Differential Geometry, Part 2.

February 28, 2024

Section 4: Special holonomy

For a point p in a connected Riemannian n-manifold M the holonomy group $G \subset SO(TM_p) = SO(n)$ is defined by parallel transport around contractible loops.

Special holonomy is the case $G \neq SO(n)$.

A metric with holonomy $G \subset SO(n)$ is essentially equivalent to a *torsion free G* structure. This means that the tangent bundle is associated to a principal G-bundle $P \to M$ and there is a torsion-free connection on P. In the language of Section 1 it is also equivalent to having a section of the bundle over M with fibre $GL(n)/G$ which is flat to first order.

A symmetric space H/K has holonomy K. There is a classification of which groups can appear as holonomy groups, for non-symmetric, irreducible, manifolds. There are five "classical" families associated with $\mathbf{R}, \mathbf{C}$ and the quaternions.

- $SO(n)$;
- $U(n/2), SU(n/2)$;
- $Sp(n/4).Sp(1), Sp(n/4)$;

and two exceptional cases $G_2 \subset SO(7), \mathrm{Spin}(7) \subset SO(8)$.

Some G_2 algebra.

Let V be an oriented 7-dimensional real vector space and $\phi \in \Lambda^3 V^*$. This defines a quadratic form q_ϕ on V with values in the oriented line $\Lambda^7 V^*$:

$$q_\phi(v)_= i_v(\phi) \wedge i_v(\phi) \wedge \phi.$$

Say ϕ if this form is positive definite. So a positive ϕ defines a conformal structure on V. Fix a Euclidean structure in the conformal class by the normalisation $|\phi|^2 = 7$.

Basic fact: the positive forms make up a single orbit under the action of $GL^+(V)$—-they are all equivalent. The stabiliser of any one is a model for the exceptional Lie group $G_2 \subset SO(7)$ which has dimension $49 - 35 = 14$.
A positive 3-form ϕ and its associated Euclidean structure define a cross product $V \times V \to V$:

$$\phi(x, y, z) = \langle x \times y, z \rangle.$$

The group G_2 can also be obtained as the automorphisms of $(V, \times)$.
The cross-product can be viewed as the main part of octonion multiplication on $\mathbf{R}1 \oplus V$ (in the same way as the cross-product on $\mathbf{R}^3$ is related to quaternion multiplication.)

A G_2 structure on and oriented 7-manifold M^7 is given by a 3-form ϕ which is positive at each point and this defines a Riemannian metric g_ϕ with Levi-Civita connection ∇_ϕ and a 4-form $*_\phi\phi$.

The structure is torsion free if $\nabla_\phi \phi = 0$.

Theorem (Fernandez and Gray) The structure is torsion-free if and only if $d\phi = 0$ and $d(*_{\phi}\phi) = 0$.
We have $d\phi = a(\nabla\phi)$ and $*d*\phi = b(\nabla\phi)$ where a is the wedge product map

$$a : \Lambda^1 \otimes \Lambda^3 \to \Lambda^4,$$

and b is the contraction map

$$b : \Lambda^1 \otimes \Lambda^3 \to \Lambda^2.$$

So we want to show that $a(\nabla\phi) = 0, b(\nabla\phi) = 0$ implies $\nabla\phi = 0$.

We consider the decomposition of Λ^3 as a representation of G_2.
There is a one dimensional piece Λ^3_0 spanned by ϕ.
There is a copy of Λ^1 defined by the contraction $i_v(*\phi)$ for vectors v.
The orthogonal complement is an irreducible representation Λ^3_{27}.
So

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}.$$

An infinitesimal deformation of the 3-form defines a deformation of the metric.

So we have a linear map $\Lambda^3 \to s^2$.

This is projection onto $\Lambda^3_1 \oplus \Lambda^3_{27} = s^2$.

Since ∇ is the Levi-Civita connection we have $\nabla g_\phi = 0$.

This implies that $\nabla \phi \in \Lambda^3_7 \otimes \Lambda^1$.

To prove the Theorem, one has to check that

$$a \oplus b : \Lambda^3_7 \otimes \Lambda^1 \to \Lambda^4 \oplus \Lambda^2$$

is injective.

Now we consider the case of a hyperkähler 4-manifold X i.e. holonomy $Sp(1) \subset SO(4)$. (Equivalently $SU(2) \subset SO(4)$.) This means that we have an action of the quaternions on TX, compatible with the Riemannian metric and preserved by parallel transport.

