GEOMETRIC ANALYSIS SECTIONS 5,6

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Section 5. Calabi-Yau metrics

- The equations we have studied so far in this course have a simple nonlinear structure $\text{linear} + \text{lower order}$.
- We have considered situations with a favourable sign.

Recall that if $\mathcal{F} = 0$ is any PDE and u is a solution we have a linearised operator $\mathcal{F}(u + tf) = tL(f) + O(t^2)$. The nonlinear PDE is said to be elliptic *at the solution u* if L is a linear elliptic operator (of the same order as $\mathcal{F}$).

For example a *Monge-Ampère equation*

$$\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = 1.$$

The linearised operator is

$$L(f) = \sum U^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

where U^{ij} is the matrix of co-factors of the Hessian u_{ij} of u. This is elliptic if and only of u_{ij} is positive or negative definite: i.e. $\pm u$ is strictly convex.

The nonlinear PDE is not elliptic at a solution like $x_1^2 - x_2^2 - x_3^2$ on $\mathbf{R}^3$.

In this section we discuss complex Monge-Ampère equations which, in local complex co-ordinates, involve the analogous expression

$$\det\left(\frac{\partial^2 u}{\partial z_a \partial \overline{z}_b}\right).$$

But we consider a global setting, on a compact complex n-manifold M.

Review of some complex geometry.

- A Hermitian metric on M corresponds to a positive $(1,1)$ form ω. The volume form of the metric is $\omega^n/n!$.
- The canonical line bundle of M is $K_M = \Lambda^n T^*M$, so sections are $(n,0)$ forms.
- Giving a Hermitian metric on the canonical line bundle is equivalent to giving a volume form on M.

- A Kähler metric is one with $d\omega = 0$. The Kähler metrics in a given cohomology class $[\omega_0] \in H^2(M; \mathbf{R})$ are parametrised by Kähler potentials

$$\omega_\phi = \omega_0 + i\overline{\partial}\partial\phi.$$

- The total volume of M with a Kähler metric is determined by the cohomology class.
- We have the Laplacian formula

$$-\frac{1}{2}\Delta f\ (\omega^n) = n\ i\overline{\partial}\partial f \wedge \omega^{n-1}.$$

- The Ricci curvature of a Kähler metric ω is Hermitian, so can be identified with a $(1,1)$ form ρ. This is $-i$ times the curvature form of the connection on K_M induced by the volume form $\omega^n/n!$.

We consider the question of prescribing the volume form of a Kähler metric.
(Calabi, 1954.)

Fix a cohomology class $[\omega_0]$ and let V be the corresponding total volume. Write $d\mu_0 = \omega_0^n/n!$. Given a positive function F with

$$\int_M F d\mu_0 = V$$

we want to solve the PDE

$$(\omega_0 + i\overline{\partial}\partial\phi)^n = F d\mu_0. \qquad (CY)$$

In local coordinates this has the shape

$$\det\left(g_{ab} + \frac{\partial^2 \phi}{\partial z_a \partial \overline{z}_b}\right) = gF$$

where g_{ab} is the matrix corresponding to ω_0 and g is its determinant.

The main result (Yau's Theorem, 1978) is that there is always a solution ϕ, unique up to a constant.

The most important application is to the case when M is a Calabi-Yau manifold, i.e. K_M is trivial so there is a nowhere vanishing holomorphic n-form Θ. The curvature associated to a Hermitian metric on K_M with $|\Theta| = 1$ is zero. Regarding this metric on K_M as a volume form and scaling suitably we get a metric in any Kähler class with zero Ricci curvature.

When K_M is a negative or positive line bundle one can seek Kähler-Einstein metrics, with $\mathrm{Ricci} = \lambda g$ where $\lambda = \pm 1$. This leads to equations

$$(\omega_0 + i\overline{\partial}\partial\phi)^n = \Gamma_0 e^{\lambda\phi}, \qquad (KE\pm)$$

for a positive function Γ_0 determined by ω_0.

Digression. Geometry of the space of Kähler potentials

Let $\mathcal{H}$ be the set of Kähler potentials $\{\phi : \omega_\phi > 0\}$.
(If the Kähler class is 2π times an integral class, $\mathcal{H}$ can be identified with a set of metrics on a holomorphic line bundle $L \to M$.)
Let H be the set of positive Hermitian forms on $\mathbf{C}^N$. This has two natural geometries:

1. H is a convex set in the vector space of Hermitian matrices.
2. H is a symmetric space $GL(N, \mathbf{C})/U(N)$ and splits as a product $H = \mathbf{R} \times H_0$ where the projection to $\mathbf{R}$ is given by $\log\det$ and $H_0 = SL(N, \mathbf{C})/SU(N)$.

