are: $\nabla_X s$ for a vector field X on M and $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$.

From this point of view the curvature arises from the commutator of covariant derivatives

$$[\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s = \rho(F(X,Y))s,$$

for vector fields X, Y.

In this course we will mainly be concerned with the case when $E = TM$. A covariant derivative on TM is called *torsion free* if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all vector fields X, Y. This is equivalent to saying that around each point there are local co-ordinates such that $\nabla(\frac{\partial}{\partial x_i})$ vanishes *at that point*.

Let (M, g) be a Riemannian manifold, so there is a principle $O(n)$-bundle $P \to M$ of orthonormal frames on M and the tangent bundle is associated to the fundamental representation of $O(n)$. The "fundamental lemma of Riemannian geometry is that there is a unique connection, the Levi-Civita connection on P which is torsion free. There are various points of view on this.

- The lemma is equivalent to our statement about 1-jets of metrics. In a co-ordinate system adapted the metric at a point covariant differentiation, at that point, is the ordinary derivative of vector valued functions.

- The connection is characterised in terms of 2-vector fields X, Y, Z by

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(\nabla_X \langle Y, Z\rangle + \nabla_Y \langle X, Z\rangle - \nabla_Z \langle X, Y\rangle + \langle [X,Y], Z\rangle - \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle\right).$$

- In local co-ordinates a vector field is given by $v = \sum v^i \frac{\partial}{\partial x^i}$ and

$$\nabla_j v = v^i_j + \sum_k \Gamma^i_{jk} v^k,$$

where the *Christoffel symbols* Γ^i_{jk} are

$$\Gamma^i_{jk} = \frac{1}{2}\sum_a g^{ia}\left(g_{aj,k} + g_{ak,j} - g_{jk,a}\right),$$

where the comma notation means partial derivative.

- We work with covariant derivative on T^*M. This

$$\nabla : \Gamma(T^*M) \to \Gamma(T^*M \otimes T^*M).$$

Write

$$T^* \otimes T^*M = s^2T^*M \oplus \Lambda^2 T^*M.$$

Then

1. The component mapping to $\Lambda^2 T^*M$ is the exterior derivative d.
2. The component mapping to s^2T^*M is the *Killing operator* which takes a vector field v to the Lie derivative $\mathcal{L}_v g$. Here we use the metric to identify TM and T^*M.

The *Riemann curvature tensor* is the curvature of this Levi-Civita connection. Since the Lie algebra of $O(n)$ can be identified with $\Lambda^2\mathbf{R}^n$ the curvature tensor is a section of $\Lambda^2T^*M \otimes \Lambda^2 T^*M$. In a local co-ordinate system adapted to the metric at a point the curvature at that point is given by

$$R_{ijkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k}(*)$$

Using the formula above for the Christoffel symbols this gives

$$R_{ijkl} = g_{ik,jl} - g_{jk,il} - g_{il,jk} + g_{jl,ik}.$$

The curvature satisfies the Bianchi identity

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

In invariant terms this expression defines a map

$$\Lambda^2 \otimes \Lambda^2 \to \Lambda^1 \otimes \Lambda^3.$$

We write $\mathcal{R}' \subset \Lambda^2 \otimes \Lambda^2$ for the kernel of this map: the space of curvature tensors. This is compatible with the previous definition: the formula (*) gives an isomorphism from $\mathcal{R}$ to $\mathcal{R}'$. That is, the curvature is equivalent to the 2-jet of the metric, modulo diffeomorphisms. In fact $\mathcal{R}'$ lies in the symmetric part $s^2\Lambda^2$ and can be identified with the kernel of the map

$$s^2\Lambda^2 \to \Lambda^4,$$

defined by the wedge product.

A basic fact is that a metric with Riem $= 0$ is locally Euclidean (or "flat"). To see this we first use the Frobenius Theorem in the frame bundle to construct a local orthonormal frame ϵ_i of cotangent vectors with $\nabla \epsilon_i = 0$. Then the torsion free condition gives $d\epsilon_i = 0$ so locally the ϵ_i are the derivatives of functions x_i and these give the desired Euclidean co-ordinates.

For a pair of vectors X, Y we put

$$K(X,Y) = \langle R(X,Y)Y, X\rangle = -\langle R(X,Y)X, Y\rangle.$$

For vector in a fixed plane this is proportional to $|X \wedge Y|^2$ an in particular is the same for all orthonormal pairs. It defines the *sectional curvature* in the plane. If we think of the Plucker embedding

$$\mathrm{Gr}_2\mathbf{R}^n \to \mathbf{P}(\mathbf{R}^N)$$

with $\mathbf{R}^N = \Lambda^2\mathbf{R}^n$, the sectional curvature is the function on Gr_2 given in the usual way by the quadratic functions on $\mathbf{R}^N$. This function uniquely determines the curvature tensor, since the quadratic functions $s^2\Lambda^2$ which vanish on the Grassmannian are exactly $\Lambda^4 \subset s^2\Lambda^2$.

The Ricci curvature is obtained from the curvature tensor by the contraction

$$\Lambda^2 \otimes \Lambda^2 \to s^2,$$

$$R_{jk} = \sum_i R_{ijki}.$$

So regarded as a quadratic function on tangent vectors the Ricci curvature in the unit vector e_1 is the sum of the sectional curvatures in the (n-1) planes (e_1, e_i), where e_i is an orthonormal basis. We have an orthogonal direct sum

$$\mathcal{R} = s^2 \oplus \mathcal{W}$$

and the component of the curvature in $\mathcal{W}$ is the Weyl tensor. The Weyl tensor depends only on the conformal class of the metric and if $n > 3$ a metric with zero Weyl curvature is conformal to a flat metric. In dimension 3 we have $\mathcal{R} = s^2$, the Ricci curvature determined the full curvature and there is no Weyl term.

The principle bundle approach is useful for studying other structures. For a general Lie group G with a fixed representation on $\mathbf{R}^n$ a G-structure on a manifold M is a principle G-bundle $P \to M$ and an isomorphism from TM to the associated vector bundle. A torsion-free G-structure is a G-connection on P which induces a torsion free covariant derivative. For example if $G = GL(m, \mathbf{C})$ with its standard action on $\mathbf{R}^{2m} = \mathbf{C}^m$ a torsion free G-structure makes M a complex manifold.

2 Constructions in a Riemannian manifold

Geodesics

The geodesic equation in a Riemannian manifold is the Euler-Lagraange equation for the energy functional on paths $\gamma : (a, b) \to M$:

$$E(\gamma) = \int_a^b |\gamma'|^2 dt.$$

It is equivalent to work with the length functional

$$L(\gamma) = \int_a^b |\gamma'| dt;$$

the difference is that the energy controls the parametrisation of the path. The geodesic equation is $\nabla_T T = 0$ where T is the tangent vector regarded as a vector field along the geodesic. (Note: More strictly we should think of T as a section of the pull-back bundle $\gamma^*(TM)$ over (a, b) and use the fact that connections pull back in a natural way. But we will use a more informal notation. The same applies at a number of points later where we shall be vague about exactly where our vector fields are defined.) In local co-ordinates the geodesic equation is given by

$$x_i'' = -\sum_{jk} \Gamma^i_{jk} x_j' x_k'.$$

A Riemannian metric on M gives a metric structure via the infimum of lengths of paths between points. The manifold is called complete if it is so as a metric space. This is equivalent to saying that geodesics are infinitely extendable, so at each $p \in M$ we have an *exponential map*

$$\exp_p : TM_p \to M$$

. **Examples**

- The Riemannian manifold $\mathbf{R}^2 \setminus \{0\}$ is not complete: of course the completion is $\mathbf{R}^2$.
- Let M be the quotient of the example above by the map $x \mapsto -x$. Then M is not complete and the metric space completion is a cone, which is not a Riemannian manifold.