If I_1, I_2, I_3 is a standard basis for the imaginary quaternions we get 2-forms $\omega_1, \omega_2, \omega_3$:

$$\omega_a(v, w) = \langle I_a v, w \rangle.$$

with $\nabla \omega_i = 0$.

In a similar manner to the G_2 case, a hyperkähler structure can be defined by a triple of closed 2-forms $\omega_1, \omega_2, \omega_3$ with

$$\omega_i \wedge \omega_j = \delta_{ij} \mu,$$

where μ is a volume form on X.

The metric is determined by the ω_i via the formula

$$|v|^2\mu = \sum_{\text{cyclic}} i_v(\omega_i) \wedge i_v(\omega_j) \wedge \omega_k.$$

This can be regarded as a dimension reduction of the G_2-theory.
Take $M^7 = X \times \mathbf{R}^3$ with co-ordinates t_i on $\mathbf{R}^3$.
Given such a triple we define

$$\phi = dt_1 dt_2 dt_3 - \sum \omega_i dt_i,$$

Then

$$*\phi = \mu + \sum_{\text{cyclic}} \omega_i \, dt_j dt_k$$

from which we see that $d\phi = 0, d * \phi = 0$.
From the point of view of holonomy groups, this corresponds to an embedding $Sp(1) \subset G_2$.

Digression
$\pm$-self-duality and the curvature tensor of an oriented Riemannian 4-manifold.
Recall that the curvature tensor of a Riemannian manifold lies in the kernel of $s^2(\Lambda^2) \to \Lambda^4$.
On an oriented Riemannian 4-manifold we have $* : \Lambda^2 \to \Lambda^2$ with $*^2 = 1$ so there is a decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-.$$

So the curvature tensor has components in
$s^2(\Lambda^+), \Lambda^+ \otimes \Lambda^-, s^2(\Lambda^-)$.
The condition that it lies in the kernel of the map to Λ^4 is that the traces of the components in $s^2(\Lambda^\pm)$ are the same: they are equal to $1/6$ times the scalar curvature.

The component in $\Lambda^+ \otimes \Lambda^-$ can be identified with the trace-free part of the Ricci curvature.

The components in $s_0^2(\Lambda^\pm)$ are the $\pm$ self-dual parts $W^\pm$ of the Weyl curvature.

On a hyperkähler 4-manifold the bundle Λ^+ is flat.

It follows that $\mathrm{Ricci} = 0$, $W^+ = 0$. The curvature tensor is just given by $W^- \in s_0^2(\Lambda^-)$.

The Gibbons-Hawking construction

Let X be a hyperkähler 4-manifold with an action of S^1, preserving the ω_i. Let v be the vector field generating the action. The ω_i are symplectic forms so at least locally there are Hamiltonian functions H_i defining a map $X \to \mathbf{R}^3$. Away from the fixed points of the action this is a fibration over $U \subset \mathbf{R}^3$, with fibres given by the S^1-orbits.

The span of $I_1 v, I_2 v, I_3 v$ defines a connection on this S^1-bundle, with a connection 1-form α on X.

Take standard co-ordinates x_i on $\mathbf{R}^3$. By construction,

$$\omega_i = \alpha \wedge dx_i + \Omega_i,$$

where Ω_i is the lift of a 2-form on $U \subset \mathbf{R}^3$.
The orthogonality condition means that

$$\Omega_i = \phi \, dx_j dx_k,$$

for a positive function ϕ on U.

Let F be the curvature of the connection, a closed 2-form on U. The lift of F is $d\alpha$.
The condition that the ω_i are closed is that $F = *d\phi$. So ϕ is a *harmonic function* on U.
Conversely, given a positive harmonic function on a domain $U \subset \mathbf{R}^3$ satisfying the integrality condition

$$\int_\Sigma *df \in 2\pi\mathbf{Z}$$

for all 2-cycles Σ in U, we can construct a hyperkähler 4-manifold with S^1-action.

The Riemannian metric is

$$g = \phi^{-1}\alpha^2 + \phi \sum dx_i^2.$$

In particular $|v|_g^2 = \phi^{-1}$.