The function $\log\det$ is *concave* on the set of positive Hermitian matrices.

There are detailed analogies between $\mathcal{H}$ and H. We will just consider one part of this story, in which $\mathcal{H}$ appears as a convex subset of $C^\infty(M)$. We define a functional I on $\mathcal{H}$ (up to a constant) by its derivative

$$\delta I = \int_M \delta\phi \, d\mu_\phi.$$

It is an exercise to see that there is such a functional. (You can write down an explicit formula.)

This functional I is the analogue of the function $\log\det$ on H. One computes that I is concave, so the set $\mathcal{K} = \{\phi \in \mathcal{H} : I(\phi) \geq 0\}$ is convex.

A volume form $d\nu$ on M defines a linear functional $\nu^* : C^\infty(M) \to \mathbf{R}$;

$$\nu^*(f) = \int_M f d\nu.$$

Consider the problem of minimising ν^* on the convex set K. If we find such a minimum ϕ then μ_ϕ is a constant multiple of $d\nu$ and we have solved equation (CY).

End of digression

To solve the equation (CY) we use the continuity method. We consider a 1-parameter family F_s to give equations CY_s for $s \in [0,1]$.

We have

$$(\omega + ti\overline{\partial}\partial f)^n = \omega^n + int\omega^{n-1} \wedge \overline{\partial}\partial f + O(t^2)$$

The linear term is $-\frac{1}{2}\Delta_\omega f\ \omega^n$. It follows that the map taking $\mathcal{H}$ to volume forms of total volume V has surjective derivative and this gives openness in our continuity path.

We need *a priori* estimates for a solution ϕ of equation CY, depending only on ω_0 and the right hand side F.

Set $\omega = \omega_\phi, \eta = \omega_0$ and write:

$$\tau = \mathrm{Tr}_\omega \eta \quad \sigma = \mathrm{Tr}_\eta \omega.$$

At a point in M we can choose standard coordinates

$$\omega_0 = \eta = (1/2) \sum_a dz_a d\overline{z}_a, \omega = (i/2) \sum_a \lambda_a dz_a d\overline{z}_a,$$

with $\lambda_a > 0$.

Then $\tau = \sum \lambda_a^{-1}, \sigma = \sum \lambda_a$.

The product $\prod \lambda_a$ is the rqatio of the volume forms, which is prescribed
So control of *either* σ, τ gives control of all the eigenvalues λ_a.

We seek a maximum principle argument for τ. (The more standard approach considers σ.)

It is a simple general fact that a holomorphic map $f:(M,\omega)\to(N,\theta)$ between Kähler manifolds is harmonic. This follows from the fact that (in the compact case) we can write the energy as a topological quantity given by a multiple of

$$E=\frac{2}{(n-1)!}\int_M f^*(\theta)\wedge\omega^{n-1},$$

which depends only on the homotopy class of f. For any map in the homotopy class we get an inequality, so f minimises energy in the homotopy class.

So for $f : (M,\omega) \to (N,\theta)$ we have the Eells-Sampson formula

$$\Delta_\omega |df|^2 = |\nabla df|^2 + \mathrm{Ric}_\omega(df^2) - \mathrm{Riem}_\theta df^4.$$

If we compute $\Delta \log |df|^2$ we get a term which is dominated by the first term on the RHS above and we find that

$$\Delta_\omega \log |df|^2 \geq |df|^{-2}\mathrm{Ric}_\omega(df^2) - |df|^{-2}\mathrm{Riem}_\theta df^4 \quad (CL).$$

This is known as the *Chern-Lu inequality*.

Apply this to the identity map from (M, ω) to (M, η). Then $|df|^2 = \mathrm{Tr}_\omega \eta = \tau$. We have

$$\mathrm{Riem}_\eta df^4 = R^\eta_{ijkl} \omega^{ik} \omega^{jl},$$

which is bounded by $C_1 \tau^2$ for some C_1.
The Ricci $(1, 1)$ form ρ_ω is determined by the volume form, which is controlled. So $\rho_\omega \geq -C_2 \eta$ for some C_2. We have

$$\mathrm{Ric}_\omega(df^2) = R^\omega_{ij} \omega^{ik} \omega^{jl} \eta_{kl} \geq -C_2 \eta_{ij} \eta_{kl} \omega^{ik} \omega^{jl} \geq -C_2 \tau^2.$$

So we get

$$\Delta_\omega \log \tau \geq -C\tau.$$

By itself, this is not very helpful. However we have

$$-\frac{1}{2}\Delta_\omega \phi = \mathrm{Tr}_\omega i\overline{\partial}\partial\phi = \mathrm{Tr}_\omega(\omega - \eta) = n - \tau.$$

So

$$\Delta_\omega(\log \tau + 2K\phi) \geq -Kn + (K - C)\tau.$$

Choose $K = C + 1$ and let $p \in M$ be a point where $\log \tau - 2K\phi$ is maximal. The maximum principle gives $\tau(p) \leq Kn$.

