Errata for the book *Riemann Surfaces* (Oxford U.P, 2011)

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This is a compilation of corrections, comments and suggestions made by a number of readers: Robin Chapman, Dinesh Thakur, Thomas von Schroeter, J. David Taylor, John Milnor, Pavel Etingof. I am very grateful for their kindness in communicating these.

MINOR MATHEMATICAL ISSUES

p. 8, Proposition 2: it should be mentioned that $w_i(0) \neq 0$.

p. 24, line 10, $f(t)$ should be continuous.

p. 34, line 10, "will not make explicit use of this notion (complex manifold)" - but it is used later in the book (for example, when studying holomorphic maps from a RS to projective space).

p. 38, line -9: this should start "if $|\beta| \leq k$,"

p. 51: end, it is ignored that it can happen that $P(0, w) = 0$ has no roots and then f may not stay small and can go to infinity

p. 79, Proof of Lemma 13: I think the formulas for of A, B need to be divided by 2 and the last expression of the next displayed formula should be multiplied by 4.

p .80, line 8: a *continuous* function.

p. 86 Section 6.2 line 2: *nonconstant* meromorphic function.

p. 93, text after the third displayed formula. I think this is not correct. The oriented symmetry group of the tetrahedron has order 12, not 6, and it is A_4, which does not contain S_3. The action of S_3 in this case is the usual action by rotations of an equilateral triangle in the 3-space, it has nothing to do with the tetrahedron.

p. 100, line 6: the factor "i" is missing.

p. 107: Exercise 2, needs a and b relatively prime or some such condition, for example, if $a = b = 2$, it does not work.

p. 107 Exercise 3 is incorrect. (There can be one or two components.)

p. 135, before Lemma 19. We should say that it can be assumed that $d\phi$ does not vanish on any connected component of $X \setminus D$ where D is a closed disc containing supp ρ.

p.140, line 5. I think δ should equal $(\beta - \alpha)^2/4$.

Proof of Proposition 30 in 10.2.4. We should arrange that $X = K \cup U_0 \cup U_1$ where K is compact, U_0, U_1 have disjoint closures and are non-compact. To do this take U_1 to be a *union* of one or more connected components of $X \setminus K$.

In 11.1, 11.2, several places, the transcendence degree 1 field needs to be finitely generated over $\mathbf{C}$.

p. 148, part 2: One does not need the theorem on primitive elements here, it suffices just by definition.

p.150, rather than P irreducible in $\mathbf{C}[z, w]$, need to say as polynomial in w, and rather than any Q not identically zero, want Q not divisible by P (eg. $P = Q$ makes it false). p.151, Definition 9 needs correction: do not mean surjective to real numbers plus infinity, probably mean integers and infinity to get discrete as claimed on p. 156 for Q (but then it contradicts the statement right after the definition)

p. 154 , associating Galois group for characteristic zero extension L over K, (Gal(L/K) means for normal closure here (which is also not completely standard): extension should be finite (all that is needed here anyway). Also, in theorem 17, subscripts 1 and 2 are reversed.

p. 158, definition of absolute value needs usual conventions with non-negativity and zero (e.g., not to allow identically zero or identically one type absolute values). Also, Ostrowski theorem would need equivalence classes of absolute values.

p. 160, fourth displayed formula: square root is missing.

p. 163, last displayed formula: no square at the end.

p. 171, second and fourth displayed formula: the sign on the right hand side should be $+$.

p. 179. I think in the definition of a refinement, it should be stressed that it is already equipped with α, which is a part of data, so we don't really need to choose it.

p. 179 (the second paragraph in 12.1.1 on Cech cohomology) it is stated that the Cech cohomology of a topological space is the "inverse limit" of the cohomology over open covers but it is the direct limit / colimit / injective limit of those groups.

p. 181, Definition 11: Is S_m the stalk of S? It has not been defined before

p. 182, Definition 12: A HOLOMORPHIC rank r vector bundle...

p. 184, line 5 after Def. 13: NONZERO meromorphic function.

p. 185 and below: I think two notations are used for the same thing: $L_{[p]}$ and L_p.

pp 185-187 Proofs of Theorems 19 and 20. I think it is better to prove that if the property holds for L then it holds for $L \otimes L_p$ and vice versa (the proof is the same). Otherwise e.g. for Theorem 20 there will be a slight issue with $O(-1)$ for $\mathbf{P}^1$, which is not of the form L or KL^{-1} for L a product of L_p.

p.188, line 3: I think it should be dual to the FIRST sequence.

p.193: I think there is no need to list all monomials and their weights. It is clear that all monomials that occur in the equation must have weight $\lambda = 0$ in $\mathbf{F}_7$ since otherwise the permuted monomial by $\mathbf{Z}/3$ will have a different weight, 2λ. It suffices to find monomials of degree 0 where degree of y (of weight 4) is lowest, hence ≤ 1. This degree can't be 1 (x^2yz and xyz^2 don't have weight 0), so it is 0. So we have the monomial $x^{4-b}z^b$ of weight $4-b+2b=4+b$, so the only possibility is $b=3$.

