DIFFERENTIAL GEOMETRY: PART 1 (sections 1-3)

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Section 1. Differential Geometric structures

Subsection (1.1) **Analysis via jets.**

Much of differential geometry fits into the following general picture.
Let S be a manifold with a transitive action of $GL(n, \mathbf{R})$.
(So $S = GL(n, \mathbf{R})/G$ for a subgroup $G \subset GL(n, \mathbf{R})$.)
If M is an n-dimensional manifold we have a $GL(n, \mathbf{R})$-frame bundle $Fr(M) \to M$. We get an associated bundle $\mathcal{S} \to M$,

$$\mathcal{S} = Fr(M) \times_{GL(n,\mathbf{R})} S.$$

And we consider sections of $\mathcal{S}$.
The group of diffeomorphisms of M acts on these sections and we are interested in the sections modulo diffeomorphism.

For example we are interested in the question whether all sections are locally equivalent, modulo diffeomorphism, to the *flat model*; a constant section $\mathbf{R}^n \times \{\sigma\}$ of $\mathbf{R}^n \times S$.

Examples

Example 0. $S = Gr(p, n)$ the Grassmann manifold of p-dimensional vector subspaces of $\mathbf{R}^n$. A section is equivalent to a rank p subbundle of TM.

Example 1. $n = 2m$ is even and S is the space of non-degenerate skew symmetric forms on $\mathbf{R}^{2m}$. A section defines a 2-form ω which is non-degenerate at each point.

Example 2. $n = 2m$ is even and S is the space of complex structures $J : \mathbf{R}^{2m} \to \mathbf{R}^{2m}$, $J^2 = -1$. A section is an almost-complex structure on M.

Example 3. S is the space of positive definite quadratic forms on $\mathbf{R}^n$, a section is a Riemannian metric on M.

Example 4. S is the space of positive definite quadratic forms on $\mathbf{R}^n$ up to scale. A section is a conformal structure on M.

We can study an infinitesimal version of this using “jets” i.e. Taylor series.

Recall that if A and B are manifolds and $f : A \to B$ is a smooth map then for each $a \in A$ there is a derivative $df_a : TA_a \to TB_{f(a)}$.

Without extra structure, the higher derivatives of f are not really defined. But there is a well-defined notion of two maps f, g being equal to order k at $a \in A$.

When $k = 1$ this just means that $f(a) = g(a)$ and their derivatives at a are equal. For each k this defines an equivalence relation.

A k-jet of maps $A \to B$ is an equivalence class.

Going back to $\mathcal{S} \to M$, for $p \in M$ we can form the quotient

Q_k = k-jets of sections at p modulo $(k+1)$-jets of diffeomorphisms of M fixing p.

This is a finite-dimensional object (independent of M, p). From the definitions, Q_0 is a point.

What about Q_1?
Write $V = TM_p$. The 2-jets of diffeomorphisms with a fixed 1-jet are parametrised by the vector space $s^2(V^*) \otimes V$.
In co-ordinates we would write

$$\tilde{x}^i = x^i + \sum a^i_{jk} x^j x^k.$$

The 1-jets of sections of with the same 0-jet $\sigma \in S$ are parametrised by $V^* \otimes U$ where $U = TS_\sigma$.
The action of the diffeomorphisms defines a linear map

$$\alpha_1 : s^2(V^*) \otimes V \to V^* \otimes U.$$

Q_1 can be identified with the cokernel of α_1, divided by the action of G, the stabiliser in $GL(n, \mathbf{R})$ of σ.
The kernel of α_1 consists of 2-jets of diffeomorphisms which fix the 1-jet of the flat model.

Example 1. We have $U = \Lambda^2 V^*$. The nondegenerate form $\sigma = \omega_0$ defines an isomorphism $V = V^*$. The map α_1 is the composite of

$$s^2(V^*) \otimes V \mapsto V^* \otimes V^* \otimes V \mapsto V^* \otimes V^* \otimes V^* \mapsto V^* \otimes \Lambda^2 V^*.$$

You can check that there is an exact sequence

$$0 \to s^3(V^*) \to s^2(V^*) \otimes V^* \stackrel{\alpha_1}{\to} V^* \otimes \Lambda^2 V^* \to \Lambda^3 V^* \to 0.$$

So in this case Q_1 is the quotient by the symplectic group $Sp(m, \mathbf{R}) \subset GL(2m, \mathbf{R})$ of $\Lambda^3 V^* = \Lambda^3 T^* M_p$.
The class in Q_1 at p of a section ω is just the exterior derivative $d\omega(p)$. So a section ω is equivalent to the flat model up to first order at p if and only if $d\omega(p) = 0$.
Darboux's Theorem states that if $d\omega = 0$ on M (i.e. a *symplectic structure*) then ω is locally equivalent to the flat model. That is, there are local coordinates p^i, q^i such that $\omega = \sum dp^i dq^i$.

Now consider Example 3, Riemannian metrics.
We have $U = s^2(V^*)$ and

$$\alpha_1 : s^2(V^*) \otimes V \to V^* \otimes s^2(V^*),$$

defined similarly to the symplectic case.
The *Fundamental lemma of Riemannian geometry; Formulation I* states that this map is an isomorphism. This implies that Q_1 is a point and moreover the fact that the kernel of α_1 is zero implies that Q_2 can be identified with the $O9n$) quotient of the cokernel of the map

$$\alpha_2 : s^3(V^*) \otimes V \to s^2(V^*) \otimes s^2(V^*)$$

defined by the action of diffeomorphisms on 2-jets of sections.

In co-ordinates, we are considering the transformations of 2-jets

$$g_{ij} = \delta_{ij} + \gamma_{ij,kl} x^i x^j dx^k dx^l,$$

by co-ordinate changes

$$\tilde{x}^i = x^i + b^i_{jkl} x^j x^k x^l.$$

One finds that α_2 is injective so, counting dimensions, its cokernel has dimension

$$d(n) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n^2(n+1)(n+2)}{3!} - = \frac{n^2(n^2-1)}{12}.$$

Using the natural Euclidean structures on the spaces involved we can identify Q_2 with the $O(n)$ quotient of the kernel of the adjoint map

$$\alpha_2^* : s^2(V^*) \otimes s^2(V^*) \to s^3(V^* \otimes V^*.$$

In terms of the $\gamma_{ij,kl}$ this map is symmetrisation on the first three indices, so the kernel consists of the $\gamma_{ij,kl}$ such that

$$\gamma_{ij,kl} + \gamma_{jk,il} + \gamma_{ki,jl} = 0$$

It is a fact that the kernel of α_2^* can be identified with the kernel of the similarly-defined map

$$\beta_2^* : \Lambda^2(V^*) \otimes \Lambda^2 V^* \to \Lambda^3 V^* \otimes V^*.$$

(as evidence for this fact, note that $d(n) = \left(\frac{n(n-1)}{2}\right)^2 - \frac{n^2(n-1)(n-2)}{3!}$).