Submanifolds

Let E_1, E_2 be vector bundles over M and $E = E_1 \oplus E_2$. We have an inclusion $\iota : E_1 \to E$ and projection $p : E \to E_2$. Given a covariant derivative ∇ on E we have a map from sections of E_1 to sections of $E_2 \otimes T^*M$ given by $p \circ \nabla \circ \iota$. This map is given by multiplication by a tensor

$$\beta \in \Gamma(E_1^* \otimes E_2 \otimes T^*M);$$

the *second fundamental form* of the sub-bundle.

In particular suppose M is a Riemannian manifold and $S \subset M$ is a submanifold. Then $TM|_S = TS \oplus \nu$ where ν is the normal bundle and we have a connection on $TM|_S$ defined by the Levi-Civita connection. This gives the second fundamental form $B \in T^*S \otimes T^*S \otimes \nu$. The torsion-free condition implies that this is symmetric. In particular if S has codimension 1 (and we choose a orientation of ν) we just get $B \in s^2 T^*S$. It is useful to think of this interchangeably as a bilinear form and a linear map $B : TS \to TS$. In fact from the latter point of view

$$B(\xi) = \nabla_\xi N$$

for a vector field ξ on S and unit normal vector field N.

Extend the unit normal vector field N to a neighbourhood of S in M. Given vector fields ξ, η on S we use the flow generated by N to extend these over a neighbourhood, with $[\xi, N] = [\eta, N] = 0$. Then

$$\nabla_N \langle \xi, \eta \rangle = 2\langle B(\xi), eta \rangle.$$

In other words if we use the flow to define a 1-parameter family of maps I_t from S to M, extending the inclusion, and thenget a 1-parameter family of metrics $I_t^* g$ on S we have

$$\frac{\partial g}{\partial t}|_{t=0} = 2B.$$

A submanifold is called *totally geodesic* if the second fundamental form vanishes. In that case sectional curvature of S are the same as the corresponding sectional curvatures in M. Note that geodesics are 1-dimensional totally geodesic submanifolds.

The Laplace operator

For a function f on (M, g) we can define the gradient as the vector field corresponding to the 1-form df via the isomorphism $TM = T^*M$ given by g. (Note that we will often write ∇f for the vector field or the 1-form.)

For a vector field v on M we define the divergence via the Lie derivative of the volume form Ω induced by the metric: $L_v \Omega = \mathrm{div} v \Omega$. So $\mathrm{div} v = 0$ if and only if the flow generated by v is volume-preserving.

The Laplace operator is $\Delta f = \mathrm{div}\ \mathrm{grad} f$. In local coordinates this is

$$\Delta f = g^{-1/2} \sum_{ij} \partial_i (g^{1/2} g^{ij} \partial_j f),$$

where g^{ij} is the inverse of g_{ij} and $g = \det(g_{ij})$.

3 Symmetric spaces

In dimension two the Riemann curvature reduces to a scalar: the Gauss curvature. The reader is probably familiar with the simply connected spaces of constant curvature: +1, the round sphere; -1 the hyperbolic plane. Also that formulae from one case “analytically continue” to the other. (Both geometries can be seen as real forms of the complex projective plane with a fixed conic.) The natural higher dimensional generalisation is provided by the theory of Riemannian *symmetric spaces*.

There are a number of different notions (local, global, simply-connected…). We say (M, g) is a locally symmetric space if it satisfies either of the following equivalent conditions:

- $\nabla \mathrm{Riem} = 0$;
- For each $p \in M$, the map $e_p(\xi) \mapsto e_p(-\xi)$ defines an isometry of a neighbourhood of p.

The conclusion of the theory is that such are locally isometric to manifolds obtained in the following way. First we can reduce to the "irreducible" case when the metric is not locally a product and we can alwayd rescale our metric. Let G be a simple Lie group and $\sigma : G \to G$ an involution. Let M be the identity component of

$$\{h \in G : \sigma(h) = h^{-1}\}.$$

Then G acts transitively on M by $g(h) = \sigma(g)hg^{-1}$ and $M = G/K$ where $K = \{g \in G : \sigma(g) = g\}$. To get a metric on M recall that the Killing form of the Lie algebra $\mathbf{g}$ is

$$-\mathrm{Tr}(\mathrm{ad}\xi)^2.$$

A fundamental result from Lie theory is that if $\mathbf{g}$ is simple this is nondegenerate. It defines a bi-invariant possibly indefinite metric on the Lie group G and we require that this be positive or negative definite on $M \subset G$. So either way get a Riemannian metric on M, as desired. M is the fixed point set of the involutive isometry $\tau(h) = \sigma(h)^{-1}$ so M is totally geodesic in G and to compute the sectional curvature we can reduce to the case of a Lie group.

With its bi-invariant metric the Levi-Civita connection on G can be defined by $\nabla_X Y = \frac{1}{2}[X, Y]$ for left invariant vector fields X, Y. One finds that the sectional curvature is

$$K(X, Y) = \frac{1}{4}|[X, Y]|^2.$$

These symmetric spaces come in "dual pairs". Suppose G is noncompact and let K be a maximal compact subgroup. Then the Killing form is positive definite on $\mathbf{k}$ and we take the orthogonal complement

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{p}.$$

The map σ acting as $+1$ on $\mathbf{k}$ and -1 on $\mathbf{p}$ is an involution, which is the same as saying that $[\mathbf{p}, \mathbf{p}] \subset \mathbf{k}$. The space $\mathbf{p}$ is the tangent space of $M = G/K$ at the identity and the sectional curvature is (weakly) negative. We define a new Lie algebra structure on the vector space $\mathbf{g}$ by reversing the sign of the component $\mathbf{p} \times \mathbf{p} \to \mathbf{k}$ so we get a Lie algebra $\mathbf{g}'$ and Lie group G' which is compact. The tangent space of $M' = G'/K$ at the identity is $\mathbf{p}$ and the sectional curvature is (weakly) positive. In sum, there is one dual pair for each noncompact simple Lie algebra.

Examples.

1. $G = SL(n, \mathbf{R}), K = SO(n), G' = SU(n)$.

G/K is the set of Euclidean structures on bR^n with fixed determinant. G'/K is the set of special Lagrangian subspaces in $\mathbf{C}^n$.

2. $G = SO(n, 1), K = SO(n), G' = SO(n + 1)$

G/K is the hyperbolic n space and G'/K is S^n.

3. $G = Sp(n, \mathbf{R}), K = U(n), G' = Sp(n)$ G/K is the space of complex structures on $\mathbf{R}^{2n}$ compatible with a fixed symplectic form. This can be identified with the Siegel upper half space: the set of $n \times n$ complex symmetric matrices with positive definite imaginary part.

4. $G = SU(n, 1), K = U(n), G' = SU(n + 1)$

G'/K is $\mathbf{CP}^n$ and G/K is complex hyperbolic n-space.

One finds that the sectional curvatures of $\mathbf{CP}^n$ lie between 1 and 1/4.

The sphere theorem (Berger, Klingenberg) A complete, simply connected Riemannian manifold with sectional curvature $1 \geq K > 1/4$ is homeomorphic to a sphere.

Improved to "diffeomorphic" much later by Brendle and Schoen.

4 Comparison geometry

The *Jacobi equation* is the linearisation of the geodesic equation. Let $\gamma(t, s)$ be a map from $\mathbf{R}^2$ (or an open subset) to M such that for each fixed s $\gamma(\ , s)$ is a geodesic. Let V, T be the images of ∂_s, ∂_t. So we have $[V, T] = 0$. The geodesic equation is $\nabla_T T = 0$ so $\nabla_V \nabla_T T = 0$. Thus

$$\nabla_T \nabla_T V = \nabla_T \nabla_V T = -R(V, T)T.$$

This is a second order linear ODE for the vector field V along the geodesic. For a symmetric space $\nabla\mathrm{Riem} = 0$ and the equation has constant co-efficients.