Now *suppose* that we an L^∞ bound on ϕ:

$$\|\Phi\|_{L^\infty} \leq C_4.$$

Then at any point $q \in M$:

$$(\log \tau(q) + 2K\phi)(q) \leq (\log \tau(p) + 2K\phi)(p),$$

so

$$\log \tau(q) \leq \log \tau(p) + 4KC_4.$$

The conclusion is that, in our continuity path, an L^∞ bound on ϕ gives a bound

$$C^{-1}\omega_0 \leq \omega_{\phi_s} \leq C\omega_0.$$

The problem is to obtain an L^∞ bound on ϕ. We fix a normalisation

$$\int_M \phi d\mu_\eta = 0.$$

For simplicity we discuss the case in dimension $n = 2$.

The Sobolev inequality for $\eta = \omega_0$ is

$$\|f\|_{L^4_\eta} \leq \kappa \|\nabla f\|_{L^2_\eta} + \kappa' \|f|_{L^1_\eta}.$$

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For functions f of integral zero we can drop the second term on the right hand side.

We have $\omega = \eta + i\overline{\partial}\partial\phi$ so

$$2(\omega^2 - \eta^2) = 2i\overline{\partial}\partial\phi(\omega + \eta) = -\Delta_\omega \phi d\mu_\omega - \Delta_\eta \phi d\mu_0. \qquad (*****)$$

Multiply (*****) by ϕ and integrate over M.
We get

$$\int_M 2\phi\,(\omega^2 - \eta^2) = \int_M |\nabla\phi|^2_\omega\, d\mu_\phi + |\nabla\phi|^2_\eta\, d\mu_\eta.$$

Since ω^2 is controlled, the left hand side is bounded by a multiple of the L^1 norm of ϕ, hence by a multiple of the L^4 norm. The right hand side is obviously bounded below by $\|\nabla\phi\|^2_{L^2_\eta}$.

Applying the Sobolev inequality for the fixed metric η, using the fact that the integral of ϕ is 0, we get

$$\|\phi\|^2_{L^4_\eta} \leq \text{const.}\|\phi\|_{L^4_\eta}.$$

So we have an L^4 bound on ϕ.

Now multiply (*****) by $-\phi^3$ and integrate. We have

$$-\int \phi^3 \Delta\phi = \int \nabla(\phi^3).\nabla\phi = \int 3\phi^2|\nabla\phi|^2 = \int \frac{3}{4}|\nabla(\phi^2)|^2.$$

Using this we get an L^4 bound on ϕ^2 i.e. an L^8 bound on ϕ.

Continuing in this way, we get bounds on the L^{4k} norm of ϕ for every k,

$$\|\phi\|_{L^{4k}} \leq C_k.$$

Keeping careful track of the constants, one finds that the C_k are bounded as $k \to \infty$ and this gives the L^∞ bound.

IF we have $C^{2,\alpha}$ bounds on ϕ for some $\alpha > 0$, standard PDE theory using the Schauder estimates gives control of all higher derivatives.
The L^∞ bound on $i\overline{\partial}\partial\phi$ gives $C^{1,\beta}$ estimates for any $\beta < 1$.
So there is a gap.

This can be handled in two ways.

- A maximum principle argument and a long calculation applied to $\Delta_\omega |\nabla_\eta \overline{\partial}\partial\phi|^2_\omega$.
- A general PDE theory of Evans-Krylov.

Apart from the L^∞ bound all that we have done can be applied to the Kähler-Einstein equations. In the negative case ($\lambda = -1$) the L^∞ bound follows from an easy maximum principle argument, similar to the Riemann surface discussion in Section 2. Our equation is

$$(\omega_0 + i\overline{\partial}\partial\phi)^n = \Gamma_0 e^{-\phi},$$

and at a maximum point of ϕ we have $i\overline{\partial}\partial\phi \geq 0$.

In the positive case our continuity path is a family of equations

$$(\omega_0 + i\overline{\partial}\partial\phi)^n = \Gamma_0 e^{+s\phi}.$$

Kähler-Einstein metrics do not always exist in the positive case: there are “stability conditions”.