So Q_2 can be identified with the $O(n)$ quotient of the tensors $R_{ij,kl} \in \Lambda^2 \otimes \Lambda^2$ such that

$$R_{ij,kl} + R_{jk,il} + R_{ki,jl} = 0.$$

We can then define the curvature tensor of a Riemannian metric at a point p to be the tensor corresponding to the 2-jet of the metric. Explicitly the 2-jet is given by the formula

$$g_{ij} = \delta_{ij} - \frac{1}{3} \sum_{kl} R_{ikjl} x^k x^l.$$

Subsection (1.2) **Fields of subspaces and connections**

The above kind of jet analysis is a machine which can be applied in all situations but it is more powerful when supplemented by other points of view. Let us go back to Example 0. The basic invariant of a subbundle $H \subset TM$ is a tensor $\tau \in \Lambda^2 H^* \otimes TM/H$ which can be defined as follows. For a point $p \in M$ and $\xi_1,\ xi_2 \in H_p \subset TM_p$ choose local sections v_1, v_2 of H with those values at p. Then $\tau(\xi_1, \xi_2)$ is the reduction modulo H_p of $[v_1, v_2](p)$, where [,] is the Lie bracket on vector fields. The fact that this is independent of the choices of v_1, v_2 follows from the formula

$$[v, fw] = f[v, w] + (\nabla_w f)v.$$

The *Frobenius Theorem* states that if τ vanishes at each point then the subbundle is integrable, defined by a foliation of M.
Example.
A contact structure on a manifold M of dimensions $2m+1$ is a rank $2m$ subbundle $H \subset TM$ such that τ is nondegenerate at each point. Another way to define τ in this case is to choose locally a 1-form θ such that $H = \ker \theta$. Then

$$\tau = d\theta|_H \otimes \theta^{-1}.$$

The contact condition is that $\theta \wedge (d\theta)^m$ is a volume form.
One way in which such a subbundle arises is when M is a real hypersurface in $\mathbf{C}^{m+1}$ and $H = TM \cap ITM$. Then τ is called the Levi form. The existence of this invariant shows that a naive analogue of the Riemann mapping theorem fails for domains in $\mathbf{C}^{m+1}$, for $m > 0$.

Suppose that on a manifold M we have a pair of complementary sub-bundles $TM = H' \oplus H''$. Then we can decompose the differential forms

$$\Omega^*_M = \bigoplus \Omega^{p,q}$$

where $\Omega^{p,q}$ consists of forms with p-factors in the H'-direction and q in the H'' direction. We have

$$\tau' \in H'' \otimes (\Lambda^2 H')^*, \tau'' \in H'' \otimes (\Lambda^2 H')^*.$$

The exterior derivative has four components with respect to this decomposition:

$$d' : \Omega^{p,q} \to \Omega^{p+1,q} \quad d'' : \Omega^{p,q} \to \Omega^{p,q+1},$$

$$\nu' : \Omega^{p,q} \to \Omega^{p+2,q-1} \quad \nu'' : \Omega^{p,q} \to \Omega^{p-1,q+2}.$$

The components ν', ν'' are algebraic operators given by wedge product and contraction with τ', τ''

This gives a good way to think about almost-complex structures (Example 2).

An almost-complex structure J on M defines a decomposition of the complexified tangent bundle $TM_{\mathbf{C}} = TM' \oplus TM''$ where $J = i$ on TM' and $J = -i$ on TM''.

Using complex-valued forms we are formally in the same situation as above and we get tensors τ', τ'' which in this case are complex conjugate. The tensor

$$\tau'' \in \Omega^{0,2}(TM')$$

is the *Nijenhuis tensor* of the almost-complex structure. The *Newlander-Nirenberg Theorem* states that if this vanishes on M then the almost-complex structure is integrable i.e we have a complex manifold.

In the case when the data is real analytic this theorem can be deduced from Frobenius by complexification. For C^∞ data the proof is much harder.

An important case is when we have a fibration $\pi : E \to X$ with fibre Y and a subbundle $H \subset TE$ complementary to bundle $V \subset TE$ of tangent spaces to the fibres. So we have a decomposition of forms $\Omega^{p,q}$ (p factors in the base direction and q in the fibre). The exterior derivative has three components

$$d_Y : \Omega^{p,q} \to \Omega^{p,q+1} \quad d_H : \Omega^{p,q} \to \Omega^{p+1,q} \quad \nu : \Omega^{p,q} \to \Omega^{p+2,q-1}$$

where ν is an algebraic operator defined by a section τ of the bundle

$$\pi^*(\Lambda^2 T^* X) \otimes V.$$

The de Rham complex is filtered by subcomplexes

$$\bigoplus_{p \geq r} \Omega^{p,q}.$$

This leads to the *Leray-Serre spectral sequence* relating the cohomologies of the E, B, F.

Now consider the sub-case of a principal G-bundle $\pi : P \to M$ so there is a right action of G on P and $P/G = M$. A *connection* on P is a field of horizontal subspaces H, as above, preserved by the action of G.

Example.

S^1 acts on the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$ with quotient $\mathbf{CP}^m$. The subbundle $TS^{2m+1} \cap ITS^{2m+1}$ we considered before is a connection.

In this context, our section τ is called the *curvature* of the connection.
The derivative of the action defines a trivialisation of the bundle $V \to P$: we identify it with the trivial bundle with fibre $\mathfrak{g}$. So τ can be viewed as a 2-form on P with values in the vector space $\mathfrak{g}$ which we will denote by F.

A G-invariant section of V corresponds to an equivariant map from P to $\mathfrak{g}$, where G acts on $\mathfrak{g}$ by the adjoint action.

This implies that the curvature of the connection can also be regarded as 2-form on the base M with values in the vector bundle $\mathrm{ad}\, P$ associated to P by the adjoint action. We will write this as $\underline{F} \in \Omega^2(M, \mathrm{ad} P)$.