Take $U = \mathbf{R}^3 \setminus \{0\}$ and $\phi(x) = \frac{1}{2|x|}$. Then you find the the 4-manifold is $\mathbf{C}^2 \setminus \{0\}$ with the flat metric and the circle action

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

In standard complex co-ordinates z, w one can take

$$\omega_1 = \frac{i}{2}(dzd\overline{z} + dwd\overline{w}) \quad \omega_2 + i\omega_3 = dzdw.$$

If we take $\phi(x) = \frac{\mu}{2|x|}$ for a positive integer ν then the 4-manifold we construct is the quotient of $\mathbf{C}^2 \setminus \{0\}$ by the cyclic subgroup $C_\nu \subset S^1$.

Now take $\nu > 1$ distinct points $p_a \in \mathbf{R}^3$ and the function

$$\phi(x) = \sum_a \frac{1}{2|x - p_a|}.$$

The construction produces a complete hyperkähler 4-manifold X with an S^1 action having ν fixed points. We have a smooth map

$$\mu : X \to \mathbf{R}^3 = X/S^1.$$

The structure of X is asymptotic at infinity to that on the cone $\mathbf{C}^2/C_\nu$.

Suppose that the line segment l_{ab} from p_a to p_b does not contain any other of the ν points.

Then $\pi^{-1}(l_{ab}) = \Sigma_{ab}$ is a 2-sphere in X of self-intersection -2.

The complex structures on X are parametrised by the directions in $\mathbf{R}^3$.

The sphere Σ_{ab} is holomorphic with respect to the complex structure defined by the the direction of l_{ab}.

Fundamental example when $\nu = 2$. Then X is the Eguchi-Hanson manifold. It has two descriptions as a complex manifold (for different complex structures).

- The smooth affine quadric $z_1^2 + z_2^2 + z_3^2 = 1$ in $\mathbf{C}^3$. This does not contain any compact holomorphic curves.
- The blow-up at the origin of the singular quadric $z_1^2 + z_2^2 + z_3^2 = 0$. The exceptional divisor is a holomorphic sphere of self-intersection -2.

When $\nu = 2$ there is another explicit description of the metric, exhibiting more symmetry.

Let $\beta_1, \beta_2, \beta_3$ be a standard basis of left-invariant 1-forms on $SU(2)$ with $d\beta_i = 2\beta_j \wedge \beta_k$. Take a co-ordinate t on an interval $I \subset \mathbf{R}$ and functions $f_i(t)$ and define 2-forms on $I \times SU(2)$ by

$$\omega_i = d\left(f_i(t)\beta_i\right).$$

Fix the volume form $2dt\beta_1\beta_2\beta_3$. The orthonormality condition for the ω_i become $\frac{d}{dt}(f_i^2) = 1$ so $f_i(t) = \sqrt{t + \tau_i}$ for constants τ_i.

If all τ_i are 0, if we take $r = t^{1/4}$ we find that the metric is

$$dr^2 + r^2(\beta_1^2 + \beta_2^2 + \beta_3^2).$$

This is the flat metric on $\mathbf{R}^4$.
If $\tau_1 = 1$ and $\tau_2 = \tau_3 = 0$ one finds that the metric is

$$\frac{1}{16t\sqrt{t+1}}dt^2 + \frac{t}{\sqrt{t+1}}\beta_1^2 + \sqrt{t+1}(\beta_2^2 + \beta_3^2).$$

Take the quotient by ± 1. The metric extends to a smooth metric on the completion X in which the origin is replaced by a 2-sphere.

The manifold X can be regarded as T^*S^2, with isometry group $S^1 \times SO(3)$.

The hyperkahler manifolds X obtained from the Gibbons-Hawking construction are not compact but they are important in the study of singularity formation in families of compact manifolds of special holonomy.

- a sequence of quartic surfaces $V_i \subset \mathbf{CP}^3$ with limit V_∞, a surface with an ordinary double point.
- the Kummer surfaces K_i which are resolutions of $T^4/\pm 1$.

The products $X \times \mathbf{R}^3$ also appear as models for singularity formation of metrics with holonomy G_2.

Section 5; Two topics

5.1 Minimal hypersurfaces and positive scalar curvature

Does the $n+1$-dimensional torus T^{n+1} admit a Riemannian metric of strictly positive scalar curvature?
When $n = 1$: NO by the Gauss-Bonnet formula.
One might guess that in general dimensions the integral of the scalar curvature is a topological invariant, but that is not correct.

Alternative proof when $n = 1$.

Proposition A

Let M^2 be a compact oriented Riemannian surface with a metric of strictly positive scalar curvature. Then $H^1(M, \mathbf{Z}) = 0$.