For example, it can be shown that for the projective plane blown up at one point a solution in the continuity path exists exactly for $s < 6/7$ and for the plane blown up in 2 points for $s < 21/25$.

Section 6. The Yamabe problem

On a compact Riemannian manifold of dimension 2 the integral of the scalar curvature is a topological invariant. In higher dimensions it is the *Einstein-Hilbert functional* on the space of metrics:

$$I(g) = \int_M R_g d\mu_g.$$

Under an infinitesimal change of metric $\delta g = h$ the scalar curvature changes by

$$\delta R = -\Delta H + \nabla^* \nabla^* h - \langle \mathrm{Ricci}, h \rangle = -\Delta H + \sum \nabla_i \nabla_j h_{ij} - R_{ij} h_{ij},$$

where $H = \mathrm{Tr} h = \sum h_{ij} g^{ij}$.

Taking account of the variation of the volume form we get

$$\delta I = \int_M -\langle \mathrm{Ricci}, h \rangle + \frac{R}{2} H$$

since the integrals of the first two terms vanish by Stokes' Theorem.

If $\dim M = n$ it is clear that $I(\alpha^2 g) = \alpha^{n-2} I(g)$.
If we consider I as a functional on metrics of total volume 1 the Euler-Lagrange equation is

$$\mathrm{Ricci}_g + \frac{1}{2} R_g = (\lambda/2) g,$$

with a Lagrange multiplier λ. This is equivalent to the *Einstein equation*

$$\mathrm{Ricci}_g = \lambda' g$$

with $\lambda' = \lambda/(n-2)$.

In this section we consider the functional I on a fixed conformal class of metrics.

The Euler-Lagrange equation is then that the scalar curvature be constant.

The Yamabe problem is to prove that there is always a solution of this equation. More precisely:
In any conformal class there is a metric which minimises I over the conformal metrics of volume 1.
This result follows from contributions by many people (Yamabe, Trudinger, Aubin, Schoen, $\ldots$).
We call such a metric a Yamabe minimser.

For simplicity (mainly) we take $n = 3$.
It is convenient to parametrise a conformal class by $\tilde{g} = u^4 g$.
Then one finds that

$$R_{\tilde{g}} = u^{-5}\left(-8\Delta_g u + R_g u\right),$$

so, taking account of the change in volume form by a factor u^6,

$$I(\tilde{g}) = \int_M u(-8\Delta_g u + R_g u)\, d\mu_g = \int_M 8|\nabla u|^2 + Ru^2 d\mu.$$

The volume constraint is that the integral of u^6 is 1.

(Note: In general dimension n we use $u^{4/(n-2)}g$ and the formulae involve different factors.)

The Euler-Lagrange equation is

$$-8\Delta u + Ru = \lambda u^5.$$

Note that $\lambda = I(\tilde{g})$.

This is an example of a conformally invariant variational problem, similar to harmonic maps of surface.

It involves the borderline Sobolev embedding: in dimension 3,

$$L^2_1 \longrightarrow L^6,$$

but the inclusion is not compact.

Let H be the completion of compactly supported functions on $\mathbf{R}^3$ in the norm $\|\nabla f\|_{L^2}$. Let μ_0 be the best constant in the inequality

$$\mu_0 \left(\int_{\mathbf{R}^3} f^6 \right)^{1/3} \leq \|\nabla f\|_{L^2}^2. \quad (*)$$

The sphere S^3 is the conformal compactification of $\mathbf{R}^3$. Working with the Euclidean metric as our reference metric one see that the Yamabe problem for this conformal class is equivalent to minimising $\|\nabla f\|_{L^2}$ over functions f on $\mathbf{R}^3$ with the integral of f^6 equal to 1 and asymptotic to

$$\text{const. } (1+r^2)^{-1/2}$$

at infinity.
This is equivalent to finding a function realising equality in (*).

It can be shown that such a minimiser exists and corresponds to a round metric on S^3.

One can use a symmetrisation argument to reduce to functions $f(r)$ and get down to a calculus of variations argument in one dimension.

For a general compact Riemannian 3-manifold (M,g) of volume 1, let μ_g be the infimum of $I(\tilde{g})/8$ over conformal metrics of volume 1.
This is the best constant in the inequality:

$$\mu_g \left(\int_M u^6 \right)^{1/3} \leq \int_M |\nabla u|^2 + (R/8)u^2 .$$

It is clear that $\mu_g > -\infty$.

There are two main steps in the proof of the existence of a Yamabe minimiser.

- Show that if $\mu_g < \mu_0$ then there is a smooth minimiser in the conformal class.
- Show that $\mu_g \leq \mu_0$ with equality if and only g is a round metric on S^3.