Points p, q on a geodesic segment γ are *conjugate* if there is a non-trivial solution of the Jacobi equation which vanishes at p, q. A basic fact is that *a geodesic ceases to minimise length past the first conjugate point.* To see this one uses the *second variation formula* for length.

This gives a good general picture of the exponential map $\exp_p : TM_p \to M$ of a complete manifold. There is an open set $U \subset TM_p$ which is star-shaped about the origin such that $\exp_p$ is a diffeomorphism from U to a dense open set in M. There are two reasons why we can meet the boundary of U.

- We reach a conjugate point.
- The image of U starts to "overlap" itself.

Using the exponential map we get standard "geodesic coordinates" on a neighbourhood of p. For simplicity we suppose that the R-ball is contained in U. The metric is represented by a 1-parameter family of metrics g_r, for $r < R$ on the sphere S^{n-1} and this is determined by the solutions of the Jacobi equation along radial geodesics. In a manifold of constant curvature ± 1 these metric are $\sin^2 r$, $\sinh^2 r$ respectively times the standard metric.

Rauch comparison Suppose all sectional curvatures of M are $\geq c$. Let $\tilde{M}$ be the manifold of constant curvature c then $g_r \leq \tilde{g}_r$. Symmetrically for sectional curvature $\leq c$.

This can be viewed as a statement about second order linear ODE. From another point of view we consider the second fundamental forms of the spheres in M. In general let S be a codimension-1 submanifold in M with unit normal N. Let S_t be the family of hypersurfaces obtained by moving at unit speed in the normal direction. The normal geodesics give a diffeomorphism from S_t to S, so we can regard the induced metrics as a 1-parameter family g_t of metrics on S. The first fact is that

$$\partial g_t \partial t = 2B_t$$

where B_t is the second fundamental form. To see this let X be a vector field on S and extend over a neighbourhood in M using the normal flow. Then $[N, X] = 0$ and

$$\nabla_N \langle X, X\rangle = 2\langle \nabla_N X, X\rangle = 2\langle \nabla_X N, X\rangle = 2\langle B(X), X\rangle.$$

Now we compute the derivative of the second fundamental form. This time we choose a vector field ξ on S and propogate by *parallel transport* along the normals.

$$\nabla_N B(\xi) = \nabla_N \nabla_\xi N = R(N, \xi)N - \nabla_{[N,\xi]} N.$$

Now $[N, \xi] = \nabla_N \xi - \nabla_\xi N = -B(\xi)$, so we get

$$\nabla_N B(\xi) = R(N, \xi)N - B^2(\xi).$$

Write $R(N, \xi)N = -\kappa(\xi)$, so κ is a self-adjoint map on TS and the hypothesis that the sectional curvatures are $\geq c$ says that $\kappa \geq c$.

From the point of view of the Jacobi equation the nonlinear equation above is the associated *Ricatti equation*. If we have an ODE $x'' = -\kappa x$ where x is an m-vector and the matrix κ is self-adjoint we consider the matrix equation $Z'' = -\kappa Z$. Put $B = Z' Z^{-1}$ then we get $B' = -\kappa - B^2$.

Now go back to the case when submanifolds are the spheres in M.

The Rauch theorem follows easily from the corresponding statement for the second fundamental forms. Let $B(t)$ solve $B' = -\kappa - B^2$ where $\kappa \geq c$ and $\tilde{B}$ solve $\tilde{B}' = -c - \tilde{B}^2$. The behaviour for small t is given by

$$B = t^{-1} - \frac{1}{3}\kappa_0 t + \dots$$

and similarly for $\tilde{B}$. Then the statement is $B \leq \tilde{B}$. To see this reduce to the case when B_t does not have multiple eigenvalues. Let e_i be an evolving orthonormal frame of eigenvectors. The eigenvalues $\lambda_i(t)$ evolve by

$$\lambda_i' = \langle \kappa e_i, e_i\rangle - \lambda_i^2 \leq -c - \lambda_i^2,$$

then we can use an elementary comparison argument.

The **Bishop comparison theorem** involves volumes and the Ricci curvature. It is a *one-sided* result, under the hypothesis that in the manifold M we have Ric $\geq (n-1)c$—the Ricci curvature of the comparison space with sectional curvature c. Let $A(r)$ be the volume form on the sphere of radius r–we can also think of $A(r)$ as a function by comparig with the standard volume form on S^{n-1}. Then we have

$$A'(r) = HA(r),$$

where $A = \mathrm{Tr}B$ is the mean curvature. Now

$$\frac{d}{dr}\mathrm{Tr}B = -\mathrm{Tr}B^2 - \mathrm{Tr}\kappa.$$

The term $\mathrm{Tr}\kappa$ is the Ricci curvature in the radial direction. For the quadratic term we have

$$\mathrm{Tr}B^2 \geq \frac{1}{n-1}\left(\mathrm{Tr}B\right)^2.$$

For the model space this is an equality. So

$$\frac{d}{dr}H + \frac{H^2}{n-1} \leq \frac{d}{dr}\tilde{H} + \frac{\tilde{H}^2}{n-1}.$$

From this one deduces that $H \leq \tilde{H}$ and then $A \leq \tilde{A}$. In fact one obtains more, that $A/\tilde{A}$ is decreasing. Let

$$V(r) = \int_0^r A(t)dt,$$

and likewise for $\tilde{V}$. It follows that $V/\tilde{V}$ is decreasing.

Some conclusions from this discussion.

Myers Theorem If Ricci $\geq (n-1)c > 0$ then the diameter of M is at most $\pi c^{-1/2}$.

Bishop-Gromov volume monotonicity If Ricci $\geq (n-1)c$ then the volume ratio

$$\frac{\mathrm{Vol}B(r)}{\tilde{V}(r)}$$

is decreasing. This is a *global theorem*: we do not need to assume that the r-ball lies in U: conjugate points or overlapping only help in the inequality.

There is an important identity related to the discussion above involving the Laplace operator Δ. For a function f on M

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla\nabla f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla\Delta f, \nabla f\rangle.$$

If we take f to be the distance from a fixed point p then

- $|\nabla f| = 1$;
- $\nabla\nabla f$ can be identified with the second fundamental form of the sphere;
- Δf is the mean curvature and $\langle \nabla\Delta f, \nabla f\rangle$ is $\frac{dH}{dr}$.

So we get the formula for $\frac{d}{H}dr$, as before.

5 The Gauss-Bonnet formula

Readers are probable familiar with the fact that for a compact Riemannian surface M^2:

$$2\pi\chi(M) = \int_M K dA,$$

where K is the Gauss curvature and $\chi(M)$ is the Euler characteristic—a topological invariant. In this section we discuss the extension to higher dimensions.

For a closed manifold M we take as the definition of the Euler characteristic the signed count of zeros of a generic vector field. Otherwise said, this is the self-intersection number of M regarded either as the diagonal in $M \times M$ or as the zero section in TM. For simple reasons, this vanishes if the dimension of M is odd.

Our discussion goes through the notion of the *Thom class* of an oriented vector bundle. If $E \to M$ is an oriented bundle with fibre $\mathbf{R}^d$ this is a class in the compactly supported cohomology $H^d_c(E)$ which restricts to the generator on each fibre. In de Rham theory it is represented by a closed compactly supported d-form τ with integral 1 over the fibre. Suppose for simplicity that $d = n = \mathrm{dim} M$ (as for the case when $E = TM$. Then the Euler number of E is given by evaluating τ on the zero section. So if we have a way to write down a representative τ for the Thom class we get an integral formula for the Euler number. The theory of *equivariant cohomology* gives a machinery for writing down such a representative.