Suppose there is a non-trivial class in H^1. This can be represented by a length-minimising closed geodesic γ.

The normal bundle of γ is trivial, so a variation vector field can be written as fN where N is a unit normal.

The second variation formula is

$$L'' = \int_\gamma |\nabla f|^2 - K(T, N) f^2,$$

where $K(T, N)$ is the sectional curvature, which is half the scalar curvature in this case.

Taking $f = 1$ we get a variation with $L'' < 0$ which contradicts minimality.

Remark: another, related, argument uses Myers theorem.

This variational approach can be extended to higher dimensions (Schoen and Yau, Manuscripta Math. 1979).

Theorem

For $n \leq 6$ let M^{n+1} be a compact, oriented, manifold which admits a map $f : M \to T^{n+1}$ of non-zero degree. Then M does not admit a Riemannian metric of strictly positive scalar curvature.

The hypothesis is equivalent to the existence of classes $\alpha_1, \dots, \alpha_{n+1} \in H^1(M, \mathbf{Z})$ such that $\alpha_1 \alpha_2 \alpha_{n+1} =\neq 0 \in H^{n+1}(M)$.

The restriction $\mathrm{dim} M \leq 7$ is because in this range it is known that any class in H^1 is represented by an embedded, volume-minimising, hypersurface.

This is the difficult component of the proof, but we assume it.

More recent developments perhaps extend the results to all dimensions (taking account of singularities in the hypersurface).

Recall that the Euler Lagrange equation for the volume functional on hypersurfaces is $H = 0$ where $H = \mathrm{Tr}\, B$ and B is the second fundamental form. Given such a minimal hypersurface $P \subset M$ with unit normal N we need the second variation formula for volume. For a variation fN it is

$$\delta^2 \mathrm{Vol} = \int_P |\nabla f|^2 - f^2 \left(|B|^2 + \mathrm{Ricci}_M(N) \right) \qquad (**)$$

Remark The term $|B|^2$ in (**) does not appear in the case of geodesics. To see why it enters, consider a variation with $f = 1$. Then, as discussed in Section 2, the derivative of B expressed in terms of parallel transport along normal geodesics is

$$\frac{dB}{dt} = -B^2 + \mathrm{curvature},$$

and $\mathrm{Tr}\,(B^2) = |B|^2$

The obvious deduction from (**) is that if $\mathrm{Ricci}_M > 0$ then P cannot be volume minimising (taking $f = 1$). So we have

$$\mathrm{Ricci}_M > 0 \Rightarrow H^1(M) = 0;$$

but we already knew that, from Myers' Theorem.
More to the point, we only have a hypothesis on the *scalar curvature* S_M not the Ricci curvature.

Suppose for the moment that $P \subset M$ is totally geodesic, so the curvature tensor of P is given by the restriction of the curvature tensor of M. In terms of an orthonormal frame e_i with $e_0 = N$ we have

$$S_M = \sum_{i,j \geq 0} R_{ijij} = \sum_{i,j \geq 1} R_{ijij} + 2\mathrm{Ricci}_M(N) = S_P + 2\mathrm{Ricci}_M(N).$$

In general one finds that

$$S_M = S_P + 2\mathrm{Ricci}_M(N) + |B|^2,$$

(think of the case of a minimal surface in $\mathbf{R}^3$).

Consider a Euclidean vector bundle written as an orthogonal direct sum $V = V_1 \oplus V_2$. A connection ∇ on V has a decomposition

$$\begin{pmatrix} \nabla_1 & B \\ -B^T & \nabla_2 \end{pmatrix}$$

and the curvatures are related by

$$F|_{V_1} = F_1 - B \wedge B^T \quad F|_{V_2} = F_2 - B^T \wedge B.$$

So we can the second variation formula as

$$\delta^2 \mathrm{Vol} = \int_P |\nabla f|^2 - \frac{f^2}{2}\left(|B|^2 + S_M - S_P\right). \qquad (****)$$

It is now easy to prove the theorem in the case $n = \dim P = 2$.
Recall we have three classes $\alpha_0, \alpha_1, \alpha_2 \in H^1(M)$ with non-vanishing product $\alpha_0\alpha_1\alpha_2$.
Let P be an volume-minimising hypersurface representing α_0.
Apply the second variation formula (****) with $f = 1$. Since $S_M > 0$ by hypothesis we get

$$\int_P S_P > 0.$$

But $\alpha_1 \cup \alpha_2$ is non-zero in $H^2(P)$, so α_1 is non-zero in $H^1(P)$, which gives a contradiction by Gauss-Bonnet (or you can avoid Gauss-Bonnet by using Proposition A and a variant of the conformal deformation argument below.) (or use Gauss-Bonnet).