The first step uses arguments which apply to many other problems.

We will focus on the proof of a slightly weaker statement.

Theorem A

Suppose that for $s \in [0, 1]$*, we have a* 1*-parameter family of metrics* g_s *with* $\mu_{g_s} < \mu_0$*. If a Yamabe minimiser exists in the conformal class of* g_0 *then the same is true for all* s*.*

As usual, the proof has an openness part and a closedness part. We will concentrate on the closedness.

This uses the important idea of a “small energy” estimate.

Proposition 1

Suppose that (M, g) is a Riemannian 3-manifold F is a function on M and $\lambda \in \mathbf{R}$.
There are ϵ_0, ρ_0, C (depending on g, F and λ) such that if a positive function u satisfies the equation $-\Delta u = \lambda u^5 - Fu$ on M and if B is a ball with centre p of radius $\rho \leq \rho_0$ such that

$$\int_B u^6 = \epsilon \leq \epsilon_0$$

then $|u| \leq C\epsilon^{1/6}\rho^{-1/2}$ on the $\rho/2$ ball centred at p.

For simplicity we suppose that the metric is Euclidean on B, so that B is the ρ-ball in $\mathbf{R}^3$, and that $F = 0, \lambda = 1$.

If u satisfies $-\Delta u = u^5$ and we set $\tilde{u}(x) = \nu^{1/2} u(\nu x)$ for some ν then $\tilde{u}$ satisfies the same equation and

$$\int_{\nu^{-1}B} \tilde{u}^6 = \int_B u^6.$$

Thus we can suppose that $\rho = 1$ and B is the unit ball in $\mathbf{R}^3$.

Let $M = \max_{x \in B} u(x) D(x)^{1/2}$ where $D(x)$ is the distance to the boundary of B.

Let x_0 be a point where the maximum is attained and $\nu = (1/2) D(x_0)$.
Let $\psi : \mathbf{R}^3 \to \mathbf{R}^3$ be the obvious scaling map taking the unit ball to the ball of radius ν centred at x_0.

Define the function U on the unit ball by $U(y) = \nu^{1/2} u(\psi(y))$.

Then U has the following properties:

1. $-\Delta U = U^5$;
2. $\int_B U^6 = \epsilon$;
3. $U(0) = 2^{-1/2} M$;
4. $U \leq M$ on B.

The last property follows from the choice of x_0

If f is function on the unit ball B with $\Delta f \geq 0$ then the mean value formula shows that $f(0)$ is at most the average of f over B.

Set $f = u + \frac{2^{5/2}}{3} M^5 r^2$.

Then by (1) and (4) we have $\Delta f \geq 0$ and we deduce, using (2), that

$$U(0) \leq C_1 M^5 + C_2 \epsilon^{1/6}$$

for computable constants C_1, C_2.
So, using (3), we have

$$M \leq C_1 M^5 + C_2 \epsilon^{1/6}.$$

If ϵ is small (depending on C_1, C_2) the solutions of this inequality fall into disjoint sets:

- "small" with $M \leq C\epsilon^{1/6}$;
- "large" with $M \geq C' \sim C_1^{-1/4} > 0$.

This determines ϵ_0.

We can apply the whole argument starting with the ball ηB in place of B, for $\eta < 1$. So we set

$$M(\eta) = \max_{x \in \eta B} u(x) D_\eta(x)^{1/2}.$$

where D_η is the distance to the boundary of ηB.

We get either $M(\eta) \leq C\epsilon^{1/6}$ or $M(\eta) \geq C'$.

Clearly when η is sufficiently small the first alternative holds and by continuity it must be true for all $\eta \leq 1$.

This proves Proposition 1.

Once we are in the “small energy” regime on a ρ-ball as in Proposition 1 we get elliptic estimates on all derivatives of u in the interior, depending on ρ.

Proposition 2
Suppose that $\mu_g = \mu_0 - K^{-1}$ *for* $K > 0$*.*

For any $\epsilon > 0$ *there is a computable* $\delta > 0$*, depending on* K, g, ϵ*, such that if uis a Yamabe minimiser for* g *then the integral of* u^6 *over any* δ*-ball is less than* ϵ*.*

Together with Proposition 1 and the remarks above, this implies the closedness part of Theorem. A.

Since we will be working in a small neighbourhood Ω of a point there is no real loss in supporting that the metric g is Euclidean in this neighbourhood.
The function u satisfies the equation $-\Delta u = \mu u^5$ for $\mu = \mu_g$.

Let χ be any function of compact support in Ω.

