On some recent developments in Kähler geometry

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1 Introduction

The developments discussed in this document are

1. The proof of the "partial C^0-estimate", for Kähler-Einstein metrics;
2. The proof of the "Yau conjecture" (sometimes called the "YTD conjecture") concerning the existence of Kähler-Einstein metrics on Fano manifolds and stability.

These are related, because the ideas involved in (1) form one of the main foundations for (2). More precisely, the proof of the Yau conjecture relies on an extension of the partial C^0 estimate to metrics with conic singularities.

Gang Tian has made claims to credit for these results. The purpose of this document is to rebut these claims on the grounds of originality, priority and correctness of the mathematical arguments. We acknowledge Tian's many contributions to this field in the past and, partly for this reason, we have avoided raising our objections publicly over the last 15 months, but it seems now that this is the course we have to take in order to document the facts. In addition, this seems to us the responsible action to take and one we owe to our colleagues, especially those affected by these developments.

2 The partial C^0-estimate

This has been well-known as a central problem in the field for over 20 years, see for example Tian's 1990 ICM lecture. There is a more general question—not yet resolved—involving metrics with Ricci curvature bounded below but here we are just considering the Kähler-Einstein case. The results and techniques (from Riemannian convergence theory) which are used in the proof of this partial C^0 estimate have been available for 10 years or more.

The timeline for the recent developments is as follows.

- On April 11, 2012, in a lecture in a conference on Kähler geometry in Cambridge, attended by many of the leading workers in the field, Donaldson announced the proof of the partial C^0 estimate for Kähler-Einstein metrics in the case of complex dimension three. In the lecture, the main idea in the proof of the estimate was outlined and a reasonably detailed argument was provided in the case when the tangent cone has the form $\mathbf{C} \times \mathbf{C}^2/\Gamma$. The video of the lecture was created on 24th. of April, and can be found at:

[1] *Applications of the Hormander technique in Kähler-Einstein geometry.* http://www.sms.cam.ac.uk/media/1247397

- On June 12, 2012, Donaldson and Sun put the paper:

  [2] *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry.*

  http://arxiv.org/pdf/1206.2609.pdf

  on the arXiv. The paper establishes the partial C^0 estimate for Kähler-Einstein manifolds in all dimensions. After the lecture in Cambridge we found some extra arguments to deal in a simple way with the higher-dimensional case, and these were written down carefully in the paper. This paper was submitted to a leading journal on June 20th 2012 and we have received two detailed referees reports.

  In this paper [2] we gave credit to Tian for realising the importance of this partial C^0 estimate result (long before our own involvement in the field) and for the geometric consequences. In the second sentence of the paper we wrote:

  This result is essentially a verifcation of a conjecture of Tian, and Tian has, over many years, highlighted the importance of the question for the existence theory of Kähler-Einstein metrics.

- On June 13, 2012, Tian sent us a note, which was later made available in the proceedings of a conference in Australia.

  [3] *Extremal Kähler metrics and K-stability.*

  http://smp.uq.edu.au/sites/smp.uq.edu.au/files/proc-for-calabi.pdf

  In this note he claimed the partial C^0 estimate for all dimensions. But he only outlined the proof for the case when the tangent cone is of the form $\mathbf{C}^{n-2} \times \mathbf{C}^2/\Gamma$ (Page 17, Line -2), and the outline is very similar to the construction described in the lecture [1]. For the general case, he wrote (Page 20, Line 4)

  ...Since the arguments are rather lengthy and technical, we will leave details to [Ti12] .

