Extremal Kähler metrics and convex analysis

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1 A variational problem

The aim of this article is to give an impression of some contemporary developments in complex differential geometry through the particular case of toric manifolds where the constructions can be expressed in elementary terms. Our starting point is a partial differential equation for a function u of n real variables $x_1, \ldots, x_n$. We require the function to be strictly convex, by which we mean that the matrix of second derivatives

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

is positive definite at each point. Let (u^{ij}) be the inverse matrix. The partial differential equation is

$$\sum_{ij} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -A, \tag{1}$$

where A is a given function of $x_1, \ldots x_n$. (We are mainly interested in the cases when A is a constant, or a linear function.) This equation was first written down by Miguel Abreu in [1]. It is a nonlinear fourth order PDE, the nonlinearity coming from the nonlinear map which takes the matrix (u_{ij}) to its inverse. (u^{ij}). The equation is closely related to Monge-Ampère equations which arise in many parts of pure and applied mathematics. These are second order PDE which have the form

$$\det(u_{ij}) = F,$$

where F is a given function of x, u and the first derivatives of u. For example, it is an exercise to show that a solution of the Monge-Ampère equation with $F = 1$ is a solution of (1) with $A = 0$.

We want to consider a function u on a convex polytope $P \subset \mathbf{R}^n$. So P is a bounded set defined by a finite number of inequalities $\lambda_j(\mathbf{x}) < 0$, for affine-linear functions λ_j. (By affine-linear we mean a function of the form $C + \sum c_i x_i$.) We also fix a measure $d\sigma$ on the boundary of P. This is to be just a multiple of the standard $(n-1)$-dimensional volume measure on each face of the boundary. It is elementary that there is a unique affine-linear function A such that for any affine-linear function f

$$\int_P fA d\mu = \int_{\partial P} f d\sigma, \tag{2}$$

where $d\mu$ is the Lebesgue measure on $\mathbf{R}^n$. Now, with this function A, we want to solve the PDE (1) in P for a function u satisfying certain boundary conditions. These are, roughly speaking, that as we approach a point p on a face on which the measure $d\sigma$ is m_α times the volume measure the function u should behave like $m_\alpha D \log D + u(p)$ where D is the distance to the boundary. The boundary conditions can be built into a variational formulation of the problem. For a function f on the closure $\overline{P}$ define

$$L_{\overline{P}}(f) = \int_{\partial P} f d\sigma - \int_P fA d\mu.$$

Now define a functional on convex functions u on $\overline{P}$, smooth in the interior, by

$$\mathcal{M}(u) = -\int_P \log\det(u_{ij}) + L_{\overline{P}}(u). \tag{3}$$

The function $-\log\det H$ on positive symmetric matrices H is convex, so the same is true of the functional $\mathcal{M}$ and any critical point is a minimum. A variational analysis shows that a minimiser is the same as a solution of equation (1) satisfying the boundary conditions. The relevance of the condition (2) on A is clear from this variational point of view, because if it did not hold the functional is obviously not bounded below, since adding an affine-linear function to u does not change $\log\det(u_{ij})$.

2 Toric geometry

To explain where the PDE (1) comes from, we begin with the case of surfaces of revolution. Away from the fixed points we can choose "equiareal" co-ordinates (x, θ) in which the metric has the form $h dx^2 + h^{-1} d\theta^2$ where h is a function of x and the circle action rotates the θ co-ordinate. (Equiareal means that the area form of the metric is the standard form $dx d\theta$ in these co-ordinates.) The Gauss curvature is given by the formula