Formula for the curvature On our Lie group G we can identify $\mathfrak{g}^*$ with left-invariant 1-forms. Thus $1 \in \mathfrak{g} \otimes \mathfrak{g}^*$ can be viewed as a left-invariant 1-form η with values in $\mathfrak{g}$. This satisfies the Maurer-Cartan equation

$$d\eta + \frac{1}{2}[\eta, \eta] = 0,$$

where $[\eta, \eta]$ combines the bracket on $\mathfrak{g}$ with the wedge product on forms. The form η is invariant under the right action of G on G combined with adjoint action on $\mathfrak{g}$.

Note: If G is a matrix group then we can write $\eta = g^{-1}dg$ and

$$d\eta = -g^{-1}dgg^{-1}dg = -\eta \wedge \eta = -\frac{1}{2}[\eta, \eta].$$

Given a connection on P, the projection to the vertical space defined a 1-form A on P with values in the Lie algebra $\mathfrak{g}$. Then we have the important formula, for $\mathfrak{g}$-valued forms on P:

$$F = dA + \frac{1}{2}[A, A].$$

For practical calculations one normally chooses a local trvialisation of P, i.e. a section $s : U \to P$ over an open set $U \subset M$. Then write $\underline{A} = s^*(A)$—a $\mathfrak{g}$-valued 1-form on U. This section defines a local trivialisation of $\mathrm{ad}\, P$ and

$$\underline{F} = d\underline{A} + \frac{1}{2}[\underline{A}, \underline{A}]$$

which, for a matrix group G, is

$$\underline{F} = d\underline{A} + \underline{A} \wedge \underline{A}$$

for matrix-valued forms on U.

Let $\rho : G \to GL(k, \mathbf{R})$ be a representation of G. We get an associated vector bundle $W = P \times_G \mathbf{R}^k$. A connection H on P defines a *covariant derivative* ∇^H on sections of W. This is a differential operator

$$\nabla^H : \Gamma(W) \to \Gamma(T^*M \otimes W),$$

defined as follows. A section of W is the same as an equivariant map $\sigma : P \to \mathbf{R}^n$. The covariant derivative of σ in the direction of a tangent vector $v \in TM$ is induced by the usual derivative in the direction of the horizontal lift of v. If G is a matrix group the concepts of connection and covariant derivative are equivalent.

For simplicity of notation suppose ρ is faithful and identify G with its image in $GL(k, \mathbf{R})$.
Choose local coordinates x^i on $U \subset M$ and a local trivialisation of P. A section of W is then given by an $\mathbf{R}^k$-valued function on U. The covariant derivative in the x^i direction is the differential operator

$$\nabla_i = \frac{\partial}{\partial x^i} + \underline{A}_i$$

where $\underline{A} = \sum \underline{A}_i dx^i$. The curvature measures the failure of these derivatives to commute

$$[\nabla_i, \nabla_j] = \underline{F}_{ij} = \frac{\partial \underline{A}_j}{\partial x^i} - \frac{\partial \underline{A}_j}{\partial x^i} + [\underline{A}_i, \underline{A}_j].$$

For general vector fields v, w on M we have

$$[\nabla_v, \nabla_w] - \nabla_{[v,w]} = F(v, w),$$

where $F(v, w) : W \to W$ is defined using the embedding $\mathrm{ad}\ P \subset \mathrm{End}\ W$.

Suppose that $G = GL(n, \mathbf{R})$ and the principal bundle P is the frame bundle of an n-dimensional manifold M. Then taking the defining representation ρ, the associated vector bundle is the tangent bundle of TM. A connection on P is called *torsion-free* if for any vector fields v, w:

$$\nabla_v w - \nabla_w v = [v, w].$$

This is equivalent to the condition that around each point $p \in M$ there are local co-ordinates x^i such that the vector fields $v_i = \frac{\partial}{\partial x^i}$ satisfy $\nabla v_i = 0$ *at* p.

More generally, suppose $G \subset GL(n, \mathbf{R})$ and we have a section of the bundle over M with fibre $S = GL(n, \mathbf{R})/G$ as considered in Subsection 1.1. This defines a principal G-bundle over M such that TM is the associated bundle. (For example, if $G = SO(n)$ then the section is a Riemannian metric and P is the bundle of orthonormal frames.) We have the notion of a torsion-free connection on P, as above.

Fundamental Lemma of Riemannian geometry; formulation II
There is a unique torsion-free connection (the *Levi-Civita connection*) on the bundle of orthonormal frames of a Riemannian manifold.

Exercise Show that this equivalent to formulation I in Subsection 1.1.
In one direction, we can define the covariant derivative of a vector field at a point p by working in a co-ordinate system constructed using formulation I and taking the ordinary derivative in those co-ordinates.

In a general co-ordinate system x^i the connection is given by the matrix of 1-forms $\sum \underline{A}_i dx^i$ with $\underline{A}_i = \left(\Gamma^k_{ij}\right)$ where the *Christoffel symbols* Γ^k_{ij} are defined by the derivatives of the metric tensor g_{ij} in these co-ordinates

$$\Gamma^k_{ij} = \frac{1}{2} \sum_a g^{ka} \left(g_{ai,j} + g_{aj,i} - g_{ij,a} \right).$$

(Notation: the comma in the subscript denotes partial derivative.)
The *Riemann curvature tensor* $R_{ijkl} \in \Lambda^2 T^* \otimes \Lambda^2 T^*$ (which we also call Riem) is defined to be the curvature of the Levi-Civita connection. In classical notation it is

$$R_{ijkl} = \sum_\lambda g_{i\lambda} \left(\Gamma^\lambda_{jk,l} - \Gamma^\lambda_{jl,k} + \sum_a \left(\Gamma^\lambda_{ak} \Gamma^a_{jl} - \Gamma^\lambda_{al} \Gamma^a_{jk} \right) \right).$$

Theorem If a Riemannian n-manifold has $\mathrm{Riem} = 0$ it is locally isometric to $\mathbf{R}^n$ (the "flat model").

First, the Frobenius Theorem applied in the total space of the frame bundle, implies that there is a local frame of orthonormal vector fields v_i with $\nabla v_i = 0$. Let ϵ^i be the dual frame of 1-forms. The torsion-free condition gives that all Lie brackets $[v_i, v_j]$ vanish which implies that $d\epsilon^i = 0$. The Poincare Lemma tells us that there are local functions x^i with $dx^i = \epsilon^i$ and these give the desired co-ordinates.