- On September 21, 2012. Tian gave a lecture in IHP Paris, and posted on the conference homepage the document:

[4] *Partial C^0 estimate for Kähler-Einstein metrics*

http://www.ihp.fr/ckgeom/tian_gang_-_lecture.pdf

He wrote in the introduction (Page 3, Line 16): *Theorem 1.4 and 1.6 were announced with an outlined proof in our expository paper... ([3] above) ...In the next section of this note, we provide a proof of the first theorem following the arguments in [Ti12]. The second theorem follows easily as we indicated above. During the preparation of this note, we learned that Donaldson and Sun [DS12] also gave an independent and different proof of Theorem 1.4, though the two proofs share some overlapping ideas which appeared in previous works. One can also find a proof of Theorem 1.6 in [DS12].*

This is attempting to suggest to readers that he announced this result earlier than Donaldson-Sun. In reality, before June 13, there had been no mention by Tian of this result, as far as we have been able to find.

The paper [Ti12] referred to above, with "lengthy and technical" arguments, seems never to have appeared. But we recently came across a short (8 page) paper that has been published

[5] *Partial C^0-estimate for Kähler-Einstein metrics*

G. Tian Commun. Mat. Stat. I, 105-113

http://link.springer.com/article/10.1007/s40304-013-0011-9

The published data in the journal states that it was received on May 21st, 2013 and accepted on May 22nd, 2013. [1] The paper purports to give a complete proof of the result. Mathematically, we believe that there is a serious difficulty with the argument presented. In the first line of the proof of Lemma 2.4 it is claimed, without explanation, that a recent result of Colding and Naber on the geodesic convexity of the regular set in a limit space can be used to deduce that, in the context at hand, the fundamental group of this regular set is finite. It seems to us that, at the very least, this would require a substantial amount of work, and we suspect that, in fact, the conclusion cannot be established in a general Riemannian context, as seems to be suggested. Another place where the argument seems incomplete involves the construction of cut-off functions (p. 112 item (ii)). But the main point is that this paper was submitted almost a year after the complete proof [2] by Donaldson and Sun was available. The submission and acceptance dates make clear that the paper has not gone through the usual critical peer reviewing process. The paper was not placed on the arxiv, thus avoiding any scutiny before publication. In connection with Section 3 below, we note also that Tian writes in this paper:

[1] While one us (Chen)serves on the editorial board of CMS as probably one of the main experts in complex geometry, he was completely unaware of the handling of this paper [5] until he was informed by others that it had been published.

It is also worth mentioning that I gave a complete solution for the YTD conjecture in the case of Fano manifolds last October. My solution relies on establishing the partial C^0-estimate for conic Kähler-Einstein metrics.

There is no reference to our work.

In sum, our fundamental objections to Tian's claim over the partial C^0-estimate are:

- It seems to us highly improbable that Tian independently came on the proof, involving exactly the same ideas, in the short time interval (roughly April-June 2012) in question. Here we have in mind that, as noted above, the techniques which underpin the proof have been available for ten years or more.
- Even given that it is not impossible that such a coincidence occurred, we have clear priority in the presentation of both outline and detailed proofs.
- Even after 15 months from the appearance of Donaldson and Sun's paper [2] to the date of this writing , Tian has not produced a convincing complete proof of this result.

3 Yau's conjecture

On the 25th, October 2012, Tian gave a lecture at a conference in Stony Brook:

[6] *Conic Kähler-Einstein metrics.*
http://www.math.sunysb.edu/Videos/Cycles2012/video.php?f=14-Tian

In this lecture he claimed a proof of the Yau conjecture. The general strategy (deforming through metrics with cone singularities—a variant of the standard "continuity method") is attributed by Tian to us, and we think it should have been clear that this was a direction we were working on very actively. In fact we had written in [2]:

We finish this introduction with some words about the origins of this paper. While the question that we answer in Theorem 1.1 is a central one in the field of Kähler-Einstein geometry, it is not something that the authors have focused on until recently. The main construction in this paper emerged as an off-shoot of a joint project by the first-named author and Xiuxiong Chen, studying the slightly different problem of Kähler-Einstein metrics with cone singularities along a divisor. A companion article by the first named author and Chen, developing this related theory, will appear shortly.

Similar remarks were made in the lecture [1].

Tian's lecture [6] gave few details, and proofs of some of the key assertions made have never appeared.