Multiply the equation by $\chi^2 u$ and integrate by parts to get

$$\int \nabla(\chi^2 u).\nabla u = \mu \int \chi^2 u^6.$$

We have

$$\nabla(\chi^2 u).\nabla u = |\nabla(\chi u)|^2 - |\nabla\chi|^2 u^2.$$

So

$$\int |\nabla(\chi u)|^2 = \mu \int \chi^2 u^6 + \int |\nabla\chi|^2 u^2.$$

Applying the Euclidean Sobolev inequality to χu we get

$$\mu_0 \|\chi u\|_{L^6}^2 \leq \mu \int \chi^2 u^6 + \int |\nabla\chi|^2.$$

We estimate the two terms on the RHS using Hölder's inequality with exponents $3, 3/2$ to get

$$\mu_0 \|\chi u\|_{L^6}^2 \leq \mu \|\chi u\|_{L^6}^2 \|u\|_{L^6}^4 + \|\nabla\chi\|_{L^3}^2 \|u\|_{L^6}^2.$$

Recall that u is normalised so that the integral of u^6 is 1. Thus we have

$$(\mu_0 - \mu)\|\chi u\|^2_{L^6} \leq \|\nabla\chi\|^2_{L^3}$$

so

$$\|\chi u\|^2_{L^6} \leq K\,\|\nabla\chi\|^2_{L^3}.$$

Important fact; Exercise

For any given $\sigma > 0$ we can find $\delta < \delta'$ so that for each point $p \in M$ there is a cut-off function χ supported in the δ' neighbourhood of p, equal to 1 on the δ-neighbourhood of p and with $\|\nabla\chi\|_{L^3} \leq \sigma$.

This is a reflection of the *failure* of the Sobolev embedding $L^6_1 \to L^\infty$ in dimension 3.

This completes the proof of Proposition 2.

Leaving aside the openness in Theorem A for the moment, we return to discuss the variational problem of finding a Yamabe minimiser.

For $p < 6$ consider the modified problem of minimising

$$\int |\nabla u|^2 + R/8u^2,$$

subject to the constraint $\int u^p = 1$.

The compact inclusion $L^2_1 \to L^p$ means that a minimising sequence u_i can be chosen to converge to a limit u_∞ in L^p. We can also suppose that it converges *weakly* in L^2_1. *i.e.* $u_{p,\infty} \in L^2_1$ and for any test function ψ

$$\langle \nabla\psi, \nabla u_i \rangle \to \langle \nabla\psi, \nabla u_{p,\infty}.$$

This implies that $u_{p,\infty}$ is a weak solution of the Euler-Lagrange equation and a bootstrapping argument shows that it is smooth.

Assuming that $\mu_g < \mu_0$ a modification of the arguments above gives *a priori* estimates on all derivatives of $u_{p,\infty}$, independent of p.

Taking the limit as $p \to 6$ gives a minimiser for the original problem.

If we try to use this minimising argument directly in the critical case $p = 6$ we can still choose a weakly convergent minimising sequence but the weak limit could be zero.

Go back to the openness problem in Theorem A.

This is a *Digression* from our main thread in this section.

Openness is straightforward, using the implicit function theorem, provided that for a minimiser $\tilde{g}$ the Laplace operator does not have an eigenvalue $-(1/2)R_{\tilde{g}}$.

To handle the general case we can use the method of "reduction to finite dimensions", similar to a discussion in Section 4.

Suppose that $\mu_g < \mu_0$ and g is a Yamabe minimiser.
The arguments above show that the space K corresponding to volume 1 minimisers in the conformal class of g is compact.
We regard K as a subset of the space $\mathcal{U}$ of positive functions u with L^6 norm 1

Using the same idea as for “Kuranishi models” we can construct a compact finite dimensional manifold Σ with boundary and an immersive embedding $\iota : \Sigma \to \mathcal{U}$ such that $K = \iota(\underline{K})$ where $\underline{K}$ lies in the interior of Σ.

We make the construction so that for each $\sigma \in \Sigma$ we have a finite-codimension submanifold $N_\sigma \subset \mathcal{U}$ through $\iota(\sigma)$ and the tangent space of N_σ at $\iota(\sigma)$ is complementary to the tangent space of $\iota(\Sigma)$.

The subset K is the minimising set of the functional I on $\mathcal{U}$ with minimal value 8μ on K.

We make the construction so that $\iota(\sigma)$ is a nondegenerate minimum of the restriction of I to N_σ.

We have a finite-dimensional reduction of the functional to a function $\underline{I} = I \circ \iota$ on Σ and $\underline{K}$ is the set of minima of $\underline{I}$.