The proof of the Theorem for $n > 2$ involves an additional idea.
We want to show:
Proposition B If $S_M > 0$ and $P \subset M$ is a volume-minimising hypersurface then P admits a metric of positive scalar curvature.
Given this, the proof of the Theorem goes by induction on n.

The proof of Proposition B involves conformal deformation. In general, on an n-dimensional Riemannian manifold (P, g), let u be a positive function and consider the conformal metric $g' = u^{4/n-2}g$. Then one finds that

$$4(n-1)/(n-2)\Delta u + S_g u = S_{g'} u^{n+2/n-2} \qquad (******),$$

where $\Delta = \nabla^* \nabla$.

Write L_g for the linear operator $4(n-1)/(n-2)\Delta + S_g$. Assume now P is compact. Then L_g is a self-adjoint linear operator with a discrete spectrum. In particular if $L_g > 0$ as an operator (i.e. $\langle L_g\phi, \phi\rangle > 0$ for all nonzero ϕ) then the first eigenvalue λ_0 is positive and there is an eigenfunction ϕ_0 with $L_g\phi_0 = \lambda_0\phi_0$. A "rounding corners" argument shows that ϕ_0 cannot change sign so we can suppose $\phi_0 > 0$. Taking $u = \phi_0$ in (******) we see that $S_{g'} > 0$.

Conclusion (Kazdan and Warner, 1975): *if P is a compact manifold with a metric g such that $L_g > 0$ then there is a conformal metric with positive scalar curvature*.

To prove Proposition B go back to the second variation formula in the shape (for $S_M > 0$):

$$\delta^2 \mathrm{Vol} < \int_P |\nabla f|^2 + \frac{f^2}{2} S_g, \qquad (****)$$

where now we are writing g for the induced metric on P.
The volume-minimising condition gives

$$\int_P 2|\nabla f|^2 + f^2\, S_g > 0$$

for non-zero f. But

$$\frac{4(n-1)}{n-2} > 2$$

so

$$\int_P \frac{4(n-1)}{n-2}|\nabla f|^2 + f^2\, S_g > 0$$

which is the statement $L_g > 0$.

5.2: The Gauss-Bonnet formula, equivariant cohomology and the Mathai-Quillen form

There is a generalisation of the Gauss-Bonnet formula to compact manifolds of even dimension $2m$ but it involves an integrand which is a polynomial function of degree m in the curvature tensor.

This is easy to prove in the case of a hypersurface $M^{2m} \subset \mathbf{R}^{2m+1}$. Then, by topological arguments, the Euler characteristic $\chi(M)$ is twice the degree of the Gauss map $\gamma : M \to S^{2n}$. The second fundamental form B can be viewed as the derivative of the Gauss map so

$$\mathrm{Vol}\,(S^{2m})\,\chi(M) = 2\int_M \det B. \qquad (*******)$$

In this situation, the Riemann curvature tensor of M is just $-B \wedge B^T$: in index notation

$$R_{ijkl} = B_{ik}B_{jl} - B_{jk}B_{il}.$$

For skew-symmetric $2m \times 2m$ matrix Z the Pfaffian $\mathrm{Pfaff}(Z)$ is a polynomial of degree m in the entries of Z such that

$$\det(Z) = \mathrm{Pfaff}(Z)^2.$$

In fact if we identify these matrices with $\Lambda^2 \mathbf{R}^{2m}$ then

$$\mathrm{Pfaff}(Z)\mathrm{vol} = Z^m/m!,$$

where Z^m is computed in the exterior algebra $\Lambda^* \mathbf{R}^{2m}$ and vol is the standard volume form

More invariantly, for an oriented Euclidean vector space V of dimension $2m$ the Pfaffian is well-defined on the Lie algebra $\mathfrak{so}(V)$ of $SO(V)$.

For any oriented Riemannian $2m$-manifold M regard the curvature as a 2-form with values in the vector bundle $\mathfrak{so}(TM))$. Combining the Pfaffian polynomial with the wedge product on 2-forms we can define

$$\mathrm{Pfaff}(\mathrm{Riem}) \in \Omega^{2m}.$$

The generalised Gauss-Bonnet formula (Chern, 1948) , for compact M, is

$$(2\pi)^m \chi(M) = \int_M \mathrm{Pfaff}(\mathrm{Riem}).$$

It is an algebraic exercise to see that this reduces to (*******) in the case of a hypersurface in $\mathbf{R}^{2m+1}$.