Take $\mathbf{R}^n$ with its standard Euclidean structure and write $\Lambda^p = \Lambda^p\mathbf{R}^n$ etc. The "space of curvature tensors" is

$$\mathcal{R} = \ker \Lambda^2 \otimes \Lambda^2 \to \Lambda^3 \otimes \Lambda^1.$$

It is a representation of $O(n)$ of dimension $d(n) = \frac{1}{12}n^2(n^2 - 1)$.
The *first Bianchi identity* states that $\mathrm{Riem} \in \mathcal{R}$ (in the obvious sense).
One can show that $\mathcal{R} \subset s^2(\Lambda^2)$ i.e. $R_{ijkl} = R_{klij}$ and that $\mathcal{R}$ is the kernel of the wedge product map $s^2(\Lambda^2) \to \Lambda^4$.

The *Ricci contraction* $\mathcal{R} \to s^2$ maps R_{ijkl} to the Ricci tensor $R_{jl} = \sum_a R_{ajal}$.

The trace of the Ricci tensor is the scalar curvature $R = \sum_j R_{jj}$.

In dimension $n = 2$ we have $d(2) = 1$ and $\mathcal{R} = \mathbf{R}$, the curvature reduces to the scalar curvature (which is twice the classical "Gauss curvature").

In dimension $n = 3$ we have $d(3) = 6$, the Ricci contraction is an isomorphism so $\mathcal{R} = s^2$.

For dimensions $n > 3$ the Ricci contraction has a non-trivial kernel $\mathcal{W}$ and there is a decomposition as $O(n)$-representations

$$\mathcal{R} = \mathbf{R} \oplus s_0^2 \oplus \mathcal{W},$$

which can be shown to be irreducible representations of $O(n)$.

The component of Riem in $\mathcal{W}$ is called the *Weyl curvature W* of the Riemannian manifold.

Let us now discuss Example 4, conformal structures, briefly. Let g be a Riemannian metric and $\tilde{g} = e^{2f} g$ for a function f. Then one finds that the Ricci curvatures are related by

$$\widetilde{\mathrm{Ricci}} = \mathrm{Ricci} - (\Delta f)g - (n-2)\nabla\nabla f + (n-2)\left(\nabla f \otimes \nabla f - |\nabla f|^2 g\right),$$

where Δf is the trace of $\nabla\nabla f$.

It follows from this that for a given point $p \in M$ we can find a conformally equivalent metric $\tilde{g}$ with vanishing Ricci curvature at p. When $n = 3$ this means that the full curvature tensor of $\tilde{g}$ at p is zero. So by our previous discussion the conformal structure agrees with the flat model to second order i.e. Q_2 is a point.

When $n > 3$ the space Q_2 is non-trivial: it can be identified with the $O(n) \times \mathbf{R}^+$ quotient of the space $\mathcal{W}$ of Weyl tensors. The Weyl curvature of a metric vanishes at p if and only if the conformal structure agrees with the flat model to second order.

Suppose that $n = 3$ or $n > 3$ and the Weyl curvature of a metric g vanishes everywhere on M. We can regard the equation $\widetilde{\mathrm{Ricci}} = 0$ as a second order *PDE* for a function f. The metric is conformally flat if and only if this PDE has a solution. At each point this PDE defines $\nabla\nabla f$ in terms of ∇f and Ricci. It defines a field of subspaces H in the total space of the cotangent bundle of M, transverse to the fibres. The condition for solubility is given by the Frobenius Theorem. The tensor τ in this situation is a 2-form on M with values in T^*M depending *a priori* on $g, \nabla f$. In fact it only depends on g and is given by the *Cotton tensor*

$$C_{ijk} = \nabla_k B_{ij} - \nabla_j B_{ik}$$

where $B_{ij} = R_{ij} - \frac{1}{2(n-1)} R g_{ij}$.

If $n > 3$ one can show that the Cotton tensor can be expressed in terms of derivatives of the Weyl curvature. The conclusion is

- For $n > 3$ a Riemannian metric is conformally flat if and only if the Weyl curvature vanishes.
- For $n = 3$ a Riemannian metric is conformally flat if and only if the Cotton tensor vanishes.

Section 2: Geodesics, the Jacobi equation and comparison theorems

Let (M, g) be a connected Riemannian n-manifold.
The *energy* of a path $\gamma : [a, b] \to M$ is

$$E(\gamma) = \int_a^b |\gamma'(t)|^2 dt.$$

Geodesics are solutions of the Euler-Lagrange equation associated to the functional E. If $T = \gamma'$ is the tangent vector field, the equation can be written as $\nabla_T T = 0$.
(If we want to be really precise, we could consider T as a section as the pull back $\gamma^* TM$ of TM and ∇_T as the covariant derivative of the pulled-back connection.)

In coordinates x^i the equation is

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0,$$

where Γ^i_{jk} are the Christoffel symbols.
We get essentially the same equations using the length functional

$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

The infimum of lengths of paths makes M a metric space. If this space is complete then for any tangent vector $v \in TM_p$ there is a (unique) geodesic $\gamma_v : [0, \infty) \to M$ with $\gamma_v'(0) = v$. (Hopf-Rinow Theorem)

In that case the exponential map $\exp_p : TM_p \to M$ is defined by $\exp_p(v) = \gamma_v(1)$.

It gives a diffeomorphism from a neighbourhood of $0 \in TM_p$ to a neighbourhood of p, *i.e.* a preferred co-ordinate system around p.

Initially, we restrict the discussion to a ball on which $\exp_p$ is a diffeomorphism. We have a collection of geodesic rays emanating from p and also a collection of spheres Σ_r , points of a fixed distance r from p. The rays are orthogonal to the spheres (Gauss Lemma). Pulling back the metric g by $\exp$ we have a metric on a region in $\mathbf{R}^n$ which in "polar coordinates" $(r, \underline{\theta})$ has the schematic form

$$dr^2 + \sum g_{a,b}(r, \underline{\theta}) d\theta^a d\theta^b = dr^2 + g_r d\underline{\theta}^2$$

determined by a 1-parameter family of metrics g_r on S^{n-1}.

Suppose that $\gamma_s(t)$ is any 1-parameter family of geodesics. Let T, V be the vector fields corresponding to ∂_t, ∂_s.
Then $\nabla_T T = 0$ so $\nabla_V \nabla_T T = 0$ but $[V, T] = 0$ so this gives

$$\nabla_T \nabla_T V = \nabla_T \nabla_V T = \nabla_V \nabla_T T + R(T, V)T = R(T, V)T.$$

Along a fixed geodesic γ_0, the differential operator

$$J(V) = \nabla_T^2 - R(T, V)T$$

on variation vector fields V is called the Jacobi operator, $J(V) = 0$ is the Jacobi equation and solutions are called Jacobi fields.