Suppose that a compact Lie group G acts on a manifold X. Write $\mathbf{g}$ for the Lie algebra of G and $S(\mathbf{g}^*)$ for the polynomial functions on $\mathbf{g}$. Let

$$C_G(X) = (\Omega^*_X \otimes s^*(\mathbf{g}^*)^G,$$

where Ω^*_X is the set of forms on X, the factor $s^*(\mathbf{g}^*)$ is the symmetric product and $(\ )^G$ denotes the G-invariants. Here G acts in the obvious way on Ω^*_X and by the co-adjoint action on the other factor. At a point $x \in X$ we have the infinitesimal action $\rho_x : \mathbf{g} \to TX_x$ which we view as in $\mathbf{g}^* \otimes TX_x$. This defines

$$I_x : \Lambda^q T^* X_x \otimes s^p(\mathbf{g}^*) \to \Lambda^{q-1} T^* X \otimes s^{p+1}(\mathbf{g}^*)$$

and hence

$$I : C_G(X) \to C_G(X).$$

We define a grading on $C^G(X)$ so that elements of $\mathbf{g}^*$ have degree 2. Then I has degree $+1$. The first basic fact is that $(d + I)^2 = 0$ on $C_G(X)$ (where d is defined in the obvious way acting trivially on $s^*(\mathbf{g}^*)$). So we can define the equivariant cohomology $H^*_{G,dR}$ as the cohomology of $(C_G(X), d + I)$. We have obvious maps

$$s^*(\mathbf{g}^*) \to H^*_{G,dR}(X) \to H^*_{dR}(X),$$

where H^*_{dR} is de Rham cohomology.

For example suppose that $G = S^1$ and the action on X is generated by a vector field v. Then $s^*(\mathbf{g}^*)$ can be viewed as the polynomial ring $\mathbf{R}[t]$ and $d + I$

can be written as $d + ti_v$. The statement that $(d + I)^2 = 0$ is the fact that $di_v + i_v d$ vanishes on the invariant forms, since it gives the Lie derivative in v.

Another point of view is to regard elements of $C_G(X)$ as equivariant polynomial maps $f : \mathbf{g} \to \Omega^*_X$. Then the differential of the map f is the map Df given by

$$Df(\xi) = d(f(\xi)) + i_{\rho(\xi)}(f(\xi)).$$

Now change direction and consider a general differentiable bundle $\pi : \mathcal{X} \to M$ with fibre a manifold X. Suppose that we have a "connection" on this bundle, in the sense of a family of horizontal subspaces $H \subset T\mathcal{X}$. Recall that this gives a tensor

$$\Phi \in \Lambda^2 H^* \otimes V.$$

We can decompose the forms on $\mathcal{X}$ in the familiar way $\Omega^*_{\mathcal{X}} = \bigoplus \Omega^{p,q}$ (with "q factors in the fibre direction"). The exterior derivative on $\Omega^{p,q}$ has components d_V mapping to $\Omega^{p,q+1}$,d_H mapping to $\Omega^{p+1,q}$ and $\tilde{Phi}$ mapping to $\Omega^{p+2,q-1}$ where $\tilde{\Phi}$ is the algebraic action of Φ by wedge product and contraction.

Specialise the above to the case when $P \to M$ is a principal G-bundle and $\mathcal{X} \to M$ is the bundle associated to the action of G on X. Fix a connection on P; the curvature F is a section of $\mathrm{ad}P \otimes \Lambda^2 T^* M$. At a point m in M choose a trivialisation of the fibre of P so the fibre of $\mathcal{X}$ is identified with X and the curvature with an element of $\Lambda^2 T^* M_m \otimes \mathbf{g}$. Then we get a map from $\Omega^*(X) \otimes s^*(\mathbf{g}^*)$ to sections of $\Lambda^* T\mathcal{X}$ restricted to the fibre by evaluating a polynomial on the curvature and using the wedge product on $\Lambda^2 T^* M$. For the invariant elements, in $C_G(X)$ the result is independent of the choice of trivialisation so we get a map

$$C_G(X) \to \Omega^*(\mathcal{X})$$

and one checks that this is a cochain map with respect to $d+I$ and the ordinary exterior derivative on $\Omega^*(\mathcal{T}\mathcal{X})$. This comes down to the fact that the component $\tilde{\Phi}$ corresponds to I.

The conclusion is that if we can extend an ordinary closed form to an "equivariantly closed" form in $C^*_G(X)$ then we can extend over the total space of any bundle $\mathcal{X}$.

With this background in place we return to the Thom class. Take $X = \mathbf{R}^n$ and $G = SON(n)$. We identify the Lie algebra $\mathbf{g}$ with $\Lambda^2 \mathbf{R}^n$ and define $f : \mathbf{g} \to \Omega^*(X)$ by

$$f(\xi) = *e^{-r^2/2} \exp(\xi),$$

where the exponential is defined in the algebra Λ^{even}. This the *Matthai-Quillen* form. One checks that $Df = 0$ in the sense above. So if $E \to M$ is an oriented Euclidean rank n vector bundle with connection the discussion above gives a closed n-form τ_0 in the total space of E. Strictly this is not a Thom form since it is not compactly supported. One can get around that by choosing a suitable diffeomorphism $g : B^n \to \mathbf{R}^n$ equal to the identity near the origin and considering $g^*(\tau_0)$, extended by zero over the complement of B^n. The very rapid decay of $e^{-r^2/2}$ implies that this is a smooth form and nothing is changed over the zero section.

The upshot is a formula, with n even and for an oriented $\mathbf{R}^n$ bundle with connection over an oriented n-manifold M

$$\mathrm{Euler}(E) = c \int_M \mathrm{Pfaff}(F),$$

where the Pfaffian Pfaff is the polynomial of degree n on Λ^2 given by $\xi^n/n!$. Specialising to the tangent bundle of a Riemannian n-manifold (n even) this is the Gauss-Bonnet formula

$$\chi(M) = c \int_M \mathrm{Pfaff}(\mathrm{Riem}).$$

Here we take $\mathrm{Riem} \in \Lambda^2 \otimes \Lambda^2$ and

$$n!\mathrm{Pfaff}(\mathrm{Riem}) = (\mathrm{Riem})^n \in \Lambda^{2n} \otimes \Lambda^{2n} = \mathbf{R}.$$

The curvature tensor in dimension 4

Let M be an oriented Riemannian 4-manifold. We decompose the 2-forms into self-dual and anti-self-dual parts $\Lambda^2 + \Lambda^+ \oplus \Lambda^-$. The curvature lies in $s^2(\Lambda^2)$ so it has components

$$V_\pm \in s^2(\Lambda^\pm), Z \in \Lambda^+ \otimes \Lambda^-.$$

The component Z can be identified with the trace-free part Ric_0 of the Ricci curvature under an isomorphism of representations of $SO(4)$:

$$\Lambda^+ \otimes \Lambda^- = s^2_0(\mathbf{R}^4).$$

Recall that in general the space of curvature tensors can be viewed as the kernel of the wedge product map $s^2(\Lambda^2) \to \Lambda^4$. In 4 dimensions this gives the condition that the traces of $V_\pm$ are equal: they are both a multiple of the scalar curvature S. The trace-free parts $W_\pm$ of $V_\pm$ are components of the Weyl tensor. So the curvature tensor has four pieces $S, \mathrm{Ric}_0, W_+, W_-$. When M is compact the Gauss-Bonnet formula becomes

$$\chi(M) = \frac{1}{8\pi^2} \int_M \frac{1}{24}|s|^2 + |W_+^2 + |W_-|^2 - \frac{1}{2}|\mathrm{Ric}_0|^2,$$

(take the some of the constants here with a grain of salt, there is some dependence on conventions in defining the norms).