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p.267, line 1: I think it needs to be $\sum b_i = 0$, not $\sum a_i = 0$.

MISPRINTS ETC.

p. 3, bottom: In the last two equations on this page, minus signs are missing in the arguments of the exponentials; i.e. e^t should be replaced by e^{-t}

p. 4 , centre: The power of z in the last term of $f(z)$ should be z^4, not z^5.

p. 4, text 3 lines below expression for $f(z)$: "going once around the origin" should probably be replaced by "going once around -1" since this is the branch point.

p. 5, bottom: A factor $(1/2\pi i$ is missing in the expression for $\phi(z)$.

p. 5, last displayed formula and p. 42 first displayed formula miss $1/2\pi i$ factor.

p.7, line 1: "both the sums" $\mapsto$ "both sums"

p. 8 higher-order differential equation: A term proportional to the undifferentiated function u seems to be missing; besides, as printed, the equation looks inhomogeneous. If this is to be a generalisation of (1.1), then probably P_r should be the coefficient of u , and p_{r+1} would not be needed.

p. 9. Exercise 5: I get $A = c$ and $B = 0$ in the indicial equation, which would yield $1 - c$ as the second root, not c. The cubic term in $F(z)$ has a left bracket missing from the factor $a + 2$.

p.14, line 13 from bottom: "any of one of" $\mapsto$ "any one of"

p.15, line 5: MorEover

p.16, line 2: what we should do is prove (remove "to")?

p.34, line 16 from bottom: dots missing (p_1, p_r)

p.35, line 12 from bottom: $p(z, w, 1)$; not so important, but previously $p(1, z, w)$ was used.

p.39, line 10: small enough SO that

p. 39: last but one para. 'more than' should be 'at least'. In fact, by easy argument, the fact there works for $p = 2, 3$ also. ($p = 3$ implicitly used on pa. 106 without proving here).

p. 44, line above displayed formula, "Y, connected": comma not needed.

p.47, Proposition 10(1), second statement, better "a path lift exists".

p.48, line 18: "of index then" $\mapsto$ "of index d then" (d missing).

p. 50 line 6, 'homomorphism' should be 'homeomorphism'

p. 55: two lines after displayed formula, 'i' is missing

p.59, line 4: "this is notion" $\mapsto$ "this notion"

p.60, line 3 after equation (5.2): comma in the formula missing.

p.63, line 2: I believe the derivative should be flipped $(\partial x_i/\partial y_i)$.

p.67, line 3 of example 1: aN annulus

p.73, line 8: the interval should be $[t_i, t_{i+1}]$; last displayed formula: TX should be TX_p

p.75, line 6: $\beta\bar{z}$ should be $\beta d\bar{z}$

p.76: there are sign errors in both the first two formulas. Note that $dz\, d\bar{z} = -2i\, dx\, dy$ etc.

p. 77: there is a sign error in the explicit formula for Δf which I would resolve by writing it as

$$\Delta f = \frac{2i}{4}(\cdots)(\cdots)f(d\bar{z}\, dz) = -(\cdots)dx\, dy.$$

p. 79: I reckon $\langle A, B\rangle$ should actually be

$$2i \int_X A^{1,0} \wedge B^{0,1}.$$

For if (locally) $A = B = dx$ then $A^{1,0} = dz/2$ and $B^{0,1} = d\bar{z}/2$ and then $2iA^{1,0} \wedge B^{0,1} = (i/2)dz\, d\bar{z} = (i/2)(-2i\, dx\, dy) = dx\, dy$. But $2iA^{0,1} \wedge B^{1,0} = (i/2)d\bar{z}\, dz = (i/2)(2i\, dx\, dy) = -dx\, dy$ etc.

p. 81, line 4: on compactly supported (no "a")

p. 87, last displayed formula: put derivative in parentheses. Page 89: second displayed formula: $i/1$ should be $1/2$

p. 90, line 3: notice that.