The crucial point is that there is some $\delta > 0$ so that $\underline{I} \geq 8\mu + \delta$ on $\partial\Sigma$.

Suppose that we make some small perturbation of our functional I to I'.

The nondegeneracy condition means that for each σ there is a unique nearby minimum of I' on N_σ (using the implicit function theorem in the standard way).

This defines a perturbed map $\iota' : \Sigma \to \mathcal{U}$ and hence a perturbed function $\underline{I}' = I' \circ \iota'$ on Σ.

Minima of I' on $\mathcal{U}$ correspond to minima of $\underline{I}'$ on Σ, provided that the latter do not occur on the boundary.

By compactness, there is at least one minimiser of $\underline{I}'$ on the compact manifold-with-boundary Σ.

The “crucial point” implies that that, for sufficiently small perturbations, the minimisers are not on the boundary of Σ.

So we get a minimiser of the perturbed functional I'.

End of digression

Theorem B

Any compact Riemannian 3-manifold (M, g) has $\mu_g \leq \mu_0$ with equality if and only if (M, g) is conformal to the standard sphere.

The fact that $\mu_g \leq \mu_0$ is relatively easy.

Suppose first that g is Euclidean in a small neighbourhood of a point p.

Recall that inversion $x \mapsto x/|x|^2$ is a conformal map on $R^3 \setminus \{0\}$.

Using this, it is clear that we can find a conformally equivalent metric $\hat{g}$ on $\hat{M} = M \setminus \{p\}$ which is complete and Euclidean outside a compact set.

We can arrange that $(\hat{M} \setminus K, \hat{g})$ is isometric to the complement $\mathbf{R}^3 \setminus \frac{1}{2}B$, for a suitable compact set $K \subset \hat{M}$.

Let g^S be the standard round metric on S^3 of volume 1 and fix a point $q \in S^3$.
For small ρ, conformally deform g^S slightly in an $O(\rho)$ neighbourhood of q to get a metric g^S_ρ which contains an isometric copy B_ρ of the Euclidean ρ-ball.

Let J be the $1/8$ the integral of the scalar curvature of $\hat{g}$.
Scale $\hat{g}$ to $\hat{g}_\rho = \rho^2 \hat{g}$.
Then

$$\int_{\hat{M}} R_{\hat{g}_\rho} = 8\rho J.$$

The metrics $\hat{g}_\rho$ and g^S_ρ can be glued isometrically along an annular region isometric to a neighbourhood of ∂B_ρ.
This gives a metric $g^\sharp_\rho$ on M, conformal to g.
The volume is $1 + O(\rho^3)$ and the integral of the scalar curvature is $8\mu_0 + O(\rho)$.

Letting $\rho \to 0$ shows that $\mu_g \leq \mu_0$.

If g is not Euclidean near p we get a small extra error in the gluing construction but the same argument works.

There is a similar discussion for the harmonic maps energy functional E in dimension (of the domain) 2.

For example, consider a flat torus T^2 and maps $f : T^2 \to S^2$ of degree $d > 0$.

If ω is the area form on S^2 of total area 1 we have

$$\int_{T^2} f^*(\omega) = d.$$

On the other hand there is a pointwise inequality

$$|f^*(\omega)| \leq \frac{1}{2}|df|^2 d\mu_T,$$

So $E(f) \geq d$. Equality holds if and only if f is holomorphic.

A similar construction to that above shows that for any d the infimum of the energy functional is d. (“Glue” a constant map on T^2 minus a small disc to a degree d holomorphic map from S^2 to S^2.)

When $d = 1$ there is no holomorphic map, so the infimum is not attained.

A minimising sequence will develop a “bubble”.

The essence of the Yamabe problem is to show that the analogous phenomenon does not occur.

For our discussion of Theorem B we continue to **ASSUME** that g is Euclidean in a neighbourhood of some point $p \in M$
Suppose we have a conformal metric $\hat{g}$ on $\hat{M}$ with scalar curvature $\hat{R} \leq 0$ and not identically zero and which is *asymptotically Euclidean* in the sense that outside a compact set the manifold is identified with the complement of a ball in $\mathbf{R}^3$ and

$$\hat{g}_{ij} = (1 + \phi)\delta_{ij}$$

where ϕ is $O(r^{-2})$, with corresponding estimates for derivatives. In particular the scalar curvature $\hat{R}$ is $O(r^{-4})$.

Choose a large number L and flatten $\hat{g}$ in the annulus of size $O(L)$ to get a metric $\hat{g}_L$
That is, multiply ϕ by a suitable cut-off function.