A manifold with $\mathbf{Ric}_0 = 0$ is called Einstein. We see that if M is Einstein then $\chi(M) \geq 0$ with equality if and only if M is flat. For example the only Einstein metrics on a 4-torus are flat.

Another topological invariant of a (compact, oriented) 4-manifold is the signature $\sigma(M)$. This is the signature of the quadratic form on $H^2(M, \mathbf{R})$

given by cup product. A substantially deeper theorem (of Hirzebruch) gives an integral formula for the signature:

$$\sigma(M) = \frac{1}{12\pi^2} \int_M |W^+|^2 - |W_-|^2.$$

Assuming this we get the *Hitchin-Thorpe inequality*, for an Einstein manifold

$$|\sigma(M)| \leq \frac{2}{3}\chi(M),$$

with equality holding if and only if $S = 0$ and one of W^+, W^- is everywhere zero. If M is simply connected, such metrics are *hyperkähler*. For example a K3 surface, which admits such a metric by a theorem of Yau, has $\chi = 24, \sigma = -16$.

Further remarks on equivariant cohomology

An algebraic topology point of view on the question of extending a class in $H^*(X)$ over a bundle $\mathcal{X}$ with structure group G is to consider the universal case. We have a universal principal bundle $EG \to BG$ and associated bundle $\mathcal{X}_U = EG \times_G X \to BG$ with fibre X. Any bundle $\mathcal{X} \to M$ as above is pulled back by a map $X \to BG$, so a class in $H^*(\mathcal{X}_U$ pulls back to a class in $H^*(\mathcal{X})$. In algebraic topology one defines equivariant cohomology by $H^*_G(X) = H^*(\mathcal{X}_U)$ and the "equivariant de Rham theorem" asserts that this agrees with $H^*_{G,dR}(X)$ in the case of a smooth manifold X and compact Lie group G.

From the definition, we have obvious maps

$$H^{BG} \to H^*_G(X) \to H^*(X).$$

This connects with the previous discussion by the *Chern-Weil isomorphism*

$$H^*(BG) = s^*(\mathbf{g}^*)^G,$$

the invariant polynomials on the Lie algebra. Characteristic classes of G bundles are given by elements of $H^*(BG)$ and are represented by differential forms obtained by applying the corresponding invariant polynomial to the curvature. The formula for the $\sigma(M)$ above is a composite of the Chern-Weil description of the first Pontrayagin class p_1 and the Hirzebruch signature which (in dimension 4) identifies the signature of a 4-manifold with $p_1/3$.

6 Introduction to PDE and analysis techniques

Linear theory

Let (M, g) be a compact Riemannian manifold of dimension n. We consider an operator of the form

$$Lf = \Delta f + \nabla_X f + Vf,$$

where X is a fixed vector field and V a fixed smooth function. More generally the following discussion applies to *elliptic linear operators* but we will not pause to define this condition. The adjoint operator L^* is characterised by the condition that

$$Lf, g\rangle = \langle f, L^* g\rangle,$$

for all functions f, g and the L^2 inner product. In the case at hand

$$L^* f = \Delta f - \nabla_X f + (V + \mathrm{divX}) f.$$

The fundamental result (the "Fredholm alternative") is that we can solve the equation $Lf = \rho$ for f if any only if ρ is orthogonal to the kernel of L^*. For example if $L = L^* = \Delta$ we can solve the equation if and only if the integral of ρ is zero.

We also want solutions with estimates and for these we introduce standard function spaces:

- For $p > 1$; L^p_k: functions with the first k derivatives in L^p,
- For $0 < \alpha < 1$ the Hölder spaces $C^{r,\alpha}$ with the first r derivatives α-Hölder continuous.

Let E denote one of these function spaces. There are inclusions between these spaces which are governed by two numbers $\nu(E), W(E)$. The number $\nu(E)$ is the "number of derivatives" k or $r + \alpha$ respectively. The number $W(E)$ is the scaling weight with respect to scaling the metric g. Thus $W(L^p_k) = k - n/p$ and $W(C^{r,\alpha}) = r + \alpha$. Then if $W(E) \geq W(E')$ and $\nu(E) \geq \nu(E')$ there is an inclusion $E \subset E'$. These inclusions are related to isoperimetric inequalities. For example in the case of functions of compact support on $\mathbf{R}^n$ we have an inequality

$$\|f\|_{L^{n/n-1}} \leq C_n \|\nabla f\|_{L^1},$$

and by considering smoothings of the characteristic function of a bounded domain Ω one sees that

$$\mathrm{Vol}(\Omega)^{(n-1)/n} \leq C_n \mathrm{Area}(\partial\Omega).$$

Our operator L defines a bounded map $L^p_{k+2} \to L^p_k$ and $C^{k+2,\alpha} \to C^{k,\alpha}$. The force of ellipticity is that we get estimates, for example in the L^p_k case

$$\|f\|_{L^p_{k+2}} \leq C \left(\|Lf\|_{L^p_k + \|f\|_{L^1}} \right).$$

Remarks

- . The theory is simpler for the case $p = 2$ and very often this is all one needs.
- The inequality holds in the sense that if the right hand side is finite then so is the left hand side: i.e. a function which a priori lies in some weaker space is actually in L^p_{k+2}. This is *elliptic regularity.*

  The inverse and implicit functions theorems apply to smooth maps between Banach spaces (such as those above). This means that we can often study small deformations of a nonlinear problem by applying these results in suitable functions spaces and then invoking elliptic regularity. For example if there is a closed geodesic $\gamma : S^1 \to M$ for the metric g and if there are no Jacobi fields along γ then arguments of this kind show that for small perturbations $\tilde{g}$ of g we can find a corresponding perturbed closed geodesic $\tilde{\gamma}$.

More substantial results are typically proved by combing these techniques with *a priori* estimates, as in the following examples.

Example 1: Constant negative curvature surfaces

We suppose given a compact 2-manifold with Euler characteristic $\chi < 0$ and some metric g. The problem is to find a conformally equivalent metric $= e^u g$ with curvature $K()$ equal to -1. One finds that

$$K() = e^{-u}(K(g) - \Delta u).$$

As a preliminary step it is easy (exercise) to find a conformal change making the curvature negative, so we may as well suppose that $K(g) = -F$ with $F > 0$. The equation to be solved is

$$-\Delta u + e^u = F,$$

in other words if $\mathcal{T}$ is the nonlinear operator $\mathcal{T}(u) = -\Delta u + e^u$ we want to show that $\mathcal{T}$ maps onto the set of poistive functions. We solve this using the continuity method: choose a path F_t of positive functions from $F_0 = 1$ to $F_1 = F$. When $t = 0$ there is a solution $u = 0$. Let $S \subset [0,1]$ be the set of parameters t for which a solution exists: we need to show that S is open and closed. The openness follows from the open mapping theorem: the derivative of $\mathcal{T}$ at u_0 is the linear map $v \mapsto -\Delta v + e^{u_0} v$ and this is surjective by the Fredholm alternative. To establish closedness we obtain *a priori* estimates on solutions, starting with the C^0 norm. This is done very easily using the *maximum principle*. Then we have L^2 bounds on e^u and hence an L^2_2 bound on u. Using the embedding theorems and elliptic estimates repeatedly one gets estimates on all derivatives of u.

Example 2: isometric embedding of positively curved surfaces

For this problem we are given a metric g on a manifold M, diffeomorphic to the 2-sphere, with curvature $K > 0$ and we seek an isometric embedding $(M, g) \to \mathbf{R}^3$. This problem was studied by Weyl in 1916 and the solution completed by Nirenberg and Pogerolov in the 1950's. (The conclusion is that such an isometric embedding exists and is unique up to Euclidean motions of $\mathbf{R}^3$. Note that this result is special to dimension 2: for $n > 2$ a generic Riemannian n-manifold cannot be isometrically embedded in $\mathbf{R}^{n+1}$, even locally.)