p. 91, top formula: put derivative in parentheses.

p. 92, line 12: Möbius; line before first displayed formula: delete "the".

p. 98: proposition 17 : minus sign should be dropped.

p. 99: at the end when two multiplicities are said to be the same, they are in fact negative of each other. (These two sign mistakes on pa. 98 and 99 eventually cancel out!)

p. 100: last displayed formula, k_x should be $k_x - 1$.

p. 101: last displayed formula, $k_x - 1$ should be k_x.

p. 101 : Prop. 19: X and Y are interchanged.

p. 106: $p(p^2 - p)$ should be $p(p^2 - 1)$ in two places.

p. 114: $\sigma(\alpha) = [\theta]$ rather than with minus sign.

p. 121, line 11: I would omit

$$\int_X \psi\rho - \int_X \nabla\phi.\nabla\psi$$

as ∇ hasn't been defined on a general Riemann surface and Lemma 14 already gives the equation

$$\int_X \psi\,\Delta\phi = \langle\phi,\psi\rangle\,.$$

p. 122, line 4: RepresenTAtion

p. 126: first display, inner product should have subscript D for Dirichlet norm (not to confuse with the usual one, say on pa. 128).

In 9.4, there is switch back and forth from Δ being a form or a function.

p. 127 second para., first line: ψ' should be ϕ'.

p. 129, last line: the integral should be raised to the power $1/p$.

p. 128 second display: 2π is missing.

p. 132 first para. In $\partial(A - \bar{\partial}f)$, f should be g.

p. 148, first displayed formula: the second sum is over $i < j$.

p. 151, line 3: $= 0$ is missing.

p. 155, line 15: equivariAnt

p. 157, line 7: For simplicity, assume that

11.2.4, first line of the second paragraph: no "a"

p. 167, line 2 of the second paragraph: minus in front of 0 not needed.

p.168. The identifications should be $PSL(2,\mathbf{R}) = PSU(1,1) = SO(2,1)^+$.

p.172, Lemma 28 and its proof. Should S^* be Q^*?

p. 184, zeros and poles are reversed in principal divisors compared to the usual practice, but may be consistent, so OK.

p. 195 , last line 'finitely' missing.

p. 196 , 12.2.1 first para.: there is 'non-holomorphic' meromorphic

12.2.2, line 4: meromorphic function.

12.2.4, line 9: colon after j_d

p. 197, 12.2, drop $F_i^{-1}\overline{\partial}F_i$ after χ_i

p. 203, line 7 from bottom: six lines in $\mathbf{CP}^2$ (not $\mathbf{CP}^1$).

p. 213, line 13 from bottom: brace missing after "view". Also slight latex issue in the penultimate line of this page.

p. 264, line 8 after the second displayed formula, as well as lines 4 and 6 from bottom: $\partial/\partial z$ is poorly typeset.

p. 266, end of second paragraph: this IS the upper half-plane.

p. 266: 3rd line from bottom: ratio

p. 267, Remark 1, line 3: remove comma at the end of the line.

p. 268, first display: first power should be $-1/2$ rather than $1/2$

SUGGESTIONS FOR IMPROVED/ALTERNATIVE PROOFS AND EXPOSITION.

pp. 11–12: it might be clearer to say "Start with a disc, and add g pairs of ribbons…" and state that Figure 2.3 illustrates the case $g = 2$.

p. 36, line 5 of 3.2.3. Radius 1/2 looks slightly awkward (was it originally $\mathbf{C}/$
bZ, not $\mathbf{C}/2\pi\mathbf{Z}$)? Also π has two meanings (also denotes the map). Maybe best to work with $\mathbf{C}/\mathbf{Z}$ and write $z \to e^{2\pi iz}$ on the next page?

p. 37, line 6: I think it is better to use τ rather than λ for the modulus of elliptic curve. This is more standard and also avoids a clash of notation - later you consider elliptic curves $y^2 = x(x-1)(x-\lambda)$.

p. 42, proof of Lemma 3: no need to take the Taylor expansion. Let k be the order of f at 0, then can write $f = z^k h$ with $h(0) \neq 0$, set $g := zh^{1/k}$.

p. 60, last three lines: maybe better to say more precisely that it depends only on the image for simple, oriented paths.

p. 63, Lemma 6: maybe remark that the first condition is not needed if the surface is second countable, but it is given since second countability is not assumed.

p.64, Lemma 7. (*Etingof*) Here is a simpler proof.