Note that the inner product $\langle R(T, V)T, V\rangle$ is -$|V|^2$ times the sectional curvature in the plane spanned by V, T.

The calculation above shows that the Jacobi equation is the linearisation of the geodesic equation, at γ_0.

Going back to our polar coordinates; along a fixed geodesic ray corresponding to a point the $\frac{\partial}{\partial \theta^a}$ are Jacobi fields $V_a(r)$, vanishing at $r = 0$. Knowing the length of these Jacobi fields determines the metric.

A first consequence is to confirm the formula

$$g_{ij} = \delta_{ij} - \frac{1}{3} \sum_{kl} R_{ikjl} x^k x^l + O(r^3)$$

we stated before. (Exercise)

Fix a unit vector $\sigma_0 \in TM_p$ and corresponding geodesic γ_0. Let $\sigma_1, \ldots, \sigma_{n-1}$ be an orthonormal frame for the orthogonal complement of σ_0.

Use parallel transport of the Levi-Civita connection to construct orthonormal variation fields E_i along γ_0. Then we identify variation vector fields with $\mathbf{R}^{n-1}$-valued functions $\underline{V}(r)$.

Write $A(r)$ for the symmetric matrix $\langle R(T, E_a)T, E_b \rangle$.

Then $\langle A(r)\underline{V}, \underline{V} \rangle$ *is equal to* $|\underline{V}|^2$ *times the sectional curvature in the plane spanned by* $T, \underline{V}$.

The Jacobi equation becomes

$$\underline{V}'' = -A\underline{V}.$$

We consider solutions $\underline{V}_a$ with the initial condition $\underline{V}_a(0) = 0, \underline{V}'_a(0) = e_a$ where (e_a) is the standard basis for $\mathbf{R}^{n-1}$.
Equivalently, consider the matrix equation for a matrix-valued function $L(r)$

$$L'' = -AL,$$

with initial conditions $L(0) = 0, L'(0) = \mathbf{1}$.

The matrix $L(r)$ defines the derivative of the exponential map, in terms of the bases σ_i for TM_p and E_i, T for TM_q where $q = \gamma_0(r)$.

Clearly we have

$$L(r) = r\mathbf{1} - \frac{r^2}{2}A(0) + O(r^3).$$

If the manifold has constant sectional curvature c then $A = c\mathbf{1}$ and the solutions are

- For $c = 0$, $L(r) = r\mathbf{1}$; the metric is $dr^2 + r^2 d\underline{\theta}^2$.
- For $c = \omega^2 > 0$, $L(r) = \sin(\omega r)\mathbf{1}$; the metric is $dr^2 + \sin^2(\omega r) d\underline{\theta}^2$
- For $c = -\omega^2 < 0$, $L(r) = \sinh(\omega r)\mathbf{1}$; the metric is $dr^2 + \sinh^2(\omega r) d\underline{\theta}^2$.

We denote these metrics by $dr^2 + g_{r,c} d\underline{\theta}^2$.

The Rauch Comparison Theorem (not the most general version).

Suppose that the metric $g = dr^2 + g_r d\underline{\theta}^2$ has sectional curvatures K bounded by $a \leq K \leq b$. Then $gr, b \leq g_r \leq g_{r,a}$.

Thus *negative* sectional curvature makes distances *larger* and *positive* sectional curvature makes distances *smaller* , when Riemannian manifolds are compared using exponential co-ordinates.

For simplicity we just do the proof for the case $K > 0$, the other cases are essentially the same.
We need to show that $|L(r)(e)| < r|e|$ for all vectors e.

Define $B = L' \, L^{-1}$ so $B' = -A - B^2$.

We claim first that the matrix $B(r)$ is symmetric. This follows from the fact $B \sim r^{-1}\mathbf{1}$ as $r \to 0$ and if $B(r)$ is symmetric then $B'(r)$ is also (since A is symmetric).

In the model case of constant curvature 0 we have $B(r) = r^{-1}\mathbf{1}$.

They key point is to show that for $K > 0$ we have $B(r) < r^{-1}\mathbf{1}$; in other words the largest eigenvalue $\lambda(r)$ of $B(r)$ is $< r^{-1}$.

By studying the asymptotic behaviour you see that this is true for small r.

Suppose that there is some $r_0 > 0$ for which $\lambda(r_0) = r_0^{-1}$, and let r_0 be the smallest such value.

So we have a unit eigenvector e with $B(r_0)e = r_0^{-1}e$. Define

$$f(r) = \langle B(r)e, e \rangle$$

so $f(r) \leq \lambda(r)$ Then

$$f'(r_0) = \langle B'(r_0)e, e \rangle = -\langle A + B(r_0)^2 e, e \rangle < \lambda(r_0)^2 = -r_0^{-2}.$$

Thus $\frac{d}{dr}(rf) = f + rf'$ is strictly negative at r_0 which implies that $f(r) > r^{-1}$ for r slightly less than r_0. Hence the same holds for $\lambda(r)$, which is a contradiction.

Now we know that $B(r) < r^{-1}$. Fix a unit vector e and write $f(r) = \|L(r)e\|^2$.

Then

$$f'(r) = 2\langle Le, L'e\rangle = 2\langle Le, BLe\rangle < 2\|Le\|^2 = 2f(r).$$

This means that $r^{-2}f(r)$ is decreasing and since the limit is 1 when $r \to 0$ we have $f(r) < r^2$. So $\|L(r)\| < r$, as required.

Any Riemannian metric defines a volume form, or measure. In local coordinates is is $\sqrt{\det(g_{ij})}$ times Lebesgue measure.

Let ω_r be the volume of the metric g_r on S^{n-1}, compared with the standard volume on S^{n-1}. Write $\omega_{r,c}$ for the volumes in the constant curvature spaces.

The metric with constant sectional curvature c has Ricci curvature $(n-1)c$.

The Bishop comparison theorem: version 1

If $g = dr^2 + g_r d\underline{\theta}^2$ *has* $\mathrm{Ricci} \geq (n-1)c$ *then* $\omega_r \leq \omega_{c,r}$.

As before, we will write the proof for the case $c = 0$.
We consider again the fixed geodesic γ_0 and write $\Omega(r)$. We need to show that if $\mathrm{Ricci} \geq 0$ then $\Omega(r) \leq r^{n-1}$.
We have $\Omega = \det L(r)$, so $\Omega' = \Omega \mathrm{Tr} L' L^{-1} = \Omega\, \mathrm{Tr} B$.
Write $H = \mathrm{Tr}(B)$. So $H' = -\mathrm{Tr}(A + B^2)$.