Example For a compact oriented Riemannian 4-manifold M:

$$\chi(M) = \frac{1}{8\pi^2} \int_M \frac{1}{24}|S|^2 + |W_+|^2 + |W_-|^2 - \frac{1}{2}|\mathrm{Ricci}_0|^2.$$

Thus an Einstein 4-manifold has $\chi \geq 0$ with equality if and only if the metric is flat.

The Gauss-Bonnet formula can be developed as part of the general Chern-Weil theory of characteristic classes and invariant polynomials.
We will develop it from the point of view of the Thom class and equivariant de Rham theory.

Let $V \to B$ be an oriented real vector bundle of rank $2m$.
The Thom class is the unique class in $H_c^{2m}(V)$ which restricts to the standard generator on each fibre.
Suppose that $\dim(B) = 2m$ and we have a de Rham representative τ of the Thom class: a compactly supported closed $2m$-form on the total space of V.
Let s be a section of $V \to B$. Then we can form

$$I = \int_B s^*(\tau).$$

By homotopy invariance, this does not depend on the section s. Deforming s suitably we see that I is the *Euler number* of V, the signed count of zeros of a generic section. On the other hand taking $s = 0$ we see that I is equal to the integral of τ over the zero section $B \subset V$.

Thus our problem is to construct an explicit representative of the Thom class.
Equivariant de Rham theory gives a machinery for doing that.

Let X be a manifold with an action of a compact Lie group G. For any principle G-bundle $P \to B$ we have an associated bundle $\mathcal{X} \to B$ with fibre X.
An equivariant cohomology class on X can be thought of as an ordinary cohomology class α together with a procedure for extending α over $\mathcal{X}$ for any such bundle $\mathcal{X}$.

Write s^p for the polynomials of degree p on the Lie algebra $\mathfrak{g}$. Then G acts on $s^p \otimes \Omega_X^q$. Define $C^{p,q}$ to be the G-invariant part:

$$C^{p,q}(X) = \left(\Omega_X^q \otimes s^p\right)^G .$$

Define a grading by saying that $C^{p,q}$ has degree $q + 2p$. We have $d : C^{p,q} \to C^{p,q+1}$ and an algebraic operator

$$I : C^{p,q} \to C^{p+1,q-1}$$

defined as follows. For each $x \in X$ the derivative of the action lies in $\mathfrak{g}^* \otimes TX_x$. Then I is the tensor product of the multiplication

$$\mathfrak{g}^* \otimes s^p \to s^{p+1}$$

and contraction $TX \otimes \Omega^q \to \Omega^{q-1}$.

BASIC FACT I. $(d + I)^2 = 0$ on $C^{*,*}$.

We define the *equivariant de Rham cohomology* $H^*_G(X)$ to be the cohomology of $(C^{**}, d + I)$.

Example Take $G = S^1$. Then $C^{**}(X)$ can be viewed as the invariant forms with coefficients in the polynomial ring $\mathbf{R}[t]$. The differential is $d + ti_v$ where v is the vector field generating the action.

$$(d + ti_v)^2 = d^2 + t^2 i_v^2 + t(i_v d + di_v).$$

We have $i_v d + di_v = L_v$, the Lie derivative along v, and this vanishes on the invariant forms. A class of degree q in equivariant cohomology is defined by some

$$\underline{\alpha} = \alpha_q + t\alpha_{q-2} + t^2\alpha_{q-4} + \dots$$

where α_q is a closed invariant q-form and

$$d\alpha_{q-2} = -i_v\alpha_q \qquad d\alpha_{q-4} = -i_v\alpha_{q-2},$$

etc..

In particular, suppose that ω is an S^1-invariant *symplectic form* on X. To promote it to a closed equivariant form we need a Hamiltonian function h with $dh = i_v\omega$.