1. Let us prove the lemma when U_i are disks. Let U_i' be slightly smaller disks still covering K and g_i be smooth non-negative functions supported on U_i which are strictly positive on U_i'. Then the function $\sum g_j$ is strictly positive on K. Let m be its minimal value. Let $h : \mathbf{R}_{\geq 0} \to \mathbf{R}_{>0}$ be a smooth positive

function such that $h(x) = x$ for $x \geq m$. Consider the smooth functions $f_i := g_i/h(\sum_j g_j)$. Then $\sum f_i = 1$ on K and f_i is supported in U_i, as desired.

2. Now the general case. For each $x \in K$ let B_x be a ball around x contained in some $U_{i(x)}$. These balls form an open cover which has a finite subcover $B_{x_1}, ..., B_{x_m}$. Let $f_1, ..., f_m$ be the functions constructed in part 1 for these balls. For $1 \leq k \leq n$, let $F_k := \sum_{j:i(x_j)=k} f_j$. Then $F_1, ..., F_n$ is a desired partition of 1.

p.72, Proposition 15. (*Etingof*) Here is a simpler proof. It is enough to prove the statement for simple loops (as any loop is homotopic to a concatenation of finitely many simple ones). Then we may replace S by a neighborhood of γ. So we may take S to be the cylinder $(\mathbf{R}/\mathbf{Z}) \times (-\delta, \delta)$ and γ to be the circle consisting of points $(x, 0)$. Then we may take $\theta = h(y)dy$ where h is a compactly supported function on $(-\delta, \delta)$ with $\int h(y)dy = 1$.

p. 73, maybe remark that Corollary 2 already proved on p.69, so this gives another proof.

p.84. Theorem 3.(*Etingof*) Here is a simpler proof. Suppose there is a non-vanishing holomorphic 1-form on X. Its inverse is then a holomorphic vector field on X. As X is compact, we can define the flow given by this vector field, which is an action of $\mathbf{C}$ on X. It is clear that orbits of this action are open, so there is only one orbit, as X is connected. Thus $\mathbf{C}$ acts transitively on X. Let Λ be the stabilizer of $x \in X$. Then Λ is discrete, and $\mathbf{C}/\Lambda = X$, so Λ must be a rank 2 lattice.

p. 88 6.3.1: I would use period it for $t > 0$ instead of i, this makes formulas look more natural. Also I'd take the second period to be 1 as is more customary. Then "special values" in line 4 of p.91 won't be needed.

p. 128, Proof of regularity of weak solutions of the Laplace equation (*Etingof*). Here is an alternative argument that is a bit simpler. Let ϕ be a weak solution of $\Delta\phi = 0$ on a cylinder with complex coordinate $z = t + i\theta$, $x \in (-c, c)$, $\theta \in [0, 2\pi]$ (this cylinder is conformally equivalent to an annulus in $\mathbf{C}$, so this is sufficient). Let $\phi_n(t) := \int_0^{2\pi} \phi(t + i\theta)e^{-in\theta}d\theta$. This is locally L^1 and satisfies the equation $\frac{d^2\phi_n}{dt^2} - n^2\phi_n = 0$, so $\phi_n(t) = a_n e^{|n|t} + b_n e^{-|n|t}$ and $\phi_0(t) = a_0 + b_0 t$. Also since $||\phi||_{L^2} < \infty$, we have

$$\sum_{n>0} \int_{-c}^{c} |a_n e^{nt} + b_n e^{-nt}|^2 = \sum_{n>0} (|a_n|^2 + |b_n|^2) \frac{e^{cn} - e^{-cn}}{n} + 2c \sum_{n>0} (a_n \bar{b}_n + b_n \bar{a}_n) < \infty.$$

Thus

$$\sum_{n>0} (|a_n|^2 + |b_n|^2) \frac{e^{cn}}{n} < \infty.$$

Similarly,

$$\sum_{n>0} (|a_{-n}|^2 + |b_{-n}|^2) \frac{e^{cn}}{n} < \infty.$$

So

$$\phi(z) = b_0 \log|z| + \sum_{n\in\mathbf{Z}} a_n e^{nz} + \sum_{n\in\mathbf{Z}, n\neq 0} b_n e^{n\bar{z}}$$

is an absolutely and uniformly convergent series, hence real analytic

In Chapter 8 and 12.2 it might be good to comment that assumptions p_i, q_i distinct are not really needed in Riemann-Roch and Abel-Jacobi. Also, 12.2.1 would fit even better at the end of chapter 8.