On the 28th October 2012 we posted an announcement on the arxiv:

[7] Kähler-Einstein metrics and stability. http://arxiv.org/pdf/1210.7494.pdf

This was subsequently published in Int. Math. Res. Notices 2013,

http://imrn.oxfordjournals.org/content/early/2013/01/07/imrn.rns279.refs

Our announcement [7] was followed up by three detailed papers:

[8] arxiv post 19/11/2012. Kähler-Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities. http://arxiv.org/pdf/1211.4566.pdf

[9] arxiv post 19/12/2012. Kähler-Einstein metrics on Fano manifolds, II: limits with cone angle less than 2π. http://arxiv.org/pdf/1212.4714.pdf

[10] arxiv post 01/02/2013. Kähler-Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2π and completion of the main proof. http://arxiv.org/pdf/1302.0282.pdf

These papers were submitted to a leading journal on March 8th. 2013.

The only difference between these detailed papers and the outline in the announcement [7] is that in the latter we invoked results of Jeffres, Mazzeo and Rubinstein (JMR) for an analogue of the Evans-Krylov theory in the conic setting. Later, we had some concerns about the proofs of these results, so we supplied an independent argument in [9] above. (Tian's proof depends on the validity of the JMR results.)

Tian's announcement [6] in his lecture at Stony Brook was followed up by a paper

[11] arxiv post 20/11/ 2012. *K-stability and Kähler-Einstein metrics.*
http://arxiv.org/pdf/1211.4669v1.pdf

followed by a revision:

[12] arxiv post 28/01/2013. *K-stability and Kähler-Einstein metrics.*
http://arxiv.org/pdf/1211.4669v2.pdf

We suspect that there have been further revisions, but none have been made publicly available and hence open to scrutiny. Tian has given a number of lectures on this subject during 2013, in particular in a conference in Edinburgh on July 8th.:

[13] *K-stability and Kähler-Einstein metrics,I.*
http://www.icms.org.uk/downloads/ricci/tian%20I.pdf

Our fundamental objections to Tian's claims with respect to Yau's conjecture are:

- that we feel that there is no evidence that Tian was in possession of anything approaching a complete proof at the time of his announcement [6] in Stony Brook;

- that both arXiv versions [11], [12] of his paper have serious gaps and mistakes;
- that, insofar as these gaps and mistakes have been partially filled and corrected (in comparing [11], [12], [13]), many of the changes and additions made reproduce ideas and techniques that we had previously introduced in our publicly available work [7], [8], [9], 10], without any kind of acknowledgement.

We will not attempt to take up every single gap and mistake that we see in Tian's proposed proofs (including the necessity of checking carefully the relevant results of Jeffres, Mazzeo and Rubinstein, noted above), but concentrate on three points in the subsections 3.1,3.2,3.3 below.

3.1 The construction of test configurations and reductivity

The issue here is, roughly, that one wants to pass between a sequence of manifolds converging to a (possibly singular) limit and a $\mathbf{C}^*$-equivariant degeneration (or "test configuration"), as in the algebro-geometric definition of stability.

1. In an earlier paper

   [14] *Existence of Einstein metrics on Fano manifolds.*

   http://www.springer.com/mathematics/geometry/book/978-3-0348-0256-7?changeHeader

   http://smp.uq.edu.au/sites/smp.uq.edu.au/files/proc-for-cheeger.pdf

   (which Tian had sent to us in 2010 and which was available online from 2012), there is one page in Section 4.3 discussing "relating K-stability to existence". On Page 35, Line -11, he wrote

   If φ_i is are not uniformly bounded, $\sigma_i(M)$ converges to a variety which is not bi-holomorphic to M. For each i , join $I \in SL(N+1)$ to σ_i by a the orbit O_i of a $\mathbf{C}^*$ action, without loss of generality, we may assume that O_i converge to a $\mathbf{C}^*$ orbit O_∞. Using appropriate compactification of $SL(N+1)M)$ one can show that if $\sigma(e^t)(t \in \mathbf{C})$ is the limit $\mathbf{C}^*$ action, $\sigma(e^t)(M)$ converge to the limit of $\sigma_i(M)$ as t tends to ∞.