- $\mathrm{Tr}(A) = \langle \mathrm{Ricci}(T), T \rangle \geq 0$:
- $\mathrm{Tr}(B^2) \geq (n-1)^{-1} (\mathrm{Tr} B)^2 = (n-1)^{-1} H^2$

So $(n-1)H' \leq -H^2$ and by a similar (easier) argument to that before this implies $H \leq (n-1)r^{-1}$ and $\Omega(r) \leq r^{n-1}$, which completes the proof.

Bishop comparison, Version 2

Write $V(R) = \int_0^R \Omega(r)\, dr$, and V_c for the constant curvature model metric.

Then if $\mathrm{Ricci} \geq (n-1)c$, the ratio

$$\frac{V(R)}{V_c(R)}$$

is decreasing along the geodesic ray.

This follows from what we established above and a simple calculus lemma. We have shown that V'/V_c' is decreasing. So for $r < R$

$$\frac{V'(r)}{V_c'(r)} \geq \frac{V'(R)}{V_c'(R)}.$$

This gives $V'(r)V_c'(R) - V_c'(r)V'(R) \geq 0$. So

$$\int_0^R V_c'(R)V'(r) - V_c'(r)V'(R)\, dr \geq 0,$$

Which is $V_c'(R)V(R) - V_c(R)V'(R) \geq 0$. Dividing by $V_c(R)^2$, this says that the derivative of V/V_c is decreasing.

The symmetric matrix $B(r)$ we encountered above is the *second fundamental form* of the sphere Σ_r, expressed in terms of the frame E_i.

Let $V \to X$ be a vector bundle with connection (covariant derivative) ∇. Suppose $V = V_1 \oplus V_2$ and write π_1, π_2 for the projections. Then $\pi_i \nabla$ defines a connection on V_i. The map

$$\pi_2 \nabla : \Gamma(V_1) \to \Gamma(T^* X \otimes V_2)$$

is a bundle map defined by a tensor $B_1 \in \Gamma(T^* X \otimes \mathrm{Hom}(V_1, V_2))$; the second fundamental form of V_1. Similarly we have a B_2. If V has a Euclidean structure and V_2 is the orthogonal complement of V_1 then $B_2 = -B_1^T$.

Let $N \subset M$ be a hypersurface in a Riemannian manifold (M, g) and choose a normal direction, so $TM|_N = TN \oplus \underline{\mathbf{R}}$ and the discussion above applies. We get $B \in T^*N \otimes T^*N$ and the fact that the connection is torsion-free implies that $B \in s^2(T^*N)$.

Suppose $f_t : N \to M$ is a 1-parameter family of embeddings with f_0 the inclusion and the t-derivative of f_t at $t = 0$ equal to N. For each t we have an induced metric $f_t^*(g)$ on N and

$$\frac{d}{dt}|_{t=0} f_t^*(g) = B.$$

In particular this applies to the family of spheres Σ_r. Another way of thinking about the comparison theorems is in terms of the evolution of the geometry of these spheres.

Subsection II.2. Global theory of geodesics

In a complete Riemannian manifold, any two points can be joined by a length minimising geodesic segment.

(Sketch proof. Suppose for simplicity that M is compact. Then for some sufficiently small ϵ any two points of distance $\leq \epsilon$ can be joined by a minimising geodesic. Suppose that for some L any two points of distance $\leq L$ can be joined by a minimising geodesic. Let $p, q \in M$ with $d(p, q) \leq 3L/2$. Then there are points $r \in M$ such that $d(p, r), d(r, q) \leq L$. Such a point can be joined to p and q by minimising geodesics. Now minimise $d(p, r) + d(r, q)$ over the set of such points r.)

So for a complete manifold M the exponential map $\exp_p : TM_p \to M$ is surjective.
Consider again a geodesic emanating from p: $\gamma_0(r) = \exp(r\sigma_0$.
Define $R(\sigma_0)$ to be the supremum of the r such that $\gamma_0 : [0, r] \to M$ is a length-minimising geodesic from p to $\exp(r\sigma_0$.

1. For *any* $r < R(\sigma_0)$ the geodesic segment $\gamma_0[0, r]$ is length minimising.
2. For $r_0 < R(\sigma_0)$ there is a *unique* minimising geodesic from p to $\gamma_0(r)$.
3. For $r_0 < R(\sigma_0)$ the matrix $L(r_0)$ is invertible, so the exponential map is a local diffeomorphism at $r_0\sigma_0$.

(1), (2) are reasonably straightforward. For (3) we go back to consider the Jacobi operator etc.

Let $\gamma_0 : [a, b] \to M$ be any geodesic segment and γ_s a 1-parameter family of variations, not necessarily through geodesics but with the fixed end points $\gamma_s(a), \gamma_s(b)$. Let V be the variation vector field as before (derivative of γ_s in s, at$s = 0$). We can suppose that V is normal to T.
Let $L(s)$ be the length of γ_s. We know that $L'(0)$. The "second variation" formula is $L''(0) = Q(V)$ where Q is the quadratic form

$$Q(V) = \int_a^b |\nabla_T V|^2 - \langle R(T, V)T, V\rangle.$$

The fixed end-point condition implies that V vanishes at the end-points and we can integrate by parts to write this as

$$Q(V) = \int_a^b \langle V, J(V)\rangle,$$

where $J(V)$ is the Jacobi operator we defined before.

The *index* of the segment is the dimension of a maximal subspace on which Q is negative definite. It is the number of negative eigenvalues of the operator J, on variation fields V vanishing at the end points.

Going back to item (3): if $r_0 < R(\sigma_0)$ and $L(r_0)$ is not invertible then there is a variation $V \neq 0$ vanishing at $r = 0, r_0$ with $J(V) = 0$.

Now construct a 1-parameter family γ_s of paths from p to $\gamma_0(r_0)$ as above such that $L(s) = r_0 + O(s^3)$.

By definition, there is some $r_1 > r_0$ so that $\gamma_0[0, r_1]$ is minimising.

Composing the γ_s with the segments $\gamma_0[r_0, r_1]$ we get a family of paths $\hat{\gamma}_s$ from p to $\gamma_0(r_1)$ of length $r_1 + O(s^3)$.

Choose a suitable small s and round off the corner of $\hat{\gamma}_s$ to get paths with length less than r_1, which is a contradiction.

In general, if there is a Jacobi field along a geodesic segment γ from p to q, vanishing at the end points, then p, q are called *conjugate points* (on γ).