The problem can be set up as a PDE for a map $f : M \to \mathbf{R}^3$. Alternatively, by classical results from surface theory, the existence of such a map is equivalent to finding the second fundamental form $B \in \Gamma(s^2 T^* M)$ satisfying the conditions

$$\det(B) = K \quad B = 0,$$

where is the linear operator which is the composite of teh covriant derivative

$$\nabla : \Gamma(T^* \otimes T^*) \to \Gamma(T^* \otimes T^* \otimes T^*),$$

with the skew-symmetrisation map

$$T^* \otimes T^* \otimes T^* \to T^* \otimes \Lambda^2 T^* = T^*.$$

The strategy is to first check (using the uniformisation theorem) that the space of metrics of positive curvature is connected. Then we can apply the continuity method to a 1-parameter family g_t with $K(g_t) > 0$. The pair of equations $\det B = K, B = 0$ make up an elliptic equation for B (a notion we have not defined in this course) but there are subtleties in proving the "openness". We will not go into that here but move on to the "closedness". The key step here is to derive an *a priori* bound for the mean curvature $H = \mathrm{Tr} B$. For this one uses an identity

$$\sum_{ij} (H g_{ij} - B_{ij}) \nabla_i \nabla_j H = \Delta K + |\nabla B|^2 - |\nabla H|^2 + K(H^2 - 4K).$$

The quadratic form $H g_{ij} - B_{ij}$ is positive definite so at a point where H attains its maximum we have

$$\Delta K + |\nabla B|^2 + K(H^2 - 4K) \leq 0,$$

(since ∇H vanishes at this point). This gives an upper bound on H. The "bootstrapping" arguments to obtain bounds on higher derivatives are also more delicate but we will not go into further details here.

The identity above can be understood in the following way.

7 The spectrum of a Riemannian manifold

For (M^n, g) we consider the eigenvalue equation $\Delta \phi = -\lambda \phi$ so $\lambda \geq 0$ and $\lambda = 0$ corresponds to the constants. The eigenfunctions ϕ_λ give an L^2 orthonormal basis.

There are many results on the first eigenvalue $\lambda_1 > 0$, it is the optimal constant in teh Poincare inequality

$$\|\nabla f\|^2 \geq \lambda_1 \|f\|^2,$$

for functions of integral zero.

For example:

(Obata) If Ricci $\geq (n-1)$ then $\lambda_1 \geq n$ withe equality if and only if M is the standard sphere.

(Cheeger) Let h be the best constant such that for all domains $\Omega \subset M$ with $\mathrm{Vol}(\Omega) \leq \mathrm{Vol}(M \setminus \Omega$ we have

$$\mathrm{Area}(\partial\Omega) \geq h\mathrm{Vol}(\Omega).$$

Then $\lambda_1 \geq h^2/4$.

To prove the Obata theorem recall the identity

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla\nabla f|^2(\nabla\Delta f, \nabla f) + \mathrm{Ric}(\nabla f, \nabla f).$$

Let $\nabla_0^2 f$ be the trace-free part of the $\nabla\nabla f$. Then

$$|\nabla\nabla f|^2 = |\nabla_0^2 f|^2 + n^{-1}|\Delta f|^2.$$

Suppose $\Delta f = -\lambda f$ and integrate this identity over M. We get

$$\|\nabla_0^2 f\|^2 + (n-1)\lambda\|f\|^2 \leq \lambda^2(1-n^{-1})\|f\|^2,$$

so if $\lambda > 0$ we get $\lambda \geq n$.

In the case of equality $\nabla_0^2 f = 0$. This implies that the vector field gradf is a conformal Killing field. We also have $\nabla^2 f = -g$ which implies that along any geodesic $\gamma(t)$ we have $f'' = -f$. Using these observations it is not hard to show that M is the standard sphere.

The proof of the Cheeger result uses the co-area formula. Let g be any positive function which vanishes on $\partial\Omega$ and let $\Omega_c \subset \Omega$ be the set where $g \geq c$. Then

$$\int_\Omega g = \int \mathrm{Vol}(\Omega_c)dc,$$

and

$$\int_\Omega |\nabla g| = \int \mathrm{Area}(\partial\Omega_c)dc.$$

Suppose that $\Delta f = -\lambda f$ on M and f has integral zero. We can suppose that the volume of the set $\Omega = \{f \geq 0\}$ is at most half the volume of M. Since f vanishes on $\partial\Omega$ we have

$$\int_\Omega |\nabla f|^2 = \lambda \int_\Omega f^2.$$

Now set $g = f^2$ on Ω, so

$$\int_\Omega |\nabla g| = 2\int_\Omega f\nabla f \leq 2\|f\|\|\nabla\| \leq 2\lambda^{1/2}\|f\|^2$$

where the norms are L^2 over Ω. Using the co-area formula and the definition of h, applied to the Ω_c, we get

$$\|f\|^2 = \int_\Omega g \leq h^{-1} \int_\Omega |\nabla g|,$$

and the result follows.

The heat kernel

The heat equation is $\partial_t f = \Delta f$. For initial data f_0 we can write the solution as $f_t = H_t f_0$ where H_t is an operator on $C^\infty(M)$. This has a spectral description

$$H_t \phi_\lambda = e^{-\lambda t}\phi_\lambda,$$

and an integral description

$$H_t f(x) = \int_M K_t(x,y) f_0(y) dy.$$

So

$$K_t(x,y) = \sum_\lambda \phi_\lambda(x)\phi_{lambda}(y) e^{-\lambda t},$$

and the trace of H_t has two descriptions

$$\sum_\lambda e^{-\lambda t} = \int_M K_t(x,x) dx.$$

We want to get an asymptotic expansion for $K_t(x,y)$ as $t \to 0$ and close to the diagonal $x = y$. For fixed $y_0 \in M$ the function $K_t(\ ,y_0)$ can be characterised as the solution of the heat equation which tends to the δ-function at y_0 as $t \to 0$.

On $\mathbf{R}^n$ the heat kernel is $(4\pi t)^{-n/2} e^{-r^2/4t}$. Fixing a point $y_0 \in M$ this makes sense on M with $r = d(x, y_0)$; let Ψ be the resulting function. Write

$$\Delta r^2 = n + 4S,$$

so S vanishes at y_0 and has a Taylor series expansion about y_0 (with first term given by the Ricci tensor at y_0). Now we have

$$(\partial_t - \Delta)\Psi = t^{-1} S \Psi.$$

Our strategy is to find functions $a_0(x), a_1(x) \ldots$ so that for each k

$$(\partial_t - \Delta)\left(\Psi(a_0 + a_1 t \ldots + a_k t^k)\right) = O(t^k),$$

and $a_0(y_0) = 1$. If we have done this then some fairly straightfoward analysis shows that indeed an asymptotic expansion of the true heat kernel $K_t(x, y_0)$. In particular

$$K_t(y_0, y_0) \sim (4\pi)^{-n/2} \sum a_i(y_0) t^{-n/2+i}.$$

To find the a_i we compute

$$(\partial t - \Delta)(a(x)t^p\Psi) = \left(\frac{1}{4}r\partial_r a + pa + Ea\right) t^{p-1}\Psi + \Delta a\Psi t^p.$$

The discussion is a bit different for $p = 0$ and $p > 0$. For $p = 0$ we set $a_0 = e^f$ so we want to solve the equation

$$r\partial r f = -2S.$$

Since E vanishes at y_0 there is a unique solution to this ODE with $f(y_0) = 0$. In fact since

$$r^{2q}e^{-r^2/4t} \leq C_q t^q,$$

the whole discussion only requires solving the equation as a Taylor series and everything can be done entirely algebraically. The point is that $r\partial r$ acts as multiplication by d on polynomials of degree d.