   This is the most obvious approach by which to attempt to produce a test configuration, but in general this does not work due to the possible "splitting of orbits" phenomenon when one takes the limit of 1-parameter subgroups. No proof of the claim in the passage quoted above is given in [14], the later papers [12], [13], or elsewhere.

One can get around this problem if the automorphism group of the limit is reductive. The argument for this does not seem to be widely known, and this observation is an important ingredient in our work. It features in the announcement [7] (Page 5, Line 1) and details appeared in [10]. In the light of this general observation, a crucial technical problem becomes to establish this reductivity. In the case when the limit is smooth this is essentially the standard Matsushima theorem. One of the main technical parts of our work was to develop a new, fundamentally different, approach to the reductivity which applies to the singular case, based on recent advances from pluripotential theory. In the announcement [7] page 5, line 12, we wrote

Then the uniqueness theorem of Berndtsson, as extended by Berman-Boucksom-Essydieux-Guedj-Zeriahi can be used to show that that the automorphism group is reductive.

Full details of this argument appeared in [10].

2. In the pdf file of Tian's Stony Brook announcement [6], Page 10, he writes:

   The K-stability is equivalent to the properness of the K-energy restricted to K_l for some sufficiently large l.

   No proof of this has been given in [11], [12] or elsewhere. In Tian's announcement [6] there is no mention of the automorphism group of the limit being reductive, or why this might be relevant.

3. In Tian's first written version [11], two proofs are given to finish the main theorem (i.e. to make a connection with the algebro-geometric definition of stability). In the first proof, he wrote (Page 28, Line -12)

   We need to prove that it contains a $\mathbf{C}^*$*- subgroup*

   and in Lemma 6.3 it is stated

   The Lie algebra η_∞ *of* G_∞ *is reductive.*

   This is added after the appearance of our paper [7], where the reductivity statement is announced and the reasons are explained. In the proof of Lemma 6.3 of [11], Tian wrote (Page 28, Line -8)

   the arguments are standard

and his proposed proof largely follows the proof of the Matsushima theorem in the smooth case. There are many problems with this proof, both in the form written by Tian and in the general line of attack. The formula

$$i_X\omega_\infty = \sqrt{-1}\bar{\partial}\theta_\infty, \text{ where } \theta_\infty = \theta + \frac{1}{l}\rho_{\omega_\infty,l}$$

is wrong: there should be an action of X on $\rho_{\omega_\infty,l}$. So the claim that θ_∞ is Lipschitz clearly needs more serious explanation.

In the second proposed proof, he wrote (Page 29, Line -13)

There is another way of finishing the proof of Theorem 1.1 by using the CM- stability.

After the statement of Theorem 6.5, he wrote (Page 30, Line 16)

By our discussions in Section 3, we can show that $F_{\omega_0,\mu}$ restricted to G_z is proper for any $\mu \in (0,1]$.

For $\mu = 1$ this amounts to his claim in the announcement [6] (cited in the second item above), but again no proof is provided.

4. In Tian's second written version [12], the above-mentioned incorrect formula has been corrected, but he still claims (Page 36, Line 9)

   we can show that θ_∞ is Lipschitz continuous,

   without a proof. The proposed alternative proof seems not to change from the earlier version [11].

5. Moving on to July 2013, in the talk [13] the relevant part concerning the proof of reductivity of the automorphism group is now stated as (page 36)

   This can be deduced from the uniqueness theorem due to Berndtsson and Berman. There is also a more direct proof.

   There is also a remark:

   Remark: If M_∞ is smooth, then by standard arguments, one can prove that the group is reductive. But if M_∞ is singular, one needs to pay attention to a technical problem caused by the singularity.