The *Morse Index Theorem* states that if p, q are not conjugate then the index is equal to the number of conjugate points of p in the interior of the segment.

Let $K \subset TM_p$ be the closed set given by the union over unit vectors $\sigma \in TM_p$ of the ray-segments $[0, R(\sigma)]\sigma$. Then

1. $\exp_p$ maps K onto M.
2. $\exp_p$ gives a diffeomorphism from the interior $\mathrm{int}\,(K)$ to $M \setminus \Delta$ where $\Delta \subset M$ is a closed set of measure 0.

The boundary $\partial K = K \setminus \mathrm{int}\,(K)$ is called the cut-locus at p.

Our comparison theorems apply along a geodesic γ_0 provided we do not hit any conjugate points.

If (M, g) has sectional curvature ≤ 0 then there are no conjugate points and $\exp_p$ is a covering map. (Cartan-Hadamard Theorem). So all higher homotopy groups of M vanish.

If (M, g) has Ricci curvature $\geq (n-1)c > 0$ then along γ_0 we must reach a conjugate point at distance at most $\pi/\sqrt{c}$. So the diameter of M is at most $\pi/\sqrt{c}$ (Myers Theorem) and, applying this to the universal cover, $\pi_1(M)$ is finite.

Bishop Theorem, Version 3

Given p, let $V(r)$ be the volume of the r-ball $B_{r,p}$. Let $V_c(r)$ be the volume of the r-ball in the space of constant sectional curvature c.

Then if $\mathrm{Ricci} \geq (n-1)c$ the volume ratio

$$\frac{V(r)}{V_c(r)}$$

is a decreasing function of r.

For small r this follows immediately from Version 2 by integrating over the unit sphere in TM_p.

In general one sees that the effect of the cut locus works the right way in the inequality.

Application to convergence theory.

Let $(A, d_A), (B, d_B)$ be compact metric spaces. The Gromov-Hausdorff distance between them is defined to be the infimum of $\epsilon > 0$ such that there is a metric on $A \sqcup B$ equal to the given metrics on each component and with both A, B ϵ-dense.

Roughly: if the distance between A, B is $\leq \eta$ then at scales bigger than $O(\eta)$ the metric spaces are essentially equivalent.

Gromov compactness theorem.
Let (X_i, d_i) be a sequence of compact metric spaces with diameters $\leq D$ and with the following property.

For each $\delta > 0$ there is an $N(\delta)$ such that each X_i can be covered by $N(\delta)$ balls of radius δ.

Then there is a subsequence of the (X_i, d_i) which converges in the Gromov-Hausdorff metric to some limiting space (Z, d_Z).

The proof is not too difficult. (For example, start with the case of finite spaces X_i.)

Now suppose that (M_i, g_i) is a sequence of compact Riemannian manifolds with diameters $\leq D$ and $\mathrm{Ricci} \geq (n-1)c$. We show that this satisfies the covering condition, so the compactness theorem applies. Write $M = M_i$.

Given $\epsilon > 0$ choose a collection of disjoint ϵ-balls B_α in M which is maximal in the sense that one cannot add another disjoint ϵ-ball.

Then the balls of radius $\delta = 3\epsilon$ with same centres cover M.

Let V_α be the volume of B_α. Since the balls are disjoint

$$\sum_\alpha V_\alpha \leq \mathrm{Vol}(M).$$

The Bishop Theorem gives a lower bound $V_\alpha \geq \kappa \mathrm{Vol}\, M$ where $\kappa > 0$ depends only on c, D. So if N is the number of balls we have $N \leq \kappa^{-1}$.

Section III: Symmetric spaces

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The bracket on the Lie algebra can be defined by the bracket of left-invariant vector fields. Any positive definite quadratic form on $\mathfrak{g}$ defines a left-invariant Riemannian metric on G. If the quadratic form is preserved by the adjoint action of G then this metric is bi-invariant. Any compact Lie group admits such a metric.

For a bi-invariant metric one finds that, for left-invariant vector fields X, Y,

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

The geodesics through the identity are 1-parameter subgroups. Some lines of calculation using the Jacobi identity show that the curvature is

$$R(Z, X)Y = \frac{1}{4}[Y, [Z, X]].$$

The sectional curvature in the plane spanned by orthonormal X, Y is

$$K(X, Y) = \frac{1}{4}|[X, Y]|^2.$$

For example $SU(2) = S^3$, up to a scale factor.

If K is a Lie subgroup of a Lie group G then $M = G/K$ is a homogeneous space. If we have a vector space complement $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and a positive definite form on $\mathfrak{p}$ which are both invariant under the restriction of the adjoint action of G to K, we get a Riemannian metric on M, preserved by the action of G. We get many interesting Riemannian manifolds in this way. *Symmetric spaces* are particularly important.

A Riemannian symmetric space is a (complete, connected) Riemannian manifold (M, g) with the property that for each $p \in M$ there is an isometry $\sigma_p : M \to M$ which fixes p and acts as -1 on TM_p.

Then the identity component G of the isometry group of M acts transitively so, choosing a base point p_0, we have $M = G/K$ for some compact subgroup K.

Conjugation by $\sigma_0 = \sigma_{p_0}$ induces an automorphism τ of G and hence of $\mathfrak{g}$. The $+1$ eigenspace is $\mathfrak{k}$ and the -1 eigenspace is a complement $\mathfrak{p}$ which can be identified with TM_{p_0}.
(Note that τ may or may not be an inner automorphism in G.)
We have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Conversely suppose we have

1. a Lie group G with an involutive automorphism τ and the identity component of the fixed set of τ is a compact subgroup $K \subset G$;
2. with respect to the induced decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, there is a positive definite quadratic form on $\mathfrak{p}$ invariant under the action of K;

Then $M = G/K$ becomes a Riemannian symmetric space.

Other points of view

- The symmetry forces $\nabla \mathrm{Riem} = 0$. Conversely the universal cover of a manifold satisfying this condition is a symmetric space.
- The group K acts on $TM_{p_0} = \mathfrak{p}$. Consider G as a principal K-bundle over M. The translates of $\mathfrak{p}$ define a connection. The tangent bundle TM is the associated bundle defined by the action of K on $\mathfrak{p}$ and the induced connection is the Levi-Civita connection. In the language of the next section, the manifold M has holonomy K.