The change in the scalar curvature is $O(L^{-4})$ over the annulus of volume $O(L^3)$ so the change in the integral of the scalar curvature is $O(L^{-1})$ which tends to zero as $L \to \infty$.
So we can fix L such that $I(\hat{g}_L) < 0$.
Then the same gluing construction as before shows that $\mu_g < \mu_0$.

To find a suitable metric $\hat{g}$ we use the Dirac operator.

Any oriented Riemannian 3-manifold (M, g) admits a spin structure and hence $D : \Gamma(S) \to \Gamma(S)$.

If $g' = u^4 g$ is a conformal metric we get a spin structure for g' with the same bundle S but multiply the structure map $T^*M \to \mathrm{End} S$ by u^{-2}.

Important fact The Dirac operator is conformally invariant, in the sense that

$$D's = u^{-4} D(u^2 s).$$

In general, suppose that $\mathcal{D} : \Gamma(E) \to \Gamma(F)$ is an elliptic operator of order r between sections of Hermitian bundles over a compact n-manifold X. Let $p \in X$ and $\alpha \in F_p^*$. This defines a distribution, a linear map from $\Gamma(F)$ to **C**. We can consider solutions s of the equation $\mathcal{D}s = \delta_\alpha$. Such a section satisfies $Ds = 0$ on $X \setminus \{p\}$ and has a "pole" at p.

For $r < n$ the order of growth of s is d^{r-n} where d is the distance to p.

In the case of the $\overline{\partial}$-operator on a Riemann surface we get meromorphic functions.

The general theory says that if the kernel of the adjoint operator $\mathcal{D}^*$ is trivial then such a solution exists, for any p, α.

Note that a constant spinor field on $\mathbf{R}^3$ goes over under inversion to a spinor field on $\mathbf{R}^3$ with a pole at 0.

To get quickly to the main point note that on a 3-manifold the Dirac operator is self adjoint and we expect that for typical metrics the kernel is trivial.
Also the spin bundle S has real rank 4 so generic sections have no zeros.
So **ASSUME** for the moment that $\mathrm{ker} D_g = 0$ and solve the equation $Ds = \delta_\alpha$ for some α at $p \in M$.
ASSUME also that this s does not vanish anywhere on $M \setminus \{p\}$.

We have

$$|s|^2 = Cd^{-4} + O(d^{-2}) \qquad (***)$$

near p for some $C \neq 0$.

Let g' be the conformal metric $u^4 g$ with $u = |s|^{1/2}$ so $s' = u^{-2} s$ satisfies $D's' = 0$ and by construction $|s'| = 1$ everywhere. The asymptotics (***) show that g' is asymptotically Euclidean in the sense we considered above.
The Lichnerowicz formula gives

$$\Delta' |s'|^2 = |\nabla' s'|^2 + R' |s'|^2$$

But the left hand side is zero so $R' \leq 0$ and if $R' = 0$ everywhere $\nabla' s' = 0$.
If $\nabla' s' = 0$ it is easy to show that g' is the Euclidean metric on $\mathbf{R}^3$.
(The group $SU(2)$ acts freely on the unit sphere in $\mathbf{C}^2$ so the "holonomy" is trivial.)

This completes the proof of Theorem B, under the three **ASSUMPTIONS**.

1. *The assumption that s has no zeros*
Take $u_\epsilon = F_\epsilon(|s|)$ for a suitable family of positive functions $F_\epsilon(t)$ approximating $t^{1/2}$.

Calculations show that the contribution to the integral of the scalar curvature from such a change goes to zero as $\epsilon \to 0$.
So the only problem could be when $\nabla' s' = 0$ outside the zero set of s.
A maximum principle argument shows that this cannot happen.

2. *The assumption that* $\ker D = 0$*.*

Suppose that s is a non-trivial element of the kernel. If s has no zeros we consider the same conformal deformation to get a metric with scalar curvature ≤ 0 which shows that $\mu_g \leq 0$ and hence $\mu_g < \mu_0$.

If s has zeros we argue as in (1) above.

3. *The assumption that g is Euclidean in the neighbourhood of some point.*
We can make the same constructions but now we have a slightly deformed metric on S^3. Calculations show that this is $O(\rho^2)$ and so does not affect the argument.

Higher dimensions

The same argument works in dimension n for spin manifolds which are (conformally) Euclidean in some neighbourhood.

In general:

- For manifolds which are not conformally Euclidean in any neighbourhood one has to take account of the slightly deformed metric on S^n.
- For a conformally flat manifold M there is a problem if M is not spin. The discussion is related to the "Positive Mass Theorem".