These assertions are blatant copying without attribution. This is almost half a year since the appearance of our third paper [10], in which the detailed proof of the reductivity is provided, based on the uniqueness theorems proved by Berndtsson and Berman-Boucksom-Essydieux-Guedj-Zeriahi, and the technical difficulty in extending the usual proof of the Matsushima theorem is pointed out.

3.2 Convergence theory for conical metrics, limiting cone angle less than 2π

1. In the introduction to Tian's first written version [11] , he writes (Page 1, Line -13)

   The main technical ingredient is a conic version of Cheeger-Colding-Tian's theory on compactness of Kähler-Einstein manifolds.

   But in section 3, "An extension of of Cheeger-Colding-Tian", there are only 3 pages.

   The key lemma 5.8 on the construction of a good cut-off function is described as (Page 22, Line -13)

   This is rather standard and has been known to me for quite a while

   and the proposed proof amounts to 25 lines, from line -13, page 22 to line 12, page 23.

   In Tian's second written version [12], which appeared a month and a half after our second paper [9], more than *10 pages* were added to prove Lemma 5.8, (from page 25 to page 29 and the whole appendix–pages 38 to 45). In the main context the proof of Lemma 5.8 (Page 26, Line 11) is not finished since he made an assumption **A1**. The proof in the appendix depends on a local Hörmander argument, which are very similar to Section 2.5-2.7 of our paper [9]. This is a refinement of the Hörmander argument to prove the partial C^0 estimate for smooth Kähler-Einstein metrics (Section 2 above). The latter has only appeared recently, and this clearly contradicts what he claimed above that the proof of Lemma 5.8 has been known to him for quite a while (Page 25, Line 6). Also in the appendix Page 45, Line 1, he made use of the lower bound on the volume density of the divisor and this has never been mentioned in his first written version or his announcement. In sum, what he has added in the appendix is so similar to our second paper [9] that we feel this amounts to copying.

2. In Tian's second written version [12], in the application in Section 5, the key lemma 5.5 is not proved. In particular in the sentence (Page 21, Line

-5)

Moreover, for any $x \in S \subset M_\infty$, if S_x is of complex codimension 1, then there is a closed subcone $\overline{S}_x$ of complex codimension at least 2 such that g_x is asymptotic to the product metric described above at any $y \in S_x \setminus \overline{S}_x$,

the closedness of the subcone $\overline{S}_x$ is *not* proved at all. This is a key technical hurdle one must overcome in the presence of (real) codimension 2 singularities. This closedness has also been claimed in other places of [12], see for example, Page 4, Line -10; Page 24, Line 16; Page 26, Line 6; Page 26, Line 21, without any proof.

3.3 The case when the cone angle tends to 2π

As presented in his written versions [11], [12], Tian's argument is not sufficient to prove what he needs. More specifically, Theorem 4.3 (Page 19 in [12]) states a compactness theorem which reduces to the Cheeger-Colding-Tian theorem for smooth Kähler Einstein metrics and in Section 5, he uses this theorem to prove partial C^0 estimate (Theorem 5.1) via the Hörmander argument. But we think that his proof is not complete.

1. As pointed out in subsection 3.2 above (item 2) the statement of Theorem 4.3 is not completely proved (for example, the closedness of $\bar{S}$ in the case $\beta_\infty < 1$);

2. Even if we assume that he has established Theorem 4.3, this theorem itself is not sufficient to run the Hörmander argument to draw the partial C^0 estimate. Firstly one has to prove a similar theorem for tangent cones, in order to adapt the general strategy for smooth Kähler-Einstein metrics (see Section 2 above). The present proof does not seem to extend to that setting. Secondly, as in subsection 3.2 above (item 1), since his Theorem 4.3 has only established the smooth convergence away from $\overline{S} \cup D_\infty$, one needs a suitable cut-off function adapted to the limiting divisor D_∞. This goes back to the problem with the proof of Theorem 5.8 and it seems that one again has to control the volume of neighborhoods of the divisor, and relies on a lower volume-density bound of the divisor. The latter is a key concept studied carefully in our papers [9], [10].