Any Lie group G has an $\mathrm{ad}\,G$-invariant quadratic form on its Lie algebra, the *Killing form*

$$q(\xi) = -\mathrm{Tr}(\mathrm{ad}\xi)^2.$$

The group is called semi-simple if this is nondegenerate. If the group is compact semi-simple the Killing form is positive definite. In the non-compact case we get a metric of indefinite signature on G, but the same discussion above applies to identify its curvature. If we have an involution τ as above and q is $\pm$ on the subspace $\mathfrak{p} \subset \mathfrak{g}$ then taking $\pm q$ we get a Riemannian metric on G/K.

Define $X \subset G$ to be the fixed point set of $g \in G$ of the map $\tau'(g) = (\tau(g))^{-1}$.
For $q \in M$ set $g_q = \sigma_q \sigma_0$. Then

$$\tau(g_q) = \sigma_0 \sigma_q \sigma_0 \sigma_0 = \sigma_0 \sigma_q = g_q^{-1}.$$

So $q \mapsto g_q$ gives a map $\iota : M \to G$. The derivative of ι at p_0 is the inclusion $\mathfrak{p} \subset \mathfrak{g}$ and one sees that ι gives a covering map from M to a connected component of X.

Example 1

Let M be the set of positive definite symmetric $n \times n$ real matrices with determinant 1. Then $M = SL(n, \mathbf{R})/SO(n)$ and $SO(n)$ is the fixed point set of the involution $\tau(g) = \left(g^T\right)^{-1}$ of $SL(n, \mathbf{R})$. On the other hand M is naturally embedded in $SL(n, \mathbf{R})$ as the set of g with $\tau(g) = g^{-1}$.

Example 2

Let $M = S^{n-1}$. Fix a unit vector $p_0 \in S^{n-1} \subset \mathbf{R}^n$. Then the $SO(n)$ orbit of p_0 is S^{n-1} and the stabiliser is $SO(n-1)$ so $M = SO(n)/SO(n-1)$. The subgroup $SO(n-1)$ is the identity component of fixed point set of the involution given by conjugation by the reflection in the $(n-1)$-dimensional orthogonal complement of p_0.

The map $\iota : S^{n-1} \to SO(n)$ factors through $\mathbf{RP}^{n-1}$. It takes a point $q \neq \pm p_0$ to a rotation in the plane spanned by q, p_0.

Any Lie group G has an $\mathrm{ad}\, G$-invariant quadratic form on its Lie algebra, the *Killing form*

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The group is called semi-simple if this is nondegenerate. If the group is compact semi-simple the Killing form is positive definite. In the non-compact case we get a metric of indefinite signature on G, but the same discussion above applies to identify its curvature.

The involution τ' preserves the metric on G so its fixed set X is a *totally geodesic* submanifold and the sectional curvatures of X are given by restriction from those of G.
Since the map ι is a local isometry we get a formula for the sectional curvature of the symmetric space G/K:

$$K(\xi_1, \xi_2) = \pm\frac{1}{4}|[\xi_1, \xi_2]|^2,$$

where we take the $+$ sign if q is positive on $\mathfrak{p}$ and the $-$ sign if it is negative.

Duality

Given a Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as above we can define another Lie algebra structure on the same vector space by *changing the sign* of the component $[\mathfrak{p}, \mathfrak{p}]$. This changes the sign of Killing form on $\mathfrak{p}$.

The two Lie algebras are different real forms of the same complex Lie algebra.

Example

Consider $\mathrm{Lie}(SL(n, \mathbf{R})) = \mathrm{Lie}(SO(n)) \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the set of trace-free symmetric $n \times n$ matrices.

The dual Lie algebra is $\mathrm{Lie}(SU(n))$ and they are different real forms of the complex Lie algebra $SL(n, \mathbf{C})$.

Conclusion of the theory (E. Cartan)

- Take any simple real Lie algebra $\mathfrak{g}$ such that the Killing form is indefinite.
- The automorphisms of the Lie algebra form a Lie subgroup $G^- \subset GL(\mathfrak{g})$ with Lie algebra $\mathfrak{g}$.
- There is an involution τ of G defining a (maximal) compact subgroup $K \subset G^-$.
- The manifold $M^- = G^-/K$ is a symmetric space with sectional curvatures ≤ 0.
- The dual Lie algebra is the Lie algebra of a compact group G^+ containing K.
- The manifold $M^+ = G^+/K$ is a symmetric space with sectional curvatures ≥ 0.
- Up to coverings and taking products, including flat factors $\mathbf{R}^m$, all symmetric spaces are obtained this way.

The simple real Lie algebras are classified through analysis of the real forms of simple complex Lie algebras (types A,B,C,D,E,F,G).

Examples

- $S^n = SO(n+1)/SO(n)$, non-compact dual $H^n = SO(n,1)_0/SO(n)$—the spaces of constant sectional curvature.
- Grassmann manifolds $SO(p+q)/SO(p) \times SO(q)$, noncompact dual $SO(p,q)_0/SO(p) \times SO(q)$. Similarly for the complex and quaternionic cases.
- For any compact Lie group $K = K \times K/K$. Non-compact dual K^c/K where K^c is the complexified group. e.g $SL(n,\mathbf{C})/SU(n)$.
- $SU(n)/SO(n)$ is the space of special Lagrangian subspaces. The dual is $SL(n,\mathbf{R})/SO(n)$, the space of positive definite symmetric matrices of determinant 1. Similarly for complex and quaternionic versions.

- $Sp(n, \mathbf{R})/U(n)$ is the (noncompact) space of complex structures on on $\mathbf{R}^{2n}$ compatible with a fixed symplectic form. The compact dual is $Sp(n)/U(n)$.
- $SO(2n)/U(n)$ is the (compact) space of complex structures on $\mathbf{R}^{2n}$ compatible with a fixed Euclidean structure. The noncompact dual is $SO^*(2n)/U(n)$ where $SO^*(2n) = SO(2n, \mathbf{C}) \cap GL(n, \mathbf{H})$.

Example of calculation: for $\mathbf{CP}^2$ the sectional curvatures lie between $1/4$ and 1.

Remark For any simple real Lie algebra we have an embedding $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. If we have a Euclidean structure h on $\mathfrak{g}$ we have a transposition map $T_h : \mathrm{End}(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g})$.
The whole theory (not the classification) can be derived easily from the following statement: there is a Euclidean metric h on $\mathfrak{g}$ such that the image of ad is preserved by T_h.
This has a variational description. The Lie algebra structure is a tensor $\sigma \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$. A Euclidean structure h defines a norm $|\sigma|_h$, a function $V(h)$ on the space of Euclidean structures h of fixed determinant. The desired structure is the one that minimises this norm.