$$K = -\frac{1}{2}\frac{d^2\ h^{-1}}{dx^2}. \tag{4}$$

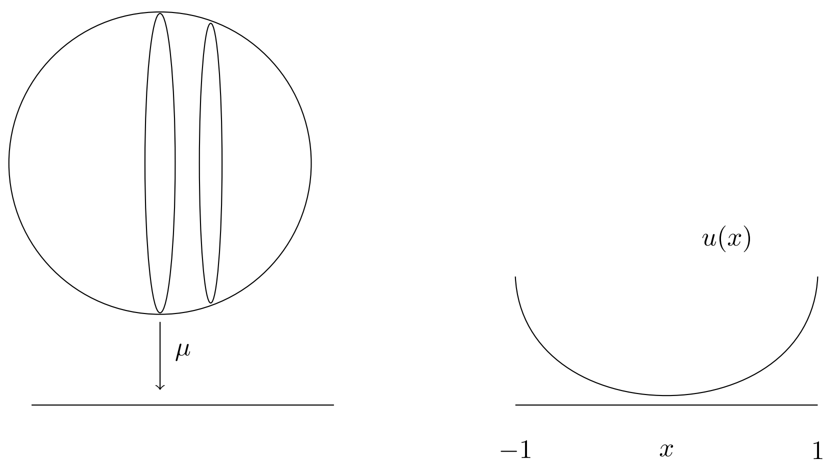
If we integrate twice to write $h = \frac{d^2u}{dx^2}$ for a convex function $u(x)$ this gives the expression on the left hand side of (1) up to a factor $-1/2$, so the equation (1) is prescribing the Gauss curvature as a given function $A(x)$. Take, for example, the case of the standard round 2-sphere in $\mathbf{R}^3$ rotating about an axis. Then, by a result of Archimedes, the equiareal co-ordinate x is the projection onto this axis and the metric is

$$(1-x^2)^{-1} dx^2 + (1-x^2) d\theta^2$$

so $h(x) = (1 - x^2)^{-1}$ and

$$u(x) = \frac{1}{2}\left((1-x)\log(1-x) + (1+x)\log(1+x)\right)$$

on the interval $(-1, 1)$, which is our polytope P in this case.



The round 2-sphere and its symplectic potential function

The introduction of u may seem artificial in this 1-dimensional case but becomes essential in higher dimensions. The general setting is a *Kähler metric* on a manifold of dimension $2n$, with an isometric action of a n-dimensional torus T^n. Thus, on the subset where the action is free, we have n angular co-ordinates $\theta_1, \dots, \theta_n$ and it can be shown that that there are additional co-ordinates $x_1, \dots, x_n$ and a "symplectic potential" function $u(x_1, \dots, x_n)$ such that the metric has the form

$$\sum u_{ij} dx_i dx_j + u^{ij} d\theta_i d\theta_j, \tag{5}$$

where u_{ij} and u^{ij} are defined as before. The expression on the left hand side of (1) gives minus the *scalar curvature* of this metric. Solutions of the equation (1) with a constant A give constant scalar curvature Kähler (CSCK) metrics. When A is an affine-linear function they give *extremal Kähler metrics*, a notion introduced by Calabi. For the purposes of this article the reader does need to know this differential geometric background: the point is that CSCK and extremal metrics are natural higher dimensional generalisations of constant

Gauss curvature surfaces. On the 2-sphere there are two points where the θ co-ordinate is not defined, the fixed points of the rotation action. We have a map $\mu : S^2 \to [-1, 1]$ mapping these two points to the endpoints of the interval and the description above is valid over the interior $(-1, 1)$. The general story for a compact $2n$-dimensional Kähler manifold X with T^n action is that there is a map $\mu : X \to \mathbf{R}^n$ with image a closed convex polytope $\overline{P}$. Over the interior P of $\overline{P}$ the fibres of μ are free T^n-orbits but over boundary points the fibres are lower dimensional tori. The polytopes that arise in this way form a special class called *Delzant polytopes*. The definition involves an integrality condition: there must be n faces meeting at each vertex and these must be equivalent, under the action of $GL(n, \mathbf{Z})$ and translations, to the standard co-ordinate hyperplanes. The integral structure defines a measure on each face of the boundary of $\overline{P}$, for the face is contained in a hyperplane $H + p$ and we have a lattice $H \cap \mathbf{Z}^n$ in H which fixes a measure.

Kähler geometry is the intersection of symplectic geometry and complex geometry and the discussion above is the symplectic picture. We could go on to write down a complex structure on X in which the action of the torus T^n extends to a holomorphic action of $T^n_{\mathbf{C}} = (\mathbf{C}^*)^n$, with an open dense orbit—just as for $\mathbf{C}^* \subset S^2$. But to keep things short let us move on to an algebro-geometric point of view.