Now we have found a_0 so at the beginning of the step $p = 1$ we have an error term

$$(\partial_t - \Delta)(a_0\Phi) = E_1\Phi + O(t)\Phi.$$

We want to choose a_1 so that

$$(\partial_t - \Delta)a_1(x)t\Phi) = E_1\Phi + O(t)\Phi,$$

which is to say

$$\frac{1}{2}r\partial_r a_1 + a_1 + Sa_1 = E_1.$$

We write $a_1 = e^f b_1$ and the equation for b becomes

$$\frac{1}{2}r\partial_r b_1 + b_1 = e^{-f}E_1.$$

The operator on the left acts as multiplication by $d/2 + 1$ on polynomials of degree d so we can solve this equation to complete the step $p = 1$, and so on.

The conclusion is that

$$\sum_\lambda e^{-\lambda t} \sim (4\pi t)^{-n/2}(\sum A_p t^p),$$

where $A_p = \int_M a_p(y, y)dy$.

Applications

- *The Weyl asymptotic formula.*

  From the leading term of the asymptotic expansion we get

  $$\sum_\lambda e^{-\lambda t} \sim (4\pi)^{-n/2}\mathrm{Vol}(M)t^{-n/2},$$

as $t \to 0$. From this a general *Tauberian Theorem* from analysis shows that

$$N(\mu) \sim (4\pi)^{-n/2} \frac{\mathrm{Vol}(M)}{\Gamma(n/2+1)} mu^{n/2},$$

where $N(\mu)$ is the number of eigenvalues $\leq \mu$. For example let Λ be a lattice in $\mathbf{R}^n$ and $M = \mathbf{R}^n/\Lambda$. The (complex) eigenfunctions are $\exp(2\pi i \mu(x))$ where μ is in the dual lattice Λ^*, with eigenvalue $|\mu|^2$. The volume of $\mathbf{R}^n/\Lambda^*$ is $\mathrm{Vol}(M)^{-1}$ and $\Gamma(n/2+1)$ is the volume of the unit ball in $\mathbf{R}^n$. So the formula becomes the standard asymptotic result counting lattice points in a large ball.

– *Zeta functions*

For $\Re(s)$ large we can define

$$\zeta_M(s) = \sum_{\lambda>0} \lambda^{-s}.$$

This can be expressed as

$$\zeta_M(s) = \int_0^\infty \sum_{\lambda>0} e^{-\lambda t} t^{s-1} dt.$$

The asymptotic expansion for the trace of the heat kernel implies that ζ_M has a meromorphic continuation over $\mathbf{C}$ with no pole at $s = 0$. For a similar *finite* sum

$$\zeta_0(s) = \sum \lambda^{-s} = \sum \exp(-s \log \lambda)$$

we would have

$$\zeta_0'(s) = -\sum \log lambda \exp(-s \log \lambda),$$

so

$$\zeta_0'(0) = -\sum \log \lambda$$

and

$$\exp(-\zeta_0'(0)) = \Pi\lambda.$$

For the infinite set of eigenvalues the product is wildly divergent but $\zeta_M'(0)$ is defined, via analytic continuation. We can regard $\exp(-\zeta_M'(0))$ as the regularised determinant of the Laplace operator (on functions of integral 0).

– *Index formulae* (Assuming some knowledge of Hodge Theory)

The same discussion applies if we add lower order terms to the Laplacian, for example $\Delta + V$. Similarly it applies to operators of Laplace type acting on vector bundles. Suppose that $n = 2$ so M is a surface and consider the Laplace operator $\Delta_1 = -(dd^* + d^*d)$ acting on

1-forms. If ϕ is a eigenfunction of the scalar Laplacian Δ on functions with eigenvalue $\lambda > 0$ then $d\phi$ is an eigenfunction of Δ_1, with the same eigenvalue. Similarly if ψ is an eigenfunction of Δ_1 with nonzero eigenvalue then $d^*\psi$ is an eigenfunction of Δ, with the same eigenvalue. From this one sees that the non-zero spectrum of Δ_1 is two copies of that of Δ. By the Hodge Theorem the dimension of the zero eigenspace of Δ_1 is the Betti number b_1. We conclude that for any t

$$2\mathrm{Tr}e^{t\Delta} - \mathrm{Tr}e^{t\Delta_1} = 2 - b_1,$$

the Euler characteristic $\chi(M)$. Taking the t^0 term in the asymptotic expansions we see that $\chi(M)$ can be written as the integral of a computable expression constructed from the curvature tensor of M and this turns out to be the Gauss-Bonnet formula.

The same applies much more generally. For example on an oriented manifold of dimension $4k$ we decompose the $2k$ forms into self-dual and anti-self dual parts. This commutes with the Laplacian so we can write Δ^+, Δ^- for the Laplace operator on the $\pm$ self dual forms. Simple Hodge theory arguments show that the nonzero spectra of these operators agree and the difference of the zero eigenspaces is the signature $\sigma(M)$ of M. So we get

$$\sigma(M) = \mathrm{Tr}e^{t\Delta^+} - \mathrm{Tr}e^{t\Delta^-},$$

for all t and for $t \to 0$ this is given by a computable formula involving the curvature tensor.

What is harder is to identify the formula which arises, for example in dimension 4

$$\sigma(M) = (12\pi^2)^{-1} \int |W^+|^2 - |W^-|^2,$$

as we mentioned in Section 5.

8 The Selberg Trace formula

In this section we consider a compact hyperbolic surface $M = H^2/G$ where $G \subset PSL(2, \mathbf{R})$ and H^2 is the upper half space with metric

$$y^{-2}(dx^2 + dy^2).$$

To begin with we work with the heat kernel $\exp(t\Delta)$ but later we will extend to other operators. The trace formula is an *exact* formula of the shape

$$\mathrm{Tr}e^{t\Delta} = \int_M W(t) + \sum_\gamma F(l(\gamma)),$$

where γ runs over primitive closed geodesics $l(\gamma)$ is the length of γ and W, F are functions of one real variable to be found later. (A closed geodesic is *primitive* if it does not factor through a map $S^1 \to S^1$ of degree $d > 1$.)

The proof has a "topological" component which shows that there is a formula of this shape and a "calculus" component to identify the exact functions which arise. We begin with the topological part and we assume that we have found the heat kernel $\kappa_t(x, y)$ on H^2. Clearly for each t this should be a function $k_t(d(x, y))$ of the hyperbolic distance $d(x, y)$ between x, y. Now (ignoring convergence questions)

$$\sum_{g \in G} \kappa_t(x, gy),$$

is preserved by G acting on x, y so descends to $M \times M$ and this is the first formula for the heat kernel K_t on M.

Let $\tilde{M}$ be the space of pairs $(x, [\delta])$ where $x \in M$ and $[\delta] \in \pi_1(M, x)$. Equivalently we can this to be the set of pairs (x, δ) where δ is a geodesic loop based at x. We have an obvious covering map $p : \tilde{M} \to M$ so $\tilde{M}$ is also a hyperbolic surface. We also have a map $\tilde{L} : \tilde{M} \to \mathbf{R}$ given by the length of δ. Now one sees that for $x \in M$

$$K_t(x, x) = \sum \tilde{x} \in p^{-1}(x) k_t(L(\tilde{x})),$$

which implies that

$$\int_M K_t(x, x) = \int_{\tilde{M}} k_t(\tilde{L}).$$

The set $\tilde{M}$ is not connected. Let Ω be the set of *free* homotopy classes of maps $S^1 \to M$. If we fix any base point $x_0 \in M$ these correspond to conjugacy classes in $\pi_1(M, x_0)$. Then $\tilde{M}$ splits into components corresponding to elements of Ω. The trivial homotopy class gives a copy of M in $\tilde{M}$. The contribution from this to the integral above above is

$$\int_M W_t$$

where ${}_t = k_t(0)$.

The non-trivial free homotopy classes correspond to (non-constant) closed geodesics in M. Let γ be closed geodesic of length $L(\gamma) > 0$, ω the free homotopy class and $\tilde{M}_\omega$ the corresponding component. Choosing a base point and a representative $g \in G = \pi_1(M, x_0)$ for ω. One sees that $\tilde{M}_\omega$ can be identified with H^2/Z where $Z \subset G$ is the centraliser of $g \in G$. From the structure of $PSL(2, \mathbf{R})$ one sees that Z is infinite cyclic. To simplify suppose first that γ is primitive, which implies that Z is generated by g. Then as a hyperbolic surface H^2/Z is determined entirely by the length $L = L(\gamma)$ so let us call it Σ_L. Topologically, this is a cylinder $\mathbf{R} \times S^1$.

The contribution from this component is given as follows. For $z \in \Sigma_L$ let δ be the geodesic loop based at z which represents the fixed generator of $\pi_1(\Sigma_L)$, of length $L(\delta)$. Then the contribution is

$$\int_{\tilde{M}_\omega} k_t(\tilde{L}) = \int_{\Sigma_L} k_t(L(\delta)).$$

A little thought shows that the imprimitive classes can be handled as follows. For primitive γ of length $L(\gamma)$ as above and integer $m \geq 1$ and $z \in \Sigma_L$ let δ_m be the geodesic loop based at z which represents m times the generator of π_1. Let $f_m(z)$ be the length of δ_m. Then we set

$$F(L) = \sum_{m=1}^{\infty} \int_{\Sigma_L} k_t(f_m(z))dz,$$

and we obtain the formula (*) (using the fact that an imprimitive class is a multiple of a primitive class).

This completes the "topological" part of the discussion.

The calculus part involves two questions.

- Find the heat kernel function k_t for H^2.
- Find a more explicit formula for $F(L)$.

Changing point of view, we choose an arbitrary function k (with suitable decay properties etc) and consider the integral operator I on H^2 defined by the kernel $k(d(x,y))$. We claim that for each λ there is a $P(\lambda)$ with the following property: if ϕ is a solution of $\Delta\phi = -\lambda\phi$ on H^2 then $I(\phi) = P(\lambda)\phi$. This follows from the fact that there for each y in H there is a unique solution of $\Delta\psi = -\lambda\psi$ with $\psi(y) = 1$ and with $\psi(x)$ a function of the distance $d(x,y)$. It follows from this observation that if we construct an integral operator on M by summing as before it acts as $P(\lambda)$ on the λ eigenspace over M. In other words the operator is $P(-\Delta_M)$ and we get a formula for

$$\mathrm{Tr}(P(-\Delta) = \sum_{\lambda} P(\lambda).$$

So for any k we get a P but what we want is to find k given P. The analogous situation in $\mathbf{R}^n$ is given by the Fourier transform, on rotationally invariant functions.

First we find an explicit formula for P in terms of k.

For $z, w \in H^2$ set

$$D(z,w) = \frac{|z-w|^2}{\mathrm{ImzImw}}.$$

One finds that $D(z,w) = \cosh d(z,w) - 1$. It is easier to work with the function $\kappa(D)$ corresponding to $k(d)$. The Laplace operator on H^2 is

$$y^{-2}(\partial_x^2 + \partial_y^2).$$

So for any $s \in \mathbf{C}$ the function y^s is an eigenfunction with $\lambda = -s(s-1)$. Put $s = \frac{1}{2} + ir$ so $\lambda = r^2 + \frac{1}{4}$ and write $P(r^2 + \frac{1}{4}) = h(r)$. We have to evaluate

$$h(r) = \int_{H^2} \kappa(D(z,i)) y^{1/2+ir} y^{-2} dxdy,$$

which is

$$\int_{H^2} \kappa(\frac{x^2 + (y-1)^2}{y}) y^{1/2+ir} y^{-2} dxdy.$$

Writing $x = y^{1/2}u$ and $y = e^t$ this is

$$h(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \kappa(e^{-t}(e^t - 1)^2 + u^2) e^{irt} dudt.$$

Define an operator A on functions on $\mathbf{R}$ by

$$(Af)(v) = \int_{-\infty}^{\infty} f(v + u^2) du.$$

So the procedure to go from κ to h is:

- Apply A to κ;
- Make the change of variable $v = e^{-t}(e^t - 1)^2 = 2(\cosh t - 1)$;
- Take the Fourier transform.

We can invert the operator A. Let D denote the operation of differentiation. Then A commutes with D and one finds that ADA is 2π times the identity (on functions vanishing at infinity), so $A^{-1} = (2\pi)^{-1}AD$. Using Fourier inversion we get an inversion formula expressing κ in terms of h.

- Take the inverse Fourier transform

$$g(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} h(r) e^{-irt} dr.$$

- Change variable so $g(t) = Q(2(\cosh t - 1))$
- Then

$$\kappa(D) = \int_{-\infty}^{\infty} Q'(D + p^2) dp.$$

The expression for $\kappa(0)$ is simpler. We get

$$\kappa(0) = \int_0^{\infty} rh(r) dr \int_{-\infty}^{\infty} \frac{\sin rt}{\sinh(t/2)} dt.$$

The t integral can be evaluated:

$$\int_{-\infty}^{\infty} \frac{\sin rt}{\sinh tt/2} = \tanh r.$$

Thus

$$\kappa(0) = (2\pi)^{-1} \int_0^\infty h(r) r \tanh r dr.$$

Now we consider the contribution from closed geodesics. Take $L > 0$ and the action on H^2 generated by $z \mapsto e^L z$. The quotient is a model for Σ_L. A fundamental domain is given by $\{z : 1 \le \mathrm{Im} z \le e^L$. The contribution from geodesic loops representing the mth multiple of the generator is

$$\int_{-\infty}^\infty dx \int_1^{e^L} \kappa(\frac{(e^{mL}-1)^2(x^2+y^2)}{e^{mL}y^2} y^{-2} dx dy.$$

Making a change of variables this becomes

$$\frac{1}{2\sinh mL/2)} \int_{-\infty}^\infty dx \int_0^L \kappa(u^2 + \frac{(e^{mL}-1)^2}{e^{mL}} dt.$$

The t integral is trivial and by comparing with the previous calculation we can express things directly in terms of the Fourier transform g of h: the integral is

$$\frac{L}{2\sinh L/2} g(mL).$$

In conclusion we get the Selberg trace formula.

- Start with a function P so we can define $P(-\Delta_M)$ on M and set $h(r) = P(r^2 + 1/4)$.
- Let g be the Fourier transform $g(t) = \int_{-\infty}^\infty h(r) e^{irt} dr$.
- Then

$$\mathrm{Tr} P(-\Delta_M) = (4\pi)^{-1} \mathrm{Area}(M) \int_{-\infty}^\infty h(r) r \tanh r dr + \sum_\gamma l(\gamma) \sum_{m=1}^\infty \frac{g(ml(\gamma))}{\sinh ml(\gamma)/2}.$$

This can be related to at least two other subjects.

- On a general Riemannian manifold we do not an exact formula but asuymptotic relations between the spectrum and closed geodesics, in the same vein as quasi-classical approimation in qunatum mechnaics.
- The theory can be developed from the point of view of infinite dimensional representations of $SL(2, \mathbf{R})$. The parameter r labels the principle series representations and $r \tanh r dr$ is the Plancherel measure.