Topology of manifolds: problems

TO BE FORMALLY ASSESSED FOR THE COURSE PLEASE SEND SOLUTIONS (PREFERABLY IN TEX) TO s.donaldson@imperial.ac.uk BY JANUARY 7th. 2013. IT IS NOT INTENDED THAT YOU ANSWER ALL QUESTIONS (SOME ARE OPEN-ENDED). YOU SHOULD CHOOSE SOME WHICH SEEM INTERESTING AND PRACTICAL FOR YOU AND AIM TO PRODUCE A REASONABLE AMOUNT OF WORK.

Question 1. Let M be the 3-manifold obtained by $+3$ surgery on the trefoil knot.

- Show that $\pi_1(M)$ has a presentation $\langle x, y : x^3 = y^2 = (xy)^2 \rangle$.
- Let $\Gamma \subset SO(3)$ be the symmetries of a regular tetrahedron, a group of order 12. Let $\tilde{\Gamma}$ be the pre-image of Γ under the double covering $SU(2) \to SO(3)$. Thus $\tilde{\Gamma}$ has order 24. Construct a homomorphism from $\pi_1(M)$ onto $\tilde{\Gamma}$.
- Show that M is diffeomorphic to the quotient $SU(2)/\tilde{\Gamma} = SO(3)/\Gamma$.

(*Warning: one has to be careful with signs. The "right-handed" and "left handed" trefoil knots are not equivalent by an orientation-preserving diffeomorphism of S^3. I hope this sign (+3) fits with the picture of the trefoil used in the lectures.*)

Question 2.

- Let Y be the blow-up of $\mathbf{CP}^2$ at a point. Construct a fibration of X with base and fibre S^2. By considering the intersection form, or otherwise, show that X is not diffeomorphic to $S^2 \times S^2$.
- Show that the blow-up of $\mathbf{RP}^2$ (defined algebraically in the obvious way) at one point is diffeomorphic to the Klein bottle.

Question 3. Let $P_k(z_0, z_1, w_0, w_1, w_2)$ be a generic complex polynomial which is homogeneous of degree 3 in the w_i and of degree k in the z_i. (So for example a monomial $z_0 z_1 w_0^2 w_1$ would qualify as a term in P_2.) Show that P_k defines a zero set V_k in $\mathbf{CP}^1 \times \mathbf{CP}^2$.

- Show that V_1 is diffeomorphic to $X = \mathbf{CP}^2 \sharp 9 \overline{\mathbf{CP}}^2$.
- Show that V_k is diffeomorphic to the fibre sum of k copies of X, as defined in lectures.
- Investigate what happens in the real case.

Question 4.

Suppose M is a manifold and $\alpha \in H^p(M), \beta \in H^q(M), \gamma \in H^r(M)$. We want to consider a situation where the cup products $\alpha\beta$ and $\beta\gamma$ are both zero. We will work with real co-efficients using the de Rham representation of cohomology via differential forms, although this is not essential. So we choose representative differential forms a, b, c for α, β, γ and the hypothesis is that $ab = d\phi$ and $bc = d\psi$ for some forms ϕ, ψ.

- Show that $\phi c + (-1)^{p+1} a\psi$ is a closed form.
- Show that this construction gives a class $\langle \alpha; \beta; \gamma \rangle$ in $H^{p+q+r-1}(M)/L$ where $L = \gamma H^{p+q-1} + \alpha H^{q+r-1}$, which depends only on α, β, γ. (This is the "Massey product".)
- Develop a version of this definition, for classes represented by manifolds, using intersection theory.
- Let $M \subset S^3$ be the complement of the *Borromean link*. This is a well-known link with three components K_1, K_2, K_3. By Alexander Duality H^1 has rank 3 with 3 generators $\alpha_1, \alpha_2, \alpha_3$ and H^2 has rank 2 with three generators P_1, P_2, P_3 modulo the relation $P_1 + P_2 + P_3$. Show that $\alpha_i \alpha_j = 0$ and that the Massey product $\langle \alpha_1; \alpha_2; \alpha_3 \rangle$ is not zero.

Question 5

- Let $E \to B$ be a complex vector bundle so it has Chern classes $c_i(E)$. Let $E_{\mathbf{R}}$ be the same bundle regarded as a real vector bundle (forgetting the complex structure). Show that $p_1(E_{\mathbf{R}}) = 2c_2(E) - c_1(E)^2$.
- With E and B as above suppose that E has rank n and B is a closed oriented manifold of dimension $2n$. Show from the definition of the Chern classes that $\langle c_n(E), [M] \rangle$ coincides with the number of zeros of a generic section of E, counted with signs. (In other words $c_n(E)$ is the Euler class of $E_{\mathbf{R}}$.)
- Suppose X is a closed oriented 4-manifold with an *almost complex structure* (i.e. a bundle map $I : TX \to TX$ with $I^2 = -1$, compatible with the orientation); so its tangent bundle has a first Chern class $c \in H^2(X; \mathbf{Z})$. Show that $c^2 = 2\chi(X) + 3\sigma(X)$.

Question 6
Construct a fibration $\mathbf{CP}^3 \to S^4$ and use this to show that $\mathbf{CP}^3$ is an oriented boundary. (Hint: think of S^4 as $\mathbf{HP}^1$.)

Question 7

Let T be the tangent bundle of $\mathbf{CP}^n$, $H \to \mathbf{CP}^n$ the dual of the tautological bundle and $\underline{\mathbf{C}}^q$ the trivial bundle of rank q. Establish an isomorphism

$$T \oplus \underline{\mathbf{C}}^1 = H \otimes \underline{\mathbf{C}}^{n+1}.$$

Hence show that $\sum c_i(T) = (1+h)^{n+1}$, where $h = c_1(H) \in H^2(\mathbf{CP}^2)$ Use this to calculate the Pontrayagin classes of $\mathbf{CP}^n$ and thus to verify the Hirzebruch formula for $\mathbf{CP}^4, \mathbf{CP}^2 \times \mathbf{CP}^2$.

Question 8

Use a similar approach to the previous question to show that if T is the tangent bundle of $\mathbf{HP}^n$ then

$$\sum p_i(T) = (1+u)^{2n+2}/(1+4u)$$

where $u \in H^4(\mathbf{HP}^n)$ is a standard generator. Calculate the Hirzebruch polynomial L_3 and verify the signature theorem for $\mathbf{HP}^3$.

Question 9.

By considering the sphere S^n as embedded in $\mathbf{R}^{n+k}$, construct a homomorphism $J : \pi_n(SO(k)) \to \pi_{n+k}(S^k)$.

Question 10.

Let M^n be a compact manifold with trivial tangent bundle.

- Show that the manifolds underlying the groups $U(d), SO(d)$ have trivial tangent bundles.
- Show that M can be embedded in $\mathbf{R}^{n+k}$ for large k with trivial normal bundle.
- Show that a choice of trivialisation of TM yields a well defined element of the stable homotopy group $\pi_{n+k}(S^k)$ $\;k >> 0$.

Question 11.

Rohlin's Theorem (in not the most general form) states that if X is a closed oriented simply connected 4-manifold such that $\alpha.\alpha$ is even for every $\alpha \in H_2(X;\mathbf{Z})$ then the signature $\sigma(X)$ is divisible by 16.

Suppose that there is a smooth embedded 2-sphere Σ representing the class $3h \in H_2(\mathbf{CP}^2)$ (where h is the standard generator, the class of a projective line). Show that one could then define an embedded 2 sphere $\tilde{\Sigma}$ in $Y = \mathbf{CP}^2 \sharp 10 \overline{\mathbf{CP}}^2$ such that $\alpha^2 = \tilde{\Sigma}.\alpha$ modulo 2, for any class $\alpha \in H_2(Y;\mathbf{Z})$ and $\tilde{\Sigma}.\tilde{\Sigma} = -1$. Show that in that case $Y = X \sharp (\overline{\mathbf{CP}}^2)$ for a simply connected 4-manifold X and use Rohlin's Theorem to obtain a contradiction to the existence of such a surface Σ.

Question 12. Let $K \subset S^3$ be a knot and N a tubular neighbourhood. The knot is called a *fibred knot* if there is a fibration $S^3 \backslash N \to S^1$ with connected fibre F which restricts on $\partial N = T^2$ to a standard fibration $T^2 \to S^1$. In this situation we have a *monodromy* $\mu : F \to F$. This has an action $\mu_* : H_1(F) \to H_1(F)$. Set $P(t) = \det(\mu_* - t1)$.

- Show that $P(t)$ has even degree $2g$ and the co-efficients p_i satisfy $p_{2g-i} = p_i$. The Laurent polynomial $\Delta(t) = t^{-g}P(t)$ is the normalised Alexander polynomial of K.
- Show that for any coprime p, q the p,q torus knot

$$K_{p,q} = \{(z,w) \in S^3 \subset \mathbf{C}^2 : z^p = w^q\}$$

  is a fibred knot.
- When $p = 2, q = 3$ (so the knot is the trefoil) compute g, the action μ_* and the Alexander polynomial.
- More ambitiously, you could try to do the same for general p, q.

Question 13.

Let X_0, X_1 be simply connected oriented 4-manifolds and W be an h-cobordism from X_0 to X_1. Suppose that there is a Morse function f on W with critical points of indices $2, 3$ only. Suppose that for each critical point x of index 2 we have $f(x) < 1/2$ and for each critical point y of index 3 we have $f(y) > 1/2$.

- Show that the number of critical points of index 3 is the same as the number of index 2.
- Show that there are integers a, b such that $X_0 \sharp a\mathbf{CP}^2 \sharp b\overline{\mathbf{CP}}^2$ is diffeomorphic to $X_1 \sharp a\mathbf{CP}^2 \sharp b\overline{\mathbf{CP}}^2$.
- In fact if X_i have the same intersection form the hypotheses above are met. In particular we can take X_0 to be the K3 surface X and $X_1 = X_K$ to be the manifold obtained using Fintushel-Stern knot surgery based on a knot $K \subset S^3$. An interesting (ambitious) project would be to find what numbers a, b are required above. There may well be research papers which consider this.