Any convex set $\Pi \subset \mathbf{R}^n$ defines a graded algebra $R = R_\Pi$. First take the cone on Π, the set

$$C(\Pi) = \{(x, h) \in \mathbf{R}^n \times \mathbf{R} : h \geq 0, x \in h\Pi\},$$

and let Σ_Π be the intersection of $C(\Pi)$ with the integer lattice $\mathbf{Z}^n \times \mathbf{Z}$. The algebra R_Π has an additive basis s_ν corresponding to points $\nu \in \Sigma_\Pi$ and multiplication defined by $s_\lambda s_\nu = s_{\lambda+\nu}$. The grading is provided by the $\mathbf{Z}$ component of ν. Similarly, there is an obvious action of $T^n_{\mathbf{C}}$ on R. For a general convex set Π this algebra will not be finitely generated but in the case of a closed polytope $\overline{P}$ which is the convex hull of a finite number of points in the integer lattice $\mathbf{Z}^n \subset \mathbf{R}^n$ it will be. In that case, by general foundational results in algebraic geometry (the "Proj" construction), $R_{\overline{P}}$ is the coordinate ring of a "toric variety" $X \subset \mathbf{CP}^N$ with a $T^n_{\mathbf{C}}$ action on X induced by that on $R_{\overline{P}}$. If the polytope $\overline{P}$ also satisfies the Delzant condition, X will be a complex projective manifold. We will call such polytopes *integral Delzant polytopes*.

For an example, let $\overline{P}$ be the interval $[-1, 1]$ in $\mathbf{R}$. Then the ring $R_{\overline{P}}$ is generated by the three elements U, V, W corresponding to the lattice points $(-1, 1), (0, 1), (1, 1)$ with a single relation $V^2 = UW$. This is the co-ordinate ring of the conic curve in $\mathbf{CP}^2$ defined by the same equation in homogeneous co-ordinates.

The differential geometric and algebro-geometric discussions are compatible, so for an integral Delzant polytope $\overline{P}$ the T^n-invariant Kähler metrics on the complex projective manifold X we defined algebraically above correspond to convex functions u on P satisfying our boundary conditions. (More precisely, the correspondence is with Kähler metrics in the cohomology class determined by the projective embedding.)

With this background we have reached the main point. An important question in complex differential geometry is: when does a projective manifold admit an extremal metric? This includes (for the special class of manifolds with vanishing first Chern class) the question of the existence of Calabi-Yau metrics with zero Ricci curvature, which was famously answered by Yau in 1978. But in general the extremal condition is the right one to consider. By what we have said, in the case of toric manifolds X this question comes down to the solubility of our PDE (1). (The Calabi-Yau condition, in the toric case, becomes a Monge-Ampère equation.)

3 The existence theorem

Fix a base point p_0 in the interior of our polytope $\overline{P}$ and call a convex function u *normalised* if $u \geq 0$ and $u(p_0) = 0$. By adding affine-linear functions we can restrict attention to normalised functions u. Contemplating the formula (3) one sees that the minimisation problem involves two competing effects. To make the integral of $-\log\det(u_{ij})$ small we should make $\det(u_{ij})$ large, so we should make the second derivatives of u large in at least some directions, but that will make the function u large on the boundary so the term in (3) involving the integral of u over the boundary will be large. The question is whether a balance between these two effects can be achieved. An answer to this question is known, at least for Delzant polytopes.

Theorem 1 *Let $\overline{P} \subset \mathbf{R}^n$ be a Delzant polytope and $L_{\overline{P}}, \mathcal{M}_{\overline{P}}$ be the corresponding functionals. There is a minimiser of the functional $\mathcal{M}_{\overline{P}}$ on normalised functions u if and only if $L_{\overline{P}}(f) > 0$ for all non-zero normalised convex functions f on $\overline{P}$. This minimiser is unique.*

This statement combines work of many people, the final step being achieved in the recent preprint [6]. Earlier work of the author [4],[5] and B. Chen,Li, Sheng [3] dealt with the case $n = 2$. For higher dimensions, the breakthrough comes from work of X. Chen and Cheng [2], in the larger setting we discuss in the next section. The entire proof involves a mountain of analysis and we only attempt to make the statement plausible.