More generally, for a symplectic form ω invariant under a group G the data required to promote it to an closed equivariant form is a *moment map*

$$m : X \to \mathfrak{g}^*.$$

Now let $\mathcal{X} = P \times_G X$ be a bundle over B with fibre X, as discussed above. Choose a connection on the principal G-bundle P. This has curvature which is a form on the total space P ; $\quad F \in \Lambda^2 T^* B \otimes \mathfrak{g}$.
For any polynomial $\lambda \in s^p$ we can define $\lambda(F) \in \Lambda^{2p} T^* B$. This induces a map $\mu : C^{p,q} \to \Omega^{2p+q}(\mathcal{X})$.

BASIC FACT II.
μ defines a map of cochain complexes, with respect to the differential $(d + I)$ on C^{**} and the exterior derivative d on $\Omega^*(\mathcal{X})$.

Example Take $G = S^1$. The curvature of the connection is a 2-form F on B. The connection defines a horizontal subbundle $H \subset T\mathcal{X}$ and $\Omega^*(\mathcal{X}) = \bigoplus \Omega^{p,q}$. The action defines a vertical field v on $\mathcal{X}$. With respect to this decomposition $d = d_X + d_H + \nu$ where

$$\nu : \Omega^{p,q} \to \Omega^{p+2,q-1}$$

is $F \wedge i_v$.

Suppose we have a closed equivariant form $\underline{\alpha} = \alpha_q + t\alpha_{q-2} + t^2\alpha_{q-4} + \dots$ as above.

Then

$$\mu(\underline{\alpha}) = \alpha_q + F\alpha_{q-2} + F^2\alpha_{q-4} + \cdots \in \Omega^{0,q} \oplus \Omega^{2,q-2} \oplus \Omega^{4,q-4} \oplus \dots$$

Another description of C^{**} is as the G-equivariant polynomial maps

$$f : \mathfrak{g} \to \Omega_X^*$$

with the differential

$$(Df)(\xi) = d(f(\xi)) + i_{\rho(\xi)} f(\xi),$$

where $\rho : \mathfrak{g} \to TX$ is the Lie algebra action.

The Mathai-Quillen form

Now take $G = SO(2m)$ and $X = \mathbf{R}^{2m}$. Identify the Lie algebra $\mathfrak{g}$ with the 2-forms Λ^2. The Mathai-Quillen equivariant form is the map $f : \mathfrak{g} \to X$ given by

$$f(\xi) = (2\pi)^{-m} * \exp(\xi) e^{-r^2/2},$$

where $* : \Lambda^p \to \Lambda^{2m-p}$ and r is the usual radius function on $\mathbf{R}^{2m}$. The proof that $Df = 0$ comes down to the identity

$$i_{\partial_r}(\exp(\xi)) = (i_{\partial_r}\xi) \wedge \exp(\xi),$$

where ∂_r is the radial vector field .

For example in the case $m = 1$ the statement is equivalent to saying that the function $e^{-r^2/2}$ is a Hamiltonian for the circle action with respect to the symplectic form $e^{-r^2/2} r \, dr \, d\theta$.

The component of the Mathai-Quillen form in $C^{0,2m}$ is $(2\pi)^{-m}e^{-r^2/2}\mathrm{vol}$ where vol is the standard volume form on $\mathbf{R}^{2m}$. This is a $(2m)$-form on $\mathbf{R}^{2m}$ of integral 1.

The component of the Mathai-Quillen form in $C^{m,0}$ is $(2\pi)^{-m}\lambda$ where λ is the polynomial of degree m on $\mathfrak{g}$ defined by $\xi^m/m!$. This is just the Pfaffian polynomial.

Let $V \to B$ be an oriented Euclidean vector bundle of rank $2m$ associated to a principal $SO(2m)$ bundle. Choosing a connection, our general machinery defines a closed $2m$-form τ on the total space of V with integral 1 over each fibre and which restricts to $(2\pi)^m$ times the Pfaffian of the curvature on the zero section. This form has very rapid decay but is not compactly supported so is not precisely a Thom form.

However

- Suppose $\dim B = 2m$ and that s is a generic section of V. Take a large parameter λ and consider $s_\lambda = \lambda s$. Then one sees that as $\lambda \to \infty$ the form $s_\lambda^*(\tau)$ becomes concentrated around the zeros of s and one can derive the relation with the Euler number.
- Alternatively, choose a suitable diffeomorphism $\chi : B^{2m} \to \mathbf{R}^{2m}$. Then $\chi^*(\mathrm{Mathai-Quillen}))$ is a form on B^{2m} and one sees that extending this by zero defines a smooth compactly supported form on $\mathbf{R}^{2m}$. Now the general machinery gives a compactly-supported Thom form on V.