The convexity of the functional $\mathcal{M}$ gives the uniqueness part of Theorem 1. Another simple fact is that if there is a smooth convex function f with $L_{\overline{P}}(f) < 0$ then $\mathcal{M}$ is not bounded below. For if we take any convex function u satisfying the boundary conditions and set $u^{(s)} = u + sf$ then for $s \geq 0$ the function $u^{(s)}$ is convex and also satisfies the boundary conditions. We have $\mathcal{M}(u^{(s)}) \leq \mathcal{M}(u_0) + sL_{\overline{P}}(f)$ since $\det(u^{(s)})_{ij}) \geq \det(u_{ij})$, hence $\mathcal{M}(u^{(s)}) \to -\infty$ as $s \to \infty$. Turning to the existence question, consider a finite-dimensional analogue of our infinite dimensional situation, with a function F on a Euclidean space $\mathbf{R}^N$. The lack of compactness of $\mathbf{R}^N$ means that, even if F is bounded below, there may be no minimum: for example the function $F(x) = e^x$ on $\mathbf{R}$. But a "coercive inequality" of the form $F(x) \geq \epsilon\|x\| - C$ for some $\epsilon > 0$ implies that a minimiser must exist. In our problem, suppose we know that there is a bound, for some $\lambda > 0$ and all normalised convex functions f on $\overline{P}$:

$$L_{\overline{P}}(f) \geq \lambda \|f\|_{L^1(\overline{P})}. \tag{6}$$

Then it is not hard to show, using the slow growth of the logarithm function, that this implies that the nonlinear functional $\mathcal{F}$ satisfies a coercive inequality, for normalised u:

$$\mathcal{M}(u) \geq \epsilon \|u\|_{L^1(\overline{P})} - C, \tag{7}$$

for some $\epsilon > 0$. One of the main results of Chen and Cheng is that such an inequality implies the existence of a minimiser—the infinite-dimensional problem behaves like the finite-dimensional analogue. The recent work of Li, Zian and Shen [6] establishes a convex analysis result, that the positivity hypothesis in the statement of Theorem 1 is equivalent to a "uniform" inequality (6), which is an *a priori* stronger condition.

4 Stability of complex projective manifolds

The theorem of the previous section gives, in a sense, a complete answer to the question of the existence of extremal Kähler metrics on toric manifolds. It fits into a larger picture, for general projective manifolds, where the final answer is not yet known.

Recall that any convex set in $\mathbf{R}^n$ defines a graded ring. Let f be a convex function on our polytope $\overline{P}$ and define a convex subset Q of $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ by

$$Q = \{(x,h) \in \overline{P} \times \mathbf{R} : h \geq f(x)\},$$

so we get a ring R_Q. The translation $(x,h) \mapsto (x, h+1)$ induces the structure on R_Q of an algebra over the polynomial ring $\mathbf{C}[t]$. We can also obtain R_Q as the *Rees algebra* of a filtration of the graded $R_{\overline{P}}$. In general, let R be an algebra over $\mathbf{C}$ with a filtration by vector subspaces

$$0 = \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots,$$

such that

$$\mathcal{F}_a.\mathcal{F}_b \subset \mathcal{F}_{a+b}. \tag{8}$$

The Rees algebra is the algebra over $\mathbf{C}[t]$

$$\mathrm{Rees}(R, \mathcal{F}_*) = \bigoplus_a \mathcal{F}_a t^a \subset R[t].$$

In the case at hand, let Σ_a be the set of lattice points

$$\Sigma_a = \{(x,k) \in \mathbf{Z}^n \times \mathbf{Z} : k \geq 0, x \in k\overline{P}, f(x/k) \leq a/k\}.$$

This is a subset of the set of lattice points defining $R_{\overline{P}}$ and we define $\mathcal{F}_a$ to be the subspace spanned by these basis elements. The convexity of f implies that $\Sigma_a + \Sigma_b \subset \Sigma_{a+b}$ and this gives the multiplicative property (8). From the definitions, the Rees algebra of this filtration of $R_{\overline{P}}$ is canonically identified with R_Q.

Degeneration of a conic

Suppose that $\overline{P}$ is an integral Delzant polytope and that f is a piecewise linear convex function of the form

$$f(x) = \max(\mu_1(x), \ldots, \mu_r(x)) \tag{9}$$

where μ_i are affine-linear functions with integral coefficients. Then the Rees algebra R_Q has an algebro-geometric interpretation. It is finitely generated over $\mathbf{C}[t]$ and the Proj construction over $\mathbf{C}[t]$ defines a variety $\mathcal{X} \subset \mathbf{CP}^N \times \mathbf{C}$. Projection to the second factor gives a map $\pi : \mathcal{X} \to \mathbf{C}$ with the property that for $t \neq 0$ the fibre $\pi^{-1}(t)$ is a copy of our complex manifold X but the central fibre $\pi^{-1}(0)$ is a different variety: a degeneration of X. For example, if $\overline{P}$ is the interval $[-1, 1]$ in $\mathbf{R}$—so X is the Riemann sphere embedded as a conic curve in $\mathbf{CP}^2$—and f is the function $f(x) = \max(x, -x)$ the degeneration has central fibre a pair of lines in the plane; a singular conic. In general a function of the form (9) defines a decomposition of $\overline{P}$ into a union of convex pieces on each of which f is affine-linear, and the central fibre is a reducible variety with components corresponding to these pieces. From the more algebraic point of view, for any filtered algebra R one considers $\mathrm{Rees}(R, \mathcal{F}_*) \otimes_{\mathbf{C}[t]} \mathbf{C}$, where $\mathbf{C}[t]$ acts on $\mathbf{C}$ by evaluating t at some $\tau \in \mathbf{C}$. If $\tau \neq 0$ this tensor product is isomorphic to R but for $\tau = 0$ it is the associated graded ring

$$\bigoplus_a \mathcal{F}_a / \mathcal{F}_{a-1}.$$

The differential-geometric and algebro-geometric constructions we have encountered all extend beyond the toric case. For any complex projective manifold X there is a *Mabuchi functional* on the space of Kähler metrics and the problem of finding an extremal metric is the problem of minimising this functional. The work of Chen and Cheng shows that the existence of a minimiser is equivalent to a coercive inequality like (7) but a complete algebro-geometric criterion for this is not yet known. Whatever the final answer may be it must be bound up with algebro-geometric notions of "stability" which stretch back to Mumford's Geometric Invariant Theory from the 1960's and, further, to Hilbert. In place of the positivity criterion on convex functions we expect to see a criterion involving filtrations of the co-ordinate ring $R(X)$. Filtrations which satisfy a finite-generation condition correspond to degenerations of X and there is a numerical invariant of these—the Futaki invariant—which reduces in the toric case to $L(f)$. The manifold X is called *K-stable* if the Futaki invariant is positive for all non-trivial degenerations. In the toric case this corresponds to the positivity of $L(f)$ for all functions f of the form (9) where the μ_i have rational co-efficients. (Multiplying $\overline{P}$ by a scale factor one can then reduce to the case of integral co-efficients.) The extension to more general filtrations was made by Székelyhidi in [7] (whose treatment we have followed above). This leads to a strengthening of the notion of K-stability to $\hat{K}$-stability, which corresponds in the toric case to the positivity criterion in our Theorem. In another direction, there is a notion of *uniform K-stability* which corresponds in the toric case to the existence of an inequality (6). In the toric case, $\hat{K}$-stability, uniform K-stability and the existence of an extremal metric are all equivalent, and perhaps the same will turn out to be true in general. For some classes of manifolds, such as Fano manifolds, the condition of K-stability is also equivalent but this is not expected to be true in general.

In any case there is much current activity in this area and much to be done, both in proving abstract existence theorems and in understanding more deeply these interactions between algebraic and differential